Abstract. The cost functions considered are \( c(x, y) = h(x - y) \), with \( h \in C^2(\mathbb{R}^n) \) strictly convex and homogeneous of degree \( p \geq 2 \). We consider multivalued monotone maps with respect to that cost and establish fine properties of these maps such as single-valued a.e., local \( L^\infty \)-estimates, and as a consequence differentiability.

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1. Introduction

In this paper we analyze monotone maps arising in optimal transport for cost functions that are not necessarily quadratic. It is our purpose here to explore further this notion to establish that these maps are single valued a.e. and prove \( L^\infty \)-estimates for these maps minus affine functions. As a consequence we obtain differentiability properties of the maps. The cost functions considered have the form \( c(x, y) = h(x - y) \) where \( h \in C^2(\mathbb{R}^n) \) is strictly convex, nonnegative, and positively homogeneous of degree \( p \) for some \( p \geq 2 \).

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This work is continuation of our paper [GM22] that originated from the fundamental work by Goldman and Otto [GOdf] developing a variational approach to establish regularity of optimal maps for quadratic costs.

From optimal transport theory, if $c(x, y) : D \times D^* \to [0, +\infty)$ is a general cost function, then the optimal map for the Monge problem is given by $T = N_{c, \phi}$ where $\phi$ is $c$-concave and

\[ N_{c, \phi}(x) = \left\{ m \in D^* : \phi(x) + \phi^c(m) = c(x, m) \right\} \]

with $\phi^c(m) = \inf_{x \in D} \left( c(x, m) - \phi(x) \right)$, see for example [GH09, Sect. 3.2]. We say that a multivalued map $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is $h$-monotone if

\[ h(x - \xi) + h(y - \zeta) \leq h(x - \zeta) + h(y - \xi), \]

for all $x, y \in \text{dom}(T)$ and for all $\xi \in T(x)$ and $\zeta \in T(y)$; where $\text{dom}(T) = \{ x \in \mathbb{R}^n : T(x) \neq \emptyset \}$. It is then clear that the optimal map $N_{c, \phi}$ is $h$-monotone. In our analysis we will consider maps $T$ only satisfying (1.1); and that are not necessarily optimal.

The organization of the paper is as follows. Section 2 contains equivalent formulations of $h$-monotonicity, resembling the notion of standard monotone map, that are needed later. Our first result, Theorem 1, is that each $h$-monotone map $T$ is single valued a.e. and consequently the inequality (1.1) holds for a.e. $x, y \in \text{dom}(T)$. Then $T$ induces a push forward measure, Corollary 2. In addition, if $T$ is maximal and locally bounded, it induces a measure Theorem 5. Section 4 contains our main second result, Theorem 6, concerning estimates of an $h$-monotone mapping $T$ minus affine functions. Since weak differentiability is related to approximation by affine functions, the estimates in Section 4 lead to differentiability properties of $h$-monotone maps which is the contents of Section 5. Subsection 5.1 shows that if $T$ is an $h$-monotone map, then the map $Dh(x - Tx)$ has bounded deformation and from [ACDM97, Theorem 7.4] it is weakly differentiable a.e. Finally, Section 6 is a brief appendix containing results used in the proofs.

2. Preliminaries on $h$-Monotone maps

In this section we present equivalent formulations of the definition (1.1) of $h$-monotonicity and integral representation formulas that will be needed to prove our results. From (1.1)
and since $h \in C^2$ we have
\[
0 \leq h(y - \xi) - h(y - \zeta) - (h(x - \xi) - h(x - \zeta))
\]
\[
= \int_0^1 \langle Dh(y - \zeta + s(\zeta - \xi)), \zeta - \xi \rangle ds - \int_0^1 \langle Dh(x - \zeta + s(\zeta - \xi)), \zeta - \xi \rangle ds
\]
\[
= \int_0^1 \int_0^1 \langle D^2h(x - \zeta + s(\zeta - \xi) + t(x - y))(y - x), (\zeta - \xi) \rangle dt ds
\]
\[
= \langle A(x, y; \xi, \zeta)(x - y), \xi - \zeta \rangle, \quad \forall x, y \in \text{dom}(T), \xi \in T(x), \zeta \in T(y).
\]

Therefore (1.1) is equivalent to
\[
(2.1) \quad \langle A(x, y; \xi, \zeta)(x - y), \xi - \zeta \rangle \geq 0, \quad \forall x, y \in \text{dom}(T), \xi \in T(x), \zeta \in T(y)
\]

with
\[
(2.2) \quad A(x, y; \xi, \zeta) = \int_0^1 \int_0^1 D^2h(y - \zeta + s(\zeta - \xi) + t(x - y))dt ds.
\]

The matrix $A(x, y; \xi, \zeta)$ is clearly symmetric, and satisfies $A(x, y; \xi, \zeta) = A(y, x; \zeta, \xi)$ by making the change of variables $t = 1 - t', s = 1 - s'$ in the integral. If $h$ is homogenous of degree $p$ with $p \geq 2$, then $D^2h(z)$ is homogeneous of degree $p - 2$, i.e., $D^2h(\mu z) = \mu^{p-2}D^2h(z)$ for all $\mu > 0$. In addition, if $h$ is strictly convex, then $D^2h(x)$ is positive definite for each $x \in S^{n-1}$, i.e., there is a constant $\lambda > 0$ such that
\[
\langle D^2h(x) v, v \rangle \geq \lambda |v|^2
\]
for all $x \in S^{n-1}$ and all $v \in \mathbb{R}^n$. Since $h$ is $C^2$, there is also a positive constant $\Lambda$ such that
\[
(2.3) \quad \lambda |v|^2 \leq \langle D^2h(x) v, v \rangle \leq \Lambda |v|^2, \quad \forall x \in S^{n-1}, v \in \mathbb{R}^n.
\]

We then have
\[
A(x, y; \xi, \zeta) = \int_0^1 \int_0^1 |y - \zeta + s(\zeta - \xi) + t(x - y)|^{p-2}D^2h\left(\frac{y - \zeta + s(\zeta - \xi) + t(x - y)}{|y - \zeta + s(\zeta - \xi) + t(x - y)|}\right) dt ds
\]
and
\[
(2.4) \quad \lambda \Phi(x, y; \xi, \zeta) |v|^2 \leq \langle A(x, y; \xi, \zeta) v, v \rangle \leq \Lambda \Phi(x, y; \xi, \zeta) |v|^2 \quad \forall v \in \mathbb{R}^n,
\]
with
\[
(2.5) \quad \Phi(x, y; \xi, \zeta) = \int_0^1 \int_0^1 |y - \zeta + s(\zeta - \xi) + t(x - y)|^{p-2} dt ds.
\]

\[\text{One has} h(x) = |x|^2, \text{this is the well-known notion of monotone map that has been the subject of large amount of research and applications, see for example the classical and polished book by H. Brézis [B73].}\]
We also have that \( \Phi(x, y; \xi, \zeta) = 0 \) if and only if \( y - \zeta + s(\zeta - \xi) + t(x - y) = 0 \) for all \( s, t \in [0, 1] \). That is, \( \Phi(x, y; \xi, \zeta) = 0 \) if and only if \( y - \zeta = 0 \), \( \zeta - \xi = 0 \) and \( x - y = 0 \). Therefore \( \Phi(x, y; \xi, \zeta) > 0 \) if and only if \( \zeta \neq y \) or \( \zeta \neq \xi \) or \( x \neq y \).

If \( T \) is \( h \)-monotone, then from (1.1) we obviously have

\[
h(y - Ty) - h(x - Ty) \leq h(y - Tx) - h(x - Tx).
\]

Adding \( h(x - Ay - b) - h(y - Ay - b) \), with \( A \) any \( n \times n \) matrix and \( b \in \mathbb{R}^n \), to the last inequality yields

\[
\begin{align*}
h(y - Ty) - h(x - Ty) + h(x - Ay - b) - h(y - Ay - b) \\
\leq h(y - Tx) - h(x - Tx) + h(x - Ay - b) - h(y - Ay - b) \\
= h(y - Tx) - h(y - Ax - b) + h(x - Ay - b) - h(x - Tx) \\
+ h(y - Ax - b) - h(y - Ay - b) + h(x - Ay - b) - h(x - Ax - b)
\end{align*}
\]

by re writing terms. If we set

\[
G(z_1, z_2, z_3) = h(z_2 - z_3) - h(z_1 - z_3) - h(z_2) + h(z_1),
\]

then (1.1) is equivalent to

\[
(2.6) \ G(x - Ay - b, y - Ay - b, Ty - Ay - b) \leq G(x - Ax - b, y - Ax - b, Tx - Ax - b) + P_{A,b}(x, y),
\]

where

\[
P_{A,b}(x, y) = h(y - Ax - b) - h(y - Ay - b) + h(x - Ay - b) - h(x - Ax - b).
\]

To establish the desired estimate of \( Tx - Ax - b \), the idea is to use the potential theory formula (6.1) with \( v(x) \) equals the left hand side of (2.6) on an appropriate ball and majorize the resulting integrals using (2.6). Recall the formula

\[
(2.7) \ h(\alpha) - h(\beta) = \int_0^1 Dh(\beta + s(\alpha - \beta)) \cdot (\alpha - \beta) \, ds.
\]
Theorem 1. If the assumptions in the previous section, then
\[ T(z_{2} - z_{3}) - h(z_{1} - z_{3}) - (h(z_{2}) - h(z_{1})) \]
\[ = \int_{0}^{1} Dh(z_{1} - z_{3} + s(z_{2} - z_{1})) \cdot (z_{2} - z_{1}) \, ds - \int_{0}^{1} Dh(z_{1} + s(z_{2} - z_{1})) \cdot (z_{2} - z_{1}) \, ds \]
\[ = \int_{0}^{1} \{ Dh(z_{1} - z_{3} + s(z_{2} - z_{1})) - Dh(z_{1} + s(z_{2} - z_{1})) \} \cdot (z_{2} - z_{1}) \, ds \]
\[ = \int_{0}^{1} \left\{ \int_{0}^{1} D^{2}h(z_{1} + s(z_{2} - z_{1}) - t z_{3}) (z_{3}) \, dt \right\} \cdot (z_{2} - z_{1}) \, ds \]
\[ = \int_{0}^{1} \int_{0}^{1} \left\{ D^{2}h(z_{1} + s(z_{2} - z_{1}) - t z_{3}, z_{3}, z_{1} - z_{2}) \right\} \, ds \, dt. \tag{2.8} \]

And we can also write a similar integral expression for \( P_{A,b}(x, y) \):
\[ P_{A,b}(x, y) = h(y - Ax - b) - h(y - Ay - b) - (h(x - Ax - b) - h(x - Ay - b)) \]
\[ = \int_{0}^{1} Dh(y - Ay - b + s(Ay - Ax)) \cdot (Ay - Ax) \, ds - \int_{0}^{1} Dh(x - Ay - b + s(Ay - Ax)) \cdot (Ay - Ax) \, ds \]
\[ = \int_{0}^{1} \{ Dh(y - Ay - b + s(Ay - Ax)) - Dh(x - Ay - b + s(Ay - Ax)) \} \cdot (Ay - Ax) \, ds \]
\[ = \int_{0}^{1} \int_{0}^{1} \left\{ D^{2}h(x - Ay - b + s(Ay - Ax) + t(y - x)) (y - x), Ay - Ax \right\} \, ds \, dt. \tag{2.9} \]

And we can also write a similar integral expression for \( P_{A,b}(x, y) \):
\[ P_{A,b}(x, y) = h(y - Ax - b) - h(y - Ay - b) - (h(x - Ax - b) - h(x - Ay - b)) \]
\[ = \int_{0}^{1} Dh(y - Ay - b + s(Ay - Ax)) \cdot (Ay - Ax) \, ds - \int_{0}^{1} Dh(x - Ay - b + s(Ay - Ax)) \cdot (Ay - Ax) \, ds \]
\[ = \int_{0}^{1} \{ Dh(y - Ay - b + s(Ay - Ax)) - Dh(x - Ay - b + s(Ay - Ax)) \} \cdot (Ay - Ax) \, ds \]
\[ = \int_{0}^{1} \int_{0}^{1} \left\{ D^{2}h(x - Ay - b + s(Ay - Ax) + t(y - x)) (y - x), Ay - Ax \right\} \, ds \, dt. \tag{2.9} \]

3. \textit{h-monotone maps are single valued a.e.}

The main result of this section is the following.

**Theorem 1.** If \( T : \mathbb{R}^{n} \to \mathcal{P}(\mathbb{R}^{n}) \) is a multivalued map that is \( h \)-monotone, with \( h \) satisfying the assumptions in the previous section, then \( T(x) \) is a singleton for all points \( x \in \text{dom}(T) \) except on a set of measure zero.

**Proof.** Let
\[ S = \{ x \in \text{dom}(T) : T(x) \text{ is not a singleton} \}, \]
we will prove that \( S \) has Lebesgue measure zero. For each \( k \geq 1 \) integer, let
\[ S_{k} = \{ x \in \text{dom}(T) : \text{diam}(T(x)) > 1/k \}. \]
We have \( S = \bigcup_{k=1}^{\infty} S_{k} \). Also for each \( k \geq 1 \) integer, let \( \{ B_{j} \}_{j=1}^{\infty} \) be a family of Euclidean balls in \( \mathbb{R}^{n} \) each one with radius \( \epsilon/k \) with \( \epsilon > 0 \) small to be chosen later after inequality (3.9), depending only \( \Lambda/\lambda \), the constants in (2.3), and such that \( \mathbb{R}^{n} = \bigcup_{j=1}^{\infty} B_{j} \). Let
\[ B_{j}^{*} = \{ x \in \text{dom}(T) : T(x) \cap B_{j} \neq \emptyset \}, \]
and

$$S_{kj} = S_k \cap B_j^*.$$  

We have $\bigcup_{j=1}^{\infty} B_j^* = \text{dom}(T)$ and so $S = \bigcup_{k,j=1}^{\infty} S_{kj}$. We shall prove that $|S_{kj}| = 0$ for all $k$ and $j$. In order to do this we recall that a point $x \in \mathbb{R}^n$ is a density point for the set $E \subset \mathbb{R}^n$, with $E$ not necessarily Lebesgue measurable, if

$$\limsup_{r \to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = 1$$

where $| \cdot |$ denotes the Lebesgue outer measure and $B_r(x)$ is the Euclidean ball with center $x$ and radius $r$. We shall prove in Lemma [16] that if $E \subset \mathbb{R}^n$ is any set, then almost all points in $E$ are density points for $E$. In view of this, if we prove that each $x_0 \in S_{kj}$ is not a density point for $S_{kj}$, then it follows that $|S_{kj}| = 0$.

Let us fix $x_0 \in S_{kj}$. Then $\text{diam}(T(x_0)) > 1/k$ and $T(x_0) \cap B_j \neq \emptyset$, so we can pick $y_1 \in T(x_0) \cap B_j$. There exists $y_2 \in T(x_0)$ such that $|y_1 - y_2| \geq 1/2k$, because otherwise $T(x_0) \subset B_{1/2k}(y_1)$ which would imply that $\text{diam}(T(x_0)) \leq 1/k$. Also notice that $y_2 \notin B_j$ if $\epsilon < 1/4$, since $B_j$ has radius $\epsilon/k$. In addition, if $x_j$ is the center of $B_j$, it follows that $|x_j - y_2| \geq |y_2 - y_1| - |y_1 - x_j| \geq 1 - \frac{\epsilon}{2k} \geq \frac{2 \epsilon}{k}$ if $\epsilon < 1/6$. And so for each $\xi \in B_j$, $|\xi - y_2| \geq |x_j - y_2| - |\xi - x_j| \geq \frac{1}{2k} - \frac{\epsilon}{k} \geq \frac{2 \epsilon}{k}$ if $\epsilon < 1/8$.

Set

$$y_2 - y_1 = e.$$

Given $x \neq x_0$, $x \in \text{dom}(T)$, let $z = x - x_0 - \left((x - x_0) \cdot \frac{e}{|e|}\right) \frac{e}{|e|}$. Then $z \cdot \frac{e}{|e|} = 0$, $(x - x_0) \cdot \frac{z}{|z|} = |z|$ and

$$x - x_0 = \left((x - x_0) \cdot \frac{e}{|e|}\right) \frac{e}{|e|} + \left((x - x_0) \cdot \frac{z}{|z|}\right) \frac{z}{|z|}.$$  

If $\delta$ is the angle between the unit vectors $\frac{x - x_0}{|x - x_0|}$ and $\frac{e}{|e|}$, then we have $0 < \delta < \pi$ and

$$\frac{x - x_0}{|x - x_0|} = \cos \delta \frac{e}{|e|} + \sin \delta \frac{z}{|z|}.$$  

Let $\xi \in T(x)$ and consider the matrix $A(x, x_0; \xi, y_2)$ defined by (2.2). From (2.4),

$$\lambda \Phi(x, x_0; \xi, y_2) Id \leq A(x, x_0; \xi, y_2) \leq \Lambda \Phi(x, x_0; \xi, y_2) Id$$

and since $x \neq x_0$, $\Phi(x, x_0; \xi, y_2) > 0$. To simplify the notation we write $A(x, x_0; \xi, y_2) = A$ and $\Phi(x, x_0; \xi, y_2) = \Phi$.

To show that $x_0$ is not a density point for $S_{kj}$ we analyze the sizes of the angles between various vectors using the $h$-monotonicity. For the sake of clarity we divide the proof into three steps.

---

2 A priori we do not know if the set $S_{kj}$ is Lebesgue measurable and therefore we cannot apply the Lebesgue differentiation theorem directly.
Step 1. We shall estimate

\[ F(\delta) := \angle \left( A^{1/2} \frac{x-x_0}{|x-x_0|}, A^{1/2} \frac{e}{|e|} \right), \]

proving (3.4).

We have

\[ \cos F(\delta) = \frac{\left( A^{1/2} \frac{x-x_0}{|x-x_0|}, A^{1/2} \frac{e}{|e|} \right)}{|A^{1/2} \frac{x-x_0}{|x-x_0|}| |A^{1/2} \frac{e}{|e|}|}, \]

and

\[ \left( A^{1/2} \frac{x-x_0}{|x-x_0|}, A^{1/2} \frac{e}{|e|} \right) = \left( A^{1/2} \left( \cos \delta \frac{e}{|e|} + \sin \delta \frac{z}{|z|} \right), A^{1/2} \left( \cos \delta \frac{e}{|e|} + \sin \delta \frac{z}{|z|} \right) \right) \]

\[ = \cos \delta \left| A^{1/2} \frac{e}{|e|} \right|^2 + \sin \delta \left( A^{1/2} \frac{z}{|z|}, A^{1/2} \frac{e}{|e|} \right) \]

\[ = \cos^2 \delta \left| A^{1/2} \frac{e}{|e|} \right|^2 + 2 \sin \delta \cos \delta \left( A^{1/2} \frac{z}{|z|}, A^{1/2} \frac{e}{|e|} \right) + \sin^2 \delta \left| A^{1/2} \frac{z}{|z|} \right|^2. \]

Hence

\[ \cos F(\delta) = \frac{\cos \delta \left| A^{1/2} \frac{e}{|e|} \right|^2 + \sin \delta \left( A^{1/2} \frac{z}{|z|}, A^{1/2} \frac{e}{|e|} \right)}{\left| A^{1/2} \frac{e}{|e|} \right| \sqrt{\cos^2 \delta \left| A^{1/2} \frac{e}{|e|} \right|^2 + 2 \sin \delta \cos \delta \left( A^{1/2} \frac{z}{|z|}, A^{1/2} \frac{e}{|e|} \right) + \sin^2 \delta \left| A^{1/2} \frac{z}{|z|} \right|^2}}. \]

\[ \left( A^{1/2} \frac{x-x_0}{|x-x_0|}, A^{1/2} \frac{x-x_0}{|x-x_0|} \right) = \left( A^{1/2} \left( \cos \delta \frac{e}{|e|} + \sin \delta \frac{z}{|z|} \right), A^{1/2} \left( \cos \delta \frac{e}{|e|} + \sin \delta \frac{z}{|z|} \right) \right) \]

\[ = \cos^2 \delta \left| A^{1/2} \frac{e}{|e|} \right|^2 + 2 \sin \delta \cos \delta \left( A^{1/2} \frac{z}{|z|}, A^{1/2} \frac{e}{|e|} \right) + \sin^2 \delta \left| A^{1/2} \frac{z}{|z|} \right|^2. \]

(3.2)
From (3.1)

\[
\frac{\lambda}{\Lambda} = \frac{\lambda \Phi}{\Lambda \Phi} \leq C := \frac{A^{1/2} \frac{z}{|z|}}{|A^{1/2} \frac{e}{|e|}|^2} \leq \frac{\Lambda \Phi}{\lambda \Phi} = \frac{\Lambda}{\lambda}.
\]

If

\[
B := \left\langle \frac{A^{1/2} \frac{z}{|z|}}{|A^{1/2} \frac{e}{|e|}|}, \frac{A^{1/2} \frac{e}{|e|}}{|A^{1/2} \frac{e}{|e|}|} \right\rangle,
\]

then by Cauchy-Schwarz

\[
|B| \leq \frac{A^{1/2} \frac{z}{|z|}}{|A^{1/2} \frac{e}{|e|}|} = \sqrt{C} \leq \left(\frac{\Lambda \Phi}{\lambda \Phi}\right)^{1/2} = \sqrt{\frac{\Lambda}{\lambda}}.
\]

Then

\[
\cos F(\delta) = \begin{cases} 
1 + B \tan \delta & \text{if } 0 < \delta < \pi/2 \\
- \frac{1 + B \tan \delta}{\sqrt{1 + C \tan^2 \delta + 2 B \tan \delta}} & \text{if } \pi/2 < \delta < \pi,
\end{cases}
\]

and setting

\[
(3.3) \quad g(s) = \begin{cases} 
\frac{1 + B s}{\sqrt{1 + C s^2 + 2 B s}} & \text{if } s \in (0, \infty) \\
- \frac{1 + B s}{\sqrt{1 + C s^2 + 2 B s}} & \text{if } s \in (-\infty, 0)
\end{cases}
\]

we have

\[
\cos F(\delta) = g(\tan \delta).
\]

Notice that since \( |B| \leq \sqrt{C} \) we have \( 1 + C s^2 + 2 B s \geq 0 \) for all \( s \in \mathbb{R} \). If \( |B| < \sqrt{C} \), then \( 1 + C s^2 + 2 B s > 0 \) for all \( s \in \mathbb{R} \); and if \( |B| = \sqrt{C} \), then \( 1 + C s^2 + 2 B s = 1 + B^2 s^2 + 2 B s = (1 + B s)^2 \).
which is strictly positive except when $s = -1/B$. For $s > 0$, let us now estimate

$$\Delta(s) = 1 - g(s) = 1 - \frac{1 + B s}{\sqrt{1 + C s^2 + 2 B s}}$$

$$= \frac{\sqrt{1 + C s^2 + 2 B s} - (1 + B s)}{\sqrt{1 + C s^2 + 2 B s}}$$

$$= \frac{1 + C s^2 + 2 B s - (1 + B s)^2}{(1 + B s + \sqrt{1 + C s^2 + 2 B s}) \sqrt{1 + C s^2 + 2 B s}}$$

$$= \frac{(C - B^2) s^2}{(1 + B s + \sqrt{1 + C s^2 + 2 B s}) \sqrt{1 + C s^2 + 2 B s}}$$

Since $-\sqrt{C} \leq B \leq \sqrt{C}$, we have

$$1 + C s^2 + 2 B s \geq 1 + C s^2 - 2 \sqrt{C} s = (1 - \sqrt{C} s)^2$$

if $s > 0$, and

$$1 + C s^2 + 2 B s \geq 1 + C s^2 + 2 \sqrt{C} s = (1 + \sqrt{C} s)^2$$

if $s < 0$; and so

$$1 + C s^2 + 2 B s \geq (1 - \sqrt{C} |s|)^2$$

for $-\infty < s < \infty$. Therefore, if $|s| \leq 1/(2 \sqrt{C})$, then we get $1 - \sqrt{C} |s| \geq 1/2$. Also $1 + B s \geq 1 - \sqrt{C} s$ if $s > 0$, and $1 + B s \geq 1 + \sqrt{C} s$ for $s < 0$. So $1 + B s \geq 1 - \sqrt{C} |s|$. Hence

$$0 \leq \Delta(s) \leq 2 \left(C - B^2\right) s^2 \quad \text{for } 0 < s \leq 1/(2 \sqrt{C}).$$

In particular, since $\lambda/\Lambda \leq C \leq \Lambda/\lambda$ and $|B| \leq \sqrt{C}$, we obtain the bound

$$0 \leq \Delta(s) \leq 2 \frac{\Lambda^2}{\lambda} s^2 \quad \text{for } 0 < s \leq (1/2) \sqrt{\lambda/\Lambda}.$$

This implies that

$$1 - \cos F(\delta) \leq 4 \frac{\Lambda}{\lambda} \tan^2 \delta$$

for $\delta$ such that $0 < \tan \delta \leq (1/2) \sqrt{\lambda/\Lambda}$. Since $\tan \delta \sim \delta$ for $\delta$ sufficiently small, we then obtain the estimate

$$1 - \cos F(\delta) \leq 8 \frac{\Lambda}{\lambda} \delta^2$$

for all $0 < \delta \leq \delta_0$, with $\delta_0$ a positive number depending only on $\lambda/\Lambda$. Hence

$$F(\delta) \leq \arccos \left(1 - 8 \frac{\Lambda}{\lambda} \delta^2\right) \quad \text{for } 0 < \delta \leq \delta_0.$$

On the other hand, $\frac{\arccos(1 - x^2)}{x} \rightarrow \sqrt{2}$ as $x \rightarrow 0^+$, it follows that

(3.4) \quad \frac{\arccos(1 - x^2)}{x} \rightarrow \sqrt{2}$ as $x \rightarrow 0^+$, it follows that

$$F(\delta) \leq C(\Lambda/\lambda) \delta \quad \text{for } 0 < \delta \leq \delta_0.$$
Therefore, from the definitions of $\delta$ and $F(\delta)$ we obtain

\[(3.5) \quad \angle \left( A(x, x_0; \xi, y_2) \right)^{1/2} (x - x_0), A(x, x_0; \xi, y_2) \right)^{1/2} \leq C(\Lambda/\lambda) \angle(x - x_0, e), \]

for all $\xi \in T(x)$ if $\angle(x - x_0, e) \leq \delta_0$, $x \neq x_0$, $x \in \text{dom}(T)$. On the other hand, from the monotonicity \[(2.1) \]

\[\langle A(x, x_0; \xi, y_2)(x - x_0), \xi - y_2 \rangle \geq 0, \quad \forall \xi \in T(x) \]

that is, $\langle A(x, x_0; \xi, y_2)^{1/2}(x - x_0), A(x, x_0; \xi, y_2)^{1/2}(\xi - y_2) \rangle \geq 0$ for all $\xi \in T(x)$ and therefore

\[\angle \left( A(x, x_0; \xi, y_2) \right)^{1/2} (x - x_0), A(x, x_0; \xi, y_2) \right)^{1/2} \leq \pi/2, \forall \xi \in T(x). \]

Hence from \[(3.5)\]

\[(3.6) \quad \angle \left( A(x, x_0; \xi, y_2) \right)^{1/2} (\xi - y_2), A(x, x_0; \xi, y_2) \right)^{1/2} \leq \pi/2 + C(\Lambda/\lambda) \angle(x - x_0, e), \forall \xi \in T(x), \]

when $\angle(x - x_0, e) \leq \delta_0$, $x \neq x_0$, $x \in \text{dom}(T)$.

**Step 2.** Let $\Gamma = \{x : \angle(x - x_0, e) \leq \delta_0\}$. We shall prove that

\[(3.7) \quad T(\Gamma) \cap B_j = \emptyset, \]

for $e$ sufficiently small depending only on $\Lambda/\lambda; B_j = B_{e/k}(x_j)$.

From the choice of $y_2$ and the definition of $e$, it follows that the cone $C$ with vertex $y_2$, axis $e$, and opening $\pi + \theta_0$ does not intersect the ball $B_j$ when $\theta_0$ is small. Suppose by contradiction that \[(3.7)\] does not hold, that is, there is $\xi \in T(x) \cap B_j$ for some $x \in \Gamma$, and let $\theta$ be the angle between $\xi - y_2$ and $e$. Since $\xi \in B_j$, we have $\theta \geq \pi/2 + \delta_1$ for some $\delta_1 > 0$ depending on the initial configuration of the balls $B_j$. To obtain a contradiction, we first right down the left hand side of \[(3.6)\] in terms of the angle $\theta$ and will show that is close to $\pi$ when $\theta \sim \pi$, for $e$ sufficiently small, contradicting \[(3.6)\].

In order to do this, we proceed as before as in the writing of $F(\delta)$ letting now $\zeta = \xi - y_2 - \left( (\xi - y_2) \cdot \frac{e}{|e|} \right) \frac{e}{|e|}$. Then $\zeta \cdot \frac{e}{|e|} = 0$, $(\xi - y_2) \cdot \frac{\zeta}{|\zeta|} = |\zeta|$, and

\[
\xi - y_2 = \left( (\xi - y_2) \cdot \frac{\zeta}{|\zeta|} \right) \frac{\zeta}{|\zeta|} + \left( (\xi - y_2) \cdot \frac{e}{|e|} \right) \frac{e}{|e|}
\]

\[
= \sin \theta \frac{\zeta}{|\zeta|} + \cos \theta \frac{e}{|e|};
\]

recall $\theta$ is the angle between $\xi - y_2$ and $e$. Then the left hand side of \[(3.6)\] can be written as

\[(3.8) \quad G(\theta) = \angle \left( A(x, x_0; \xi, y_2)^{1/2} (\xi - y_2), A(x, x_0; \xi, y_2)^{1/2} \right) \]
and since \( \theta > \pi/2 \), it follows as in (3.2) that
\[
\cos G(\theta) = -\frac{1 + \tilde{B} \tan \theta}{\sqrt{1 + \tilde{C} \tan^2 \theta + 2 \tilde{B} \tan \theta}}
\]
with
\[
\tilde{B} = \frac{A^{1/2} \bar{\zeta}^{1/2} |e|}{|\zeta|}, \quad \tilde{C} = \frac{A^{1/2} \bar{\zeta}^{1/2} |e|}{|\zeta|}.
\]

Let us now analyze what happens for \( \theta \) close to \( \pi \). For \( s < 0 \) and with \( g \) defined as in (3.3) but with \( \tilde{B} \) and \( \tilde{C} \) instead of \( B \) and \( C \), we have
\[
\tilde{\Delta}(s) = -1 - g(s) = -1 + \frac{1 + \tilde{B} s}{\sqrt{1 + \tilde{C} s^2 + 2 \tilde{B} s}}
\]
\[
= -\frac{\sqrt{1 + \tilde{C} s^2 + 2 \tilde{B} s} - (1 + \tilde{B} s)}{\sqrt{1 + \tilde{C} s^2 + 2 \tilde{B} s}}
\]
\[
= -\frac{\tilde{C} - B^2}{\left(1 + \tilde{B} s + \sqrt{1 + \tilde{C} s^2 + 2 \tilde{B} s}\right) \sqrt{1 + \tilde{C} s^2 + 2 \tilde{B} s}}.
\]

Applying the estimates for the last denominator obtained in Step 1 yields
\[
\tilde{\Delta}(s) \geq -2 \left(\tilde{C} - B^2\right) s^2 \quad \text{for } -1/\left(2 \sqrt{\tilde{C}}\right) < s < 0.
\]

In particular, since \( \lambda/\Lambda \leq \tilde{C} \leq \Lambda/\lambda \), and \( |\tilde{B}| \leq \sqrt{\tilde{C}} \), we obtain the bound
\[
\tilde{\Delta}(s) \geq -2 \frac{\Lambda}{\lambda} s^2 \quad \text{for } -1/\left(2 \sqrt{\tilde{C}}\right) < s < 0.
\]

This implies that
\[
-1 - \cos G(\theta) \geq -2 \frac{\Lambda}{\lambda} \tan^2 \theta
\]
for \( \theta \) such that \(-1/2 \sqrt{\lambda/\Lambda} < \tan \theta < 0 \). Now \( \tan \theta \sim \theta - \pi \) when \( \theta \to \pi^- \), and so
\[
-1 + 8 \frac{\Lambda}{\lambda} (\pi - \theta)^2 \geq \cos G(\theta),
\]
for \( \pi - \theta_1 < \theta < \pi \) with \( \theta_1 > 0 \) small depending only on \( \Lambda/\lambda \); which implies the following lower bound for the left hand side of (3.6)
\[
(3.9) \quad G(\theta) \geq \arccos(-1 + 8 \frac{\Lambda}{\lambda} (\pi - \theta)^2),
\]
for all \( \pi - \theta_1 < \theta < \pi \). We now select the radius of the ball \( B_j \) so that for any \( \xi \in B_j \) the angle \( \theta \) satisfies \( \pi - \theta_1 < \theta < \pi \), i.e., \( B_j = B_{\epsilon/k}(x_j) \) with \( \epsilon \) to be determined in a moment.

Recall once again that \( x_0 \in S_{kj} \), \( y_1 \in Tx_0 \cap B_j \), and \( y_2 \in Tx_0 \) with \( |y_2 - y_1| \geq 1/2k \). Then
\[
|y_2 - x_j| > |y_2 - y_1| - |y_j - y_1| \geq \frac{1}{2k} - \frac{\epsilon}{k} = \left( \frac{1}{2} - \frac{\epsilon}{k} \right) \frac{1}{k} = \left( \frac{1}{2} - 1 \right) \frac{\epsilon}{k};
\]
that is, \( y_2 \notin B_{\left(\frac{1}{2} - 1\right)}(x_j) \). For each \( \delta' \) large there is \( \epsilon \) sufficiently small such that \( \frac{1}{2\epsilon} - 1 = 1 + \delta' \). Let \( \alpha \) be the angle between the vectors \( \overrightarrow{y_2 \xi} \) and \( -e = \overrightarrow{y_2 y_1} \), and consider the convex hull of \( B_j \) and \( y_2 \), i.e., the ice-cream cone containing \( B_j \). If \( \beta \) is the angle between the vector \( \overrightarrow{y_2 x_j} \) and the edge of the convex hull, we have \( \alpha \leq 2\beta \), see Figure 1. So
\[
\sin \alpha \leq 2 \sin \beta = \frac{\epsilon/k}{|y_2 - x_j|} \leq 2 \frac{\epsilon/k}{(1 - 1/2\epsilon)k} = \frac{2}{1 + \delta'},
\]
a number than can be made arbitrarily small taking \( \epsilon \) small, in particular, it can be made smaller than \( \sin \theta_1 \). Therefore, with this choice of \( \epsilon \), the ball \( B_j \) is determined and the inequality (3.9) can be applied when \( \xi \in B_j \). Then combining (3.6), (3.8), and (3.9) we get that
\[
\arccos(-1 + 8 \frac{\Lambda}{\lambda} (\pi - \theta)^2) \leq \pi/2 + C(\Lambda/\lambda) \angle(x - x_0, e)
\]
for \( \pi - \theta_1 < \theta < \pi \). Since \( x \in \Gamma \), \( \angle(x - x_0, e) \leq \delta_0 \). Thus, if \( \theta \to \pi^- \) this yields a contradiction since \( \arccos(-1) = \pi \). The proof of (3.7) is then complete.

**Step 3.** We are now in a position to prove that \( x_0 \in S_{kj} \) cannot be a point of density for \( S_{kj} \). Recall \( S_{kj} = S_k \cap B_{j}^* \) where \( B_{j}^* = \{x \in \text{dom}(T) : Tx \cap B_j \neq \emptyset \} \). From (3.7), it follows that \( B_{j}^* \cap \Gamma = \emptyset \) with \( \Gamma \) the cone in Step 2. Let \( B_r(x_0) \) be the Euclidean ball centered at \( x_0 \) with radius \( r \), then
\[
S_{kj} \cap B_r(x_0) = \left( S_{kj} \cap B_r(x_0) \cap \Gamma \right) \cup \left( S_{kj} \cap B_r(x_0) \cap \Gamma^c \right) = S_{kj} \cap B_r(x_0) \cap \Gamma^c,
\]
and therefore
\[
\frac{|S_{kj} \cap B_r(x_0)|}{|B_r(x_0)|} \leq \frac{|\Gamma^c \cap B_r(x_0)|}{|B_r(x_0)|} = c_{\delta_0} < 1,
\]
for all \( r \) and so from Lemma \(16 \) \( x_0 \) is not a point of density for \( S_{kj} \). This completes the proof of Theorem 1. \( \square \)
3.1. **Inverse maps.** Given a multivalued map $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ the domain of $T$ is the set $\text{dom}(T) = \{x \in \mathbb{R}^n : Tx \neq \emptyset\}$ and the range of $T$ the set $\text{ran}(T) = \bigcup_{x \in \mathbb{R}^n} Tx$. The inverse of $T$ is the multivalued map $T^{-1} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ defined by $T^{-1}y = \{x \in \mathbb{R}^n : y \in Tx\}$. Clearly $\text{dom}(T^{-1}) = \text{ran}(T)$.

If $T$ is $h$-monotone with $h$ even, then $T^{-1}$ is $h$-monotone and from Theorem 1 $T^{-1}$ is single-valued a.e. and we obtain the following.

**Corollary 2** (of Aleksandrov type). If $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map that is $h$-monotone, with $h$ satisfying the assumptions of Theorem 1 then the set

$$S = \{p \in \mathbb{R}^n : \text{there exist } x, y \in \mathbb{R}^n, x \neq y, \text{ such that } p \in T(x) \cap T(y)\}$$

has measure zero.

**Proof.** The map $T^{-1} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is $h$-monotone and from Theorem 1 is single valued a.e. Since $S = \{p \in \mathbb{R}^n : T^{-1}p \text{ is not a singleton}\}$, the corollary follows. □

Using the last corollary we can define the push forward measure of an $h$-monotone map. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $f \geq 0$, and let $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be an $h$-monotone map that is measurable, i.e., $T^{-1}(E)$ is a Lebesgue measurable subset of $\mathbb{R}^n$ for each $E \subset \mathbb{R}^n$ Lebesgue measurable. Define

$$\mu(E) = \int_{T^{-1}(E)} f(x) dx.$$

Then $\mu$ is $\sigma$-additive. In fact, if $\{E_i\}_{i=1}^\infty$ are disjoint Lebesgue measurable sets, then it follows from Corollary 2 that $|T^{-1}(E_i) \cap T^{-1}(E_j)| = 0$ for $i \neq j$; $T^{-1}(E) = \{x \in \mathbb{R}^n : T(x) \cap E \neq \emptyset\}$.

3.2. **Maximal $h$-monotone maps.** If $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map, then the graph of $T$ is by definition $\text{graph}(T) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in Tx\}$. If $T_1, T_2$ are two multivalued maps, then $T_1 \preceq T_2$ if $\text{graph}(T_1) \subseteq \text{graph}(T_2)$, that is, for each $x \in \mathbb{R}^n$, $T_1 x \subset T_2 x$. The relation $\preceq$ is a partial order on the set of all multivalued maps, and therefore the class of all multivalued maps with the relation $\preceq$ is a partially ordered set. Let $\mathcal{M}_h$ denote the class of all multivalued maps that are $h$-monotone. Then the set $(\mathcal{M}_h, \preceq)$ is inductive, that is, if $S \subset \mathcal{M}_h$ is a totally ordered set or chain (that is, given $T_1, T_2 \in S$ then either $T_1 \preceq T_2$ or $T_2 \preceq T_1$), then $S$ has an upper bound in $\mathcal{M}_h$, i.e., there exists $T \in \mathcal{M}_h$ such that $R \preceq T$ for all $R \in S$. Indeed, let $\hat{T}$ be the map with $\text{graph}(\hat{T}) = \bigcup_{T \in S} \text{graph}(T)$, i.e., $\hat{T}x = \bigcup_{T \in S} Tx$. Let us show that $\hat{T} \in \mathcal{M}_h$. If $x, y \in \text{dom}(\hat{T})$, $\xi \in \hat{T}x$, and $\zeta \in \hat{T}y$, then there exists $T_1, T_2 \in S$ such that $\xi \in T_1 x$ and $\zeta \in T_2 y$. Since $S$ is a chain, it follows that $T_1 \preceq T_2$ or $T_2 \preceq T_1$, that is, $T_1 z \subset T_2 z$ or $T_2 z \subset T_1 z$ for all $z$. In particular, $\xi \in T_2 x$ or $\zeta \in T_1 y$ and the $h$-monotonicity of $\hat{T}$ follows from the $h$-monotonicity of $T_1$ or $T_2$. Therefore $(\mathcal{M}_h, \preceq)$ is inductive and from Zorn’s lemma, $(\mathcal{M}_h, \preceq)$ has a maximal element, i.e., there exists $T \in \mathcal{M}_h$ such that $R \preceq T$ for all $R \in \mathcal{M}_h$.

We say $T \in \mathcal{M}_h$ is maximal $h$-monotone if whenever $T' \in \mathcal{M}_h$ satisfies $T \preceq T'$ then we must have $T' = T$. 
Given $T \in \mathcal{M}_h$, there exists $\tilde{T} \in \mathcal{M}_h$ such that $\tilde{T}$ is maximal $h$-monotone with $T \leq \tilde{T}$. Indeed, consider the class $\mathcal{R}$ of all $R \in \mathcal{M}_h$ with $T \leq R$. If $S \subset \mathcal{R}$ is any chain, then proceeding as before the map defined by $\tilde{R}x = \bigcup_{R \in S} Rx$ is $h$-monotone and is an upper bound for $S$. Therefore by Zorn’s lemma, $\mathcal{R}$ has a maximal element.

We have the following characterization: The map $T \in \mathcal{M}_h$ is maximal $h$-monotone if given $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying the condition

$$h(x - \xi) + h(y - \zeta) \leq h(x - \zeta) + h(y - \xi)$$

for each $y \in \text{dom}(T)$ and for all $\zeta \in Ty$, then we must have $\xi \in Tx$. Indeed, suppose there exists $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $h(x_0 - \xi_0) + h(y - \zeta_0) \leq h(x_0 - \zeta_0) + h(y - \xi_0)$ for all $y \in \text{dom}(T)$ and for all $\zeta \in Ty$, with $\xi_0 \notin T_{x_0}$. If we define $T'x = Tx$ for $x \neq x_0$ and $T'x = Tx_0$ for $x = x_0$, then $T'$ is $h$-monotone and $\text{graph}(T) \nsubseteq \text{graph}(T')$; so $T$ is not maximal. Reciprocally, if the condition holds and $T \leq T'$ with $T' \in \mathcal{M}_h$, then we want to prove that $T'x \subset Tx$ for all $x \in \mathbb{R}^n$. If $x \in \text{dom}(T)$, since $T \leq T'$, then $x \in \text{dom}(T')$ and let $\xi \in T'x$. Since $T'$ is $h$-monotone, $h(x - \xi) + h(y - \zeta) \leq h(x - \zeta) + h(y - \xi)$ for all $y \in \text{dom}(T')$ and for all $\zeta \in T'y$. Since $\text{dom}(T) \subset \text{dom}(T')$, the last inequality holds for all $y \in \text{dom}(T)$ and for all $\zeta \in Ty$, and hence $\xi \in Tx$. It remains to prove that if $Tx = \emptyset$, then $T'x = \emptyset$. Suppose $T'x \neq \emptyset$. Then there is $\zeta \in T'x$, and since $T'$ is $h$-monotone we have $h(x - \zeta) + h(y - \xi) \leq h(x - \xi) + h(y - \xi)$ for all $y \in \text{dom}(T')$ and for all $\zeta \in T'y$. In particular, this holds for all $y \in \text{dom}(T)$ and for all $\zeta \in Ty$. Consequently, $\xi \in Tx$ and so $Tx \neq \emptyset$.

**Theorem 3.** Let $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map that is maximal $h$-monotone and locally bounded, and let $Z \subset \mathbb{R}^n$ be a set of measure zero such that $T$ is single valued in $\mathbb{R}^n \setminus Z$. Then $T$ is continuous relative to $\mathbb{R}^n \setminus Z$.

**Proof.** Let $x_0 \in \mathbb{R}^n \setminus Z$ and suppose $T$ is discontinuous relative to $\mathbb{R}^n \setminus Z$ at $x_0$. Then there exists $\delta > 0$ such that for each $\epsilon > 0$ there exists $x_\epsilon \in \mathbb{R}^n \setminus Z$ such that $|x_\epsilon - x_0| < \epsilon$ and $|p_\epsilon - p_0| > \delta$ with $p_0 \in T(x_0)$ and $p_\epsilon \in T(x_\epsilon)$. Since $T$ is locally bounded, it follows that there exists a constant $M > 0$ such that $|p_\epsilon| \leq M$ for all $\epsilon < 1$. Then there exists a subsequence $p_{\epsilon_j} \to p_1$ as $\epsilon_j \to 0$, and so $|p_1 - p_0| \geq \delta$. On the other hand, from the $h$-monotonicity

$$h(x_{\epsilon_j} - p_{\epsilon_j}) + h(x - p) \leq h(x_{\epsilon_j} - p) + h(x - p_{\epsilon_j}) \quad \forall p \in T x.$$

Letting $\epsilon_j \to 0$ yields $h(x_0 - p_1) + h(x - p) \leq h(x_0 - p) + h(x - p_1)$ for all $p \in T x$, and since $T$ is maximal, we get $p_1 \in T(x_0)$, that is, $T$ is not single valued at $x_0$, a contradiction. \qed

**Corollary 4.** Under the assumptions of Theorem 3 we have that $T(\mathbb{R}^n)$ is a Lebesgue measurable set in $\mathbb{R}^n$.

**Proof.** Let $T_i$ be the $i$-th component of $T$. Let $Z$ be the set of measure zero such that $T$ is single valued in $\mathbb{R}^n \setminus Z$. The function $T_i$ is continuous relative to $\mathbb{R}^n \setminus Z$ and therefore for each $\alpha$ the set $\{x \in \mathbb{R}^n \setminus Z : T_i(x) \leq \alpha\}$ is relatively closed and so Lebesgue measurable. \qed
Theorem 5. Under the assumptions of Theorem 3, the class
\[ \Sigma = \{ E \subset \mathbb{R}^n : T(E) \text{ is Lebesgue measurable} \} \]
is a σ-algebra, and the set function
\[ \mu(E) = |T(E)| \]
is σ-additive in Σ, that is, \((\mathbb{R}^n, \Sigma, \mu)\) is a measure space.

Proof. If \(E_k \in \Sigma, k = 1, 2, \cdots\), then \(T\left(\bigcup_{k=1}^{\infty} E_k\right) = \bigcup_{k=1}^{\infty} T(E_k) \in \Sigma\).

For any set \(E \subset \mathbb{R}^n\) we have the following formula
\[ T(\mathbb{R}^n \setminus E) = [T(\mathbb{R}^n) \setminus T(E)] \cup [T(\mathbb{R}^n \setminus E) \cap T(E)] . \]

Suppose \(E \in \Sigma\), then from Corollary 2 the set \(T(\mathbb{R}^n \setminus E) \cap T(E)\) has measure zero and then from Corollary 4 we obtain \(\mathbb{R}^n \setminus E \in \Sigma\).

To prove the σ-additivity, the proof is the same as the proof of [G16, Theorem 1.1.13] with Lemma 1.1.12 there now replaced by Corollary 2 and \(\partial u\) replaced by \(T\).

\[ \square \]

4. \(L^\infty\)-estimates for \(Tx - Ax - b\) with \(T h\)-monotone

The purpose of this section is to prove the following local \(L^\infty\)-estimate for \(h\)-monotone maps.

Theorem 6. Let \(p \geq 2, h \in C^2(\mathbb{R}^n)\) positively homogeneous of degree \(p\), strictly convex and nonnegative. If \(T \in L^{p-1}_{\text{loc}}(\mathbb{R}^n)\) is \(h\)-monotone, \(A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n, 0 < \beta < 1, \text{and } u(x) = Tx - Ax - b\), then for each \(x_0 \in \mathbb{R}^n\) and \(R > 0\)

\[
\sup_{x \in B_R(x_0)} |u(x)| \leq \begin{cases} 
K_1 R \left(R^{-(p-1)} \int_{B_R(x_0)} |u(x)|^{p-1} \, dx\right)^{1/(p-1)} & \text{if } \left(\int_{B_R(x_0)} |u(x)|^{p-1} \, dx\right)^{1/(p-1)} \leq C(n, p, \beta) R \\
K_2 R \left(R^{-(p-1)} \int_{B_R(x_0)} |u(x)|^{p-1} \, dx\right)^{1/(p-1)} & \text{if } \left(\int_{B_R(x_0)} |u(x)|^{p-1} \, dx\right)^{1/(p-1)} \geq C(n, p, \beta) R,
\end{cases}
\]

with the constant \(K_1\) depending only on \(n, p, \lambda, \Lambda\) in (2.3), and \(||A||\); and with the constant \(K_2\) depending only on the same parameters but also on \(\beta\).

Proof. Let us set \(\omega = \frac{u(y)}{|u(y)|}\) and \(r = \delta |u(y)|\), with \(\delta > 0\) to be chosen; \(u(y) \neq 0\). Applying the identity (6.1) with \(v(x) \sim G(x - Ay - b, y - Ay - b, u(y))\) (a function that is \(C^2\) in \(x\) since...
\(p \geq 2\) in the ball \(B_r(y) \sim B_r(y + r\omega)\) yields

\[
v(y + r\omega) = G(y - Ay - b + r\omega, y - Ay - b, u(y))
\]

\[
= \int_{B_r(y + r\omega)} G(x - Ay - b, y - Ay - b, u(y)) \, dx
\]

\[
+ \frac{n}{r^n} \int_0^r \rho^{n-1} \int_{|x-r\omega| \leq \rho} (\Gamma(x - y - r\omega) - \Gamma(\rho)) \Delta_x G(x - Ay - b, y - Ay - b, u(y)) \, dx \, d\rho
\]

(4.2)

\[= I + II.\]

We first estimate the left hand side of (4.2) from below. From (2.8) write

\[
(4.3) \quad G(y - Ay - b + r\omega, y - Ay - b, u(y))
\]

\[
= \int_0^1 \int_0^1 \langle D^2 h (y - Ay - b - u(y) + t \, u(y) + s \, r\omega) \, u(y), r\omega \rangle \, ds \, dt
\]

\[
= \delta \left( \int_0^1 \int_0^1 D^2 h (y - Ay - b - u(y) + tu(y) + s\omega) \, ds \, dt \, u(y), u(y) \right)
\]

\[
= \delta \left( \int_0^1 \int_0^1 D^2 h (y - Ay - b - tu(y) + s\delta u(y)) \, ds \, dt \, u(y), u(y) \right)
\]

\[
\geq \delta \lambda \left( \int_0^1 \int_0^1 |y - Ay - b - (t - s\delta)u(y)|^{p-2} \, ds \, dt \, u(y), u(y) \right) \quad \text{from (2.3)}
\]

(4.4)

\[
\geq \delta \lambda |u(y)|^2 \int_0^1 \int_0^1 |y - Ay - b - (t - s\delta)u(y)|^{p-2} \, ds \, dt.
\]

The last double integral is estimated in the following lemma.

**Lemma 7.** Let \(v_1, v_2\) be vectors in \(\mathbb{R}^n\), \(p \geq 2\), \(\delta > 0\), and let

\[
J = \int_0^1 \int_0^1 |v_1 - (t - s\delta)v_2|^{p-2} \, ds \, dt.
\]

Then there exists \(C_p, \delta_0 > 0\), both depending only on \(p\), such that

(4.5) \[J \geq C_p \left( \max(|v_1|, |v_2|) \right)^{p-2},\]

for \(0 < \delta \leq \delta_0\).

**Proof.** First make the change of variables \((\sigma, \tau) = (s, t)\), that is, \((s, t) = \phi^{-1}(\sigma, \tau) = (\sigma, \frac{\tau - \sigma}{\delta})\) and so \(ds \, dt = \frac{1}{\delta} d\sigma \, d\tau\) and \(R\) is the region bounded by the lines \(\tau = \sigma\), \(\sigma = 1\), \(\sigma = 0\) and \(\tau = \sigma - \delta\). Hence

\[
J = \frac{1}{\delta} \int_R |v_1 - \tau v_2|^{p-2} \, d\sigma \, d\tau = \frac{1}{\delta} \int_0^1 \left( \int_{\sigma - \delta}^\sigma |v_1 - \tau v_2|^{p-2} \, d\tau \right) d\sigma.
\]
Suppose first that $|v_1| \geq |v_2|$. If $0 \leq \sigma \leq 1$ and $\sigma - \delta \leq \tau \leq \sigma$, then $-\delta \leq \tau \leq 1$ so $|\tau| \leq 1$, and $|v_1 - \tau v_2| \geq \|(v_1 - |\tau|v_2) = |v_1| \left(1 - |\tau|\frac{|v_2|}{|v_1|}\right) \geq |v_1| (1 - |\tau|)$. Hence

\[
J \geq \frac{1}{\delta} \int_0^1 \left( \int_{\sigma - \delta}^\sigma |v_1 - \tau v_2|^{p-2} \, d\tau \right) \, d\sigma
\]

\[
= \frac{|v_1|^{p-2}}{\delta} \int_0^\delta \left( \int_{\sigma - \delta}^\sigma (1 - |\tau|)^{p-2} \, d\tau \right) \, d\sigma + \frac{|v_1|^{p-2}}{\delta} \int_\delta^1 \left( \int_{\sigma - \delta}^\sigma (1 - |\tau|)^{p-2} \, d\tau \right) \, d\sigma
\]

We have

\[
I = \int_0^\delta \left( \int_{\sigma - \delta}^\sigma (1 - |\tau|)^{p-2} \, d\tau + \int_0^\sigma (1 - |\tau|)^{p-2} \, d\tau \right) \, d\sigma
\]

\[
= \int_0^\delta \left( \int_{\sigma - \delta}^\sigma (1 + \tau)^{p-2} \, d\tau + \int_0^\sigma (1 - \tau)^{p-2} \, d\tau \right) \, d\sigma
\]

\[
= \int_0^\delta \left( \frac{(1 + \tau)^{p-1}}{p-1} \bigg|_{\tau=0}^{\tau=\sigma-\delta} - \frac{(1 - \tau)^{p-1}}{p-1} \bigg|_{\tau=0}^{\tau=\sigma} \right) \, d\sigma
\]

\[
= \int_0^\delta \frac{2}{p-1} - \frac{(1 + \sigma - \delta)^{p-1}}{p-1} - \frac{(1 - \sigma)^{p-1}}{p-1} \, d\sigma
\]

\[
= \frac{2\delta}{p-1} - \frac{2}{p(p-1)} + \frac{2(1 - \delta)^p}{p(p-1)}
\]

Also

\[
II = \int_\delta^1 \int_{\sigma - \delta}^\sigma (1 - \tau)^{p-2} \, d\tau \, d\sigma
\]

\[
= \int_\delta^1 \left( \frac{(1 - \tau)^{p-1}}{p-1} \bigg|_{\tau=0}^{\tau=\sigma-\delta} \right) \, d\sigma
\]

\[
= \frac{(1 - \sigma + \delta)^{p-1}}{p(p-1)} + \frac{(1 - \sigma)^{p-1}}{p(p-1)} \bigg|_{\sigma=\delta}^{\sigma=1}
\]

\[
= \frac{1}{p(p-1)} - \frac{\delta^p}{p(p-1)} - \frac{(1 - \delta)^p}{p(p-1)}
\]

Then

\[
I + II = \frac{2\delta}{p-1} + \frac{(1 - \delta)^p}{p(p-1)} - \frac{1}{p(p-1)} - \frac{\delta^p}{p(p-1)}
\]

\[
= \frac{2\delta}{p-1} + \frac{p\xi^{p-1}(-\delta)}{p(p-1)} - \frac{\delta^p}{p(p-1)} \quad 1 - \delta < \xi < 1
\]

\[
\geq \frac{2\delta}{p-1} - \frac{\delta}{p-1} - \frac{\delta^p}{p(p-1)} = \frac{\delta}{p-1} - \frac{\delta^p}{p(p-1)}
\]
Hence \( \frac{1}{\delta} (I + II) \geq C_p > 0 \) for all \( \delta < \delta_0 \). We then obtain
\[
J \geq C_p |v_1|^{p-2}.
\]

Now suppose \(|v_1| \leq |v_2|\). Write \( v_1 = \alpha v_2 + v_2^\perp \) where \( v_2^\perp \perp v_2 \), and by Pythagoras
\[
|v_1 - \tau v_2|^2 = |(\alpha - \tau)v_2|^2 + |v_2^\perp|^2 \geq (\alpha - \tau)^2 |v_2|^2,
\]
so
\[
J \geq \frac{1}{\delta} \left| v_2 \right|^{p-2} \int_0^1 \left( \int_{\sigma - \delta}^\sigma |\alpha - \tau|^{p-2} \, d\tau \right) \, d\sigma
\]
\[
= \frac{1}{\delta} \left| v_2 \right|^{p-2} \int_0^1 \left( \int_{-\delta}^\sigma \chi_{[\sigma - \delta, \sigma]}(\tau) |\alpha - \tau|^{p-2} \, d\tau \right) \, d\sigma
\]
\[
= \frac{1}{\delta} \left| v_2 \right|^{p-2} \int_{-\delta}^1 |\alpha - \tau|^{p-2} \left( \int_0^1 \chi_{[\sigma - \delta, \sigma]}(\tau) \, d\sigma \right) \, d\tau
\]
\[
= \frac{1}{\delta} \left| v_2 \right|^{p-2} \int_{-\delta}^1 |\alpha - \tau|^{p-2} \left( \int_0^1 \chi_{[\sigma - \delta, \sigma]}(\tau) \, d\sigma \right) \, d\tau
\]
\[
= \frac{1}{\delta} \left| v_2 \right|^{p-2} \int_{-\delta}^1 |\alpha - \tau|^{p-2} \left( [0, 1] \cap [\tau, \tau + \delta] \right) \, d\tau.
\]

Now
\[
[0, 1] \cap [\tau, \tau + \delta] = \begin{cases} [0, \tau + \delta] & \text{if } -\delta \leq \tau \leq 0 \\ [\tau, \tau + \delta] & \text{if } 0 \leq \tau \leq 1 - \delta \\ [\tau, 1] & \text{if } 1 - \delta \leq \tau \leq 1. \end{cases}
\]

Hence
\[
\int_{-\delta}^1 |\alpha - \tau|^{p-2} \left( [0, 1] \cap [\tau, \tau + \delta] \right) \, d\tau
\]
\[
= \int_{-\delta}^0 |\alpha - \tau|^{p-2} (\tau + \delta) \, d\tau + \int_0^{1-\delta} |\alpha - \tau|^{p-2} \delta \, d\tau + \int_{1-\delta}^1 |\alpha - \tau|^{p-2} (1 - \tau) \, d\tau
\]
\[
\geq \delta \int_0^{1-\delta} |\alpha - \tau|^{p-2} \, d\tau \geq \delta \int_0^{1/2} |\alpha - \tau|^{p-2} \, d\tau, \quad \text{for } \delta < 1/2.
\]

Now
\[
\int_0^{1/2} |\alpha - \tau|^{p-2} \, d\tau = \int_{\alpha-1/2}^\alpha |s|^{p-2} \, ds \geq C_p > 0 \quad \text{for all } \alpha \in \mathbb{R}.
\]

Therefore combining estimates we get
\[
J \geq C_p |v_2|^{p-2}.
\]

and the lemma follows. \( \square \)
Applying Lemma 7 with \( v_1 = y - Ay - b \) and \( v_2 = u(y) \) yields from (4.3) that
\[
G(y - Ay - b + r \omega, y - Ay - b, u(y)) \geq \delta \lambda C_\rho |u(y)|^2 g(y)^{p-2},
\]
for \( 0 < \delta < \delta_0 \), where
\[
g(y) := \max \{|y - Ay - b|, |u(y)|\}.
\]
We now turn to estimate the terms \( I \) and \( II \) on the right hand side of (4.2). The first goal is to estimate \( II \) by (4.8). From the definition of \( G \) we have
\[
\Delta_s G(x - Ay - b, y - Ay - b, u(y)) = \Delta h(x - Ay - b) - \Delta h(x - Ay - b - u(y)).
\]
Hence
\[
II = \frac{n}{\rho^n} \int_0^\rho \rho^{n-1} I(\rho, r, y) \, d\rho
\]
where
\[
I(\rho, r, y) = \int_{B_r(y + r \omega)} (\Gamma(z) - \Gamma(\rho)) \left( \Delta h(z + y + r \omega - Ay - b) - \Delta h(z + y + r \omega - Ay - b - u(y)) \right) \, dx.
\]
Making the change of variables \( z = x - y - r \omega \) we have
\[
I(\rho, r, y) = \int_{|z| \leq \rho} (\Gamma(z) - \Gamma(\rho)) \left( \Delta h(z + y + r \omega - Ay - b) - \Delta h(z + y + r \omega - Ay - b - u(y)) \right) \, dz.
\]
We have that \( \Delta h \) is homogenous of degree \( p - 2 \) so
\[
\Delta h(z + y + r \omega - Ay - b) = \Delta h \left( \frac{z + y + r \omega - Ay - b}{|z + y + r \omega - Ay - b|} \right) = |z + y + r \omega - Ay - b|^{p-2} \Delta h \left( \frac{z + y + r \omega - Ay - b}{|z + y + r \omega - Ay - b|} \right).
\]
Write, with \( e_1 \) a fixed unit vector in \( S^{n-1} \),
\[
\int_{|z| \leq \rho} (\Gamma(z) - \Gamma(\rho)) \Delta h(z + y + r \omega - Ay - b) \, dz
\]
\[
= \int_{|z| \leq \rho} (\Gamma(z) - \Gamma(\rho)) \frac{|z + y + r \omega - Ay - b|^{p-2} \Delta h \left( \frac{z + y + r \omega - Ay - b}{|z + y + r \omega - Ay - b|} \right)}{z + y + r \omega - Ay - b} \, dz
\]
\[
= \int_{|z| \leq \rho} (\Gamma(z) - \Gamma(\rho)) |Ov + y - Ay - b + r Oe_1|^{p-2} \Delta h \left( \frac{Ov + y - Ay - b + r Oe_1}{|Ov + y - Ay - b + r Oe_1|} \right) \, dv, \text{ O rotation around 0 with } Oe_1 = \omega
\]
\[
= \int_{|z| \leq \rho} (\Gamma(z) - \Gamma(\rho)) |Ov + Ov' + r Oe_1|^{p-2} \Delta h \left( \frac{Ov + Ov' + r Oe_1}{|Ov + Ov' + r Oe_1|} \right) \, dv, \quad y - Ay - b = Ov'
\]
\[
= \int_{|z| \leq \rho} (\Gamma(z) - \Gamma(\rho)) |O(v + v' + r e_1)|^{p-2} \Delta h \left( \frac{O(v + v' + r e_1)}{|O(v + v' + r e_1)|} \right) \, dv
\]
\[
= \int_{|z| \leq \rho} (\Gamma(z) - \Gamma(\rho)) |v + v' + r e_1|^{p-2} \Delta h \left( \frac{O(v + v' + r e_1)}{|O(v + v' + r e_1)|} \right) \, dv.
\]
Similarly,

\[
\int_{|\rho| \leq \rho_0} (\Gamma(z) - \Gamma(p)) \, \Delta h (z + y + r \omega - Ay - b - u(y)) \, dz = \int_{|\rho| \leq \rho_0} (\Gamma(z) - \Gamma(p)) \, |z + y + r \omega - Ay - b - u(y)|^{p-2} \, \Delta h \left( \frac{z + y + r \omega - Ay - b - u(y)}{|z + y + r \omega - Ay - b - u(y)|} \right) \, dz,
\]

where \( \Delta h = \nabla^2 h \) is the Hessian of \( h \), and \( \rho_0 \) is a fixed constant. Rotation around 0 with \( \rho \) yields

\[
\int_{|\rho| \leq \rho_0} (\Gamma(v) - \Gamma(p)) \, |v + (r - |u(y)|| e_1 + v')|^{p-2} \, \Delta h \left( \frac{O(v + (r - |u(y)|| e_1 + v')}{|O(v + (r - |u(y)|| e_1 + v')|} \right) \, dv,
\]

since \( \omega = u(y)/|u(y)| \). Thus

\[
I(\rho, r, y) = \int_{|\rho| \leq \rho_0} (\Gamma(v) - \Gamma(p)) \, \left( |v + v' + r e_1|^{p-2} \, \Delta h \left( \frac{O(v + v' + r e_1)}{|O(v + v' + r e_1)|} \right) - |v + (r - |u(y)|| e_1 + v')|^{p-2} \, \Delta h \left( \frac{O(v + (r - |u(y)|| e_1 + v')}{|O(v + (r - |u(y)|| e_1 + v')|} \right) \right) \, dv.
\]

We also have

\[
II = \frac{n}{r^2} \int_0^\rho \rho^{n-1} I(\rho, r, y) \, d\rho = n \int_0^1 t^{n-1} I(t, r, y) \, dt.
\]

Now making the change of variables \( v = r \zeta \) in the integral \( I(r, t, y) \) and setting \( \omega = r \zeta' \) yields

\[
I(r, t, y)
\]

\[
= \int_{|\zeta| \leq t} (\Gamma(r \zeta) - \Gamma(r t)) \, \left( |r \zeta + r \zeta' + r e_1|^{p-2} \, \Delta h \left( \frac{O(r \zeta + r \zeta' + r e_1)}{|O(r \zeta + r \zeta' + r e_1)|} \right) - |r \zeta + (r - |u(y)|| r e_1 + r \zeta')|^{p-2} \, \Delta h \left( \frac{O(r \zeta + (r - |u(y)|| r e_1 + r \zeta')}{|O(r \zeta + (r - |u(y)|| r e_1 + r \zeta')|} \right) \right) \, r^p \, d\zeta
\]

\[
= r^p \int_{|\zeta| \leq t} (\Gamma(\zeta) - \Gamma(t)) \, \left( |\zeta + \zeta' + e_1|^{p-2} \, \Delta h \left( \frac{O(\zeta + \zeta' + e_1)}{|O(\zeta + \zeta' + e_1)|} \right) - |\zeta + (1 - |u(y)|| r e_1 + \zeta')|^{p-2} \, \Delta h \left( \frac{O(\zeta + (1 - |u(y)|| r e_1 + \zeta')}{|O(\zeta + (1 - |u(y)|| r e_1 + \zeta')|} \right) \right) \, d\zeta
\]
and since $r = \delta |u(y)|$ we get

$$II = n |u(y)|^p \delta^p \int_0^1 t^{n-1} \int_{|\zeta| \leq t} (\Gamma(\zeta) - \Gamma(t))$$

$$\left( |\zeta + \zeta' + e_1|^p \Delta h \left( \frac{O(\zeta + \zeta' + e_1)}{|O(\zeta + \zeta' + e_1)|} \right) - |\zeta + (1 - 1/\delta) e_1 + \zeta'|^p \Delta h \left( \frac{O(\zeta + (1 - 1/\delta) e_1 + \zeta')}{|O(\zeta + (1 - 1/\delta) e_1 + \zeta')|} \right) \right) d\zeta dt$$

$$= n |u(y)|^p \delta^p \int_0^1 t^{n-1} \int_{|\zeta| \leq t} (\Gamma(\zeta) - \Gamma(t)) |\zeta + \zeta' + e_1|^p \Delta h \left( \frac{O(\zeta + \zeta' + e_1)}{|O(\zeta + \zeta' + e_1)|} \right) d\zeta dt$$

$$- n |u(y)|^p \delta^2 \int_0^1 t^{n-1} \int_{|\zeta| \leq t} (\Gamma(\zeta) - \Gamma(t)) |\zeta + (\delta - 1) e_1 + \delta \zeta'|^p \Delta h \left( \frac{O(\delta \zeta + (\delta - 1) e_1 + \delta \zeta')}{|O(\delta \zeta + (\delta - 1) e_1 + \delta \zeta')|} \right) d\zeta dt$$

$$= n |u(y)|^p \delta F(\delta, \zeta'),$$

where

$$F(\delta, \zeta') = \delta^{p-1} \int_0^1 t^{n-1} \int_{|\zeta| \leq t} (\Gamma(\zeta) - \Gamma(t)) |\zeta + \zeta' + e_1|^p \Delta h \left( \frac{O(\zeta + \zeta' + e_1)}{|O(\zeta + \zeta' + e_1)|} \right) d\zeta dt$$

$$- \delta \int_0^1 t^{n-1} \int_{|\zeta| \leq t} (\Gamma(\zeta) - \Gamma(t)) |\zeta + (\delta - 1) e_1 + \delta \zeta'|^p \Delta h \left( \frac{O(\delta \zeta + (\delta - 1) e_1 + \delta \zeta')}{|O(\delta \zeta + (\delta - 1) e_1 + \delta \zeta')|} \right) d\zeta dt$$

$$= \delta^{p-1} F_1(\delta, \zeta') - \delta F_2(\delta, \zeta') = \delta \left( \delta^{p-2} F_1(\delta, \zeta') - F_2(\delta, \zeta') \right).$$

We claim that

$$|F(\delta, \zeta')| \leq C_{n,p, \Lambda} \delta \left( \frac{|y - Ay - b|}{|u(y)|} + 1 \right)^{p-2}, \quad \delta < 1/2.$$  

In fact, since $\Delta h$ is continuous, it is bounded in $S^{n-1}$. Also $Ov' = y - Ay - b, v' = r \zeta', r = \delta |u(y)|$ and so $|\zeta'| = \frac{|y - Ay - b|}{\delta |u(y)|}$. Thus

$$|F_1(\delta, \zeta')| \leq C(n, p, \Lambda) (2 + |\zeta'|)^{p-2} \leq C(n, p, \Lambda) \left( \delta + \frac{|y - Ay - b|}{|u(y)|} \right)^{p-2}.\delta^{2-p}.$$  

Also

$$|F_2(\delta, \zeta')| \leq C(n, p, \Lambda) (2 + \delta |\zeta'|)^{p-2} \leq C(n, p, \Lambda) (2 + \delta |\zeta'|)^{p-2} = C(n, p, \Lambda) \left( 2 + \frac{|y - Ay - b|}{|u(y)|} \right)^{p-2}$$

and then (4.7) follows.

This yields the following estimate of $II$:

$$|II| \leq C(n, p, \Lambda) |u(y)|^p \delta^2 \left( \frac{|y - Ay - b|}{|u(y)|} + 1 \right)^{p-2}$$

$$\leq C(n, p, \Lambda) |u(y)|^2 \delta^2 g(y)^{p-2}, \quad \text{for } \delta < 1/2.$$  

(4.8)
We next estimate $I$ in (4.2). From (2.6)

$$I = \int_{B_r(y + r \omega)} G(x - Ay - b, y - Ay - b, u(y)) \, dx$$

$$\leq \int_{B_r(y + r \omega)} G(x - Ax - b, y - Ax - b, u(x)) \, dx + \int_{B_r(y + r \omega)} P_{y,b}(x, y) \, dx := I' + I''.$$

Let us estimate $I''$. From (2.9)

$$I'' = \int_{B_r(y + r \omega)} P_{A,b}(x, y) \, dx$$

$$= \int_{B_r(y + r \omega)} \int_0^1 \int_0^1 \left( D^2 h \left( y - Ax - b + t(Ax - Ay) + s(x - y) \right) A(x - y), (x - y) \right) \, ds \, dt \, dx$$

$$\leq \int_{B_r(y + r \omega)} \int_0^1 \int_0^1 \left| y - Ax - b + t(Ax - Ay) + s(x - y) \right|^{p-2}$$

$$\left( D^2 h \left( \frac{y - Ax - b + t(Ax - Ay) + s(x - y)}{|y - Ax - b + t(Ax - Ay) + s(x - y)|} \right) (y - x), y - x \right)^{1/2}$$

$$\left( D^2 h \left( \frac{y - Ax - b + t(Ax - Ay) + s(x - y)}{|y - Ax - b + t(Ax - Ay) + s(x - y)|} \right) A(y - x), A(y - x) \right)^{1/2} \, ds \, dt \, dx$$

$$\leq \Lambda \| A \| \int_{B_r(y + r \omega)} \int_0^1 \int_0^1 \left| y - Ax - b + t(Ax - Ay) + s(x - y) \right|^{p-2} |y - x|^2 \, ds \, dt \, dx$$

$$\leq \Lambda \| A \| \int_{B_r(y + r \omega)} \int_0^1 \int_0^1 \left| y - Ay - b + t(Ay - Ax) + s(x - y) \right|^{p-2} |y - x|^2 \, ds \, dt \, dx$$

$$\leq \Lambda \| A \| \| C_p \int_{B_r(y + r \omega)} \left( |y - Ay - b|^{p-2} + (\| A \| + 1)^{p-2} |x - y|^{p-2} \right) |y - x|^2 \, dx$$

$$= \Lambda \| A \| \| C_p \left( |y - Ay - b|^{p-2} \int_{B_r(y + r \omega)} |y - x|^2 \, dx + (\| A \| + 1)^{p-2} \int_{B_r(y + r \omega)} |x - y|^p \right) \, dx.$$

If $x \in B_r(y + r \omega)$, then $|x - y| - r \leq |x - y - r \omega| \leq r$, that is, $|x - y| \leq 2r$ and we get

$$I'' \leq \Lambda \| A \| \| C_p \left( |y - Ay - b|^{p-2} r^2 + (\| A \| + 1)^{p-2} r^p \right).$$
Let us estimate $I'$. From (2.8) with $z_1 = x - Ax - b, z_2 = y - Ax - b$ and $z_3 = u(x)$ it follows that

$$I' = \int_{B_r(y+r|\omega|)}^{1} \int_{0}^{1} \left(D^2 h \left(x - Ax - b - t u(x) + s (y - x) \right) (y - x), -u(x) \right) ds dt dx$$

$$= \int_{B_r(y+r|\omega|)}^{1} \int_{0}^{1} |x - Ax - b - t u(x) + s (y - x)|^{p-2}$$

$$\left(D^2 h \left(x - Ax - b - t u(x) + s (y - x) \right) (y - x), -u(x) \right) ds dt dx$$

$$\leq \int_{B_r(y+r|\omega|)}^{1} \int_{0}^{1} |x - Ax - b - t u(x) + s (y - x)|^{p-2}$$

$$\left(D^2 h \left(x - Ax - b - t u(x) + s (y - x) \right) (y - x), -u(x) \right)^{1/2} ds dt dx$$

$$\leq \Lambda \int_{B_r(y+r|\omega|)}^{1} \int_{0}^{1} |x - Ax - b - t u(x) + s (y - x)|^{p-2} |y - x| |u(x)| ds dt dx$$

$$= \Lambda \int_{B_r(y+r|\omega|)}^{1} \int_{0}^{1} |x - Ax - b - t u(x) + s (y - x)|^{p-2} ds dt dx$$

$$\leq C_p \Lambda \int_{B_r(y+r|\omega|)}^{1} \int_{0}^{1} |y - x| |u(x)| |x - Ax - b|^{p-2} + \int_{B_r(y+r|\omega|)}^{1} |y - x| |u(x)|^{p-1} dx + \int_{B_r(y+r|\omega|)}^{1} |u(x)| |y - x|^{p-1} dx.$$ 

Set

$$A_1 = \int_{B_r(y+r|\omega|)}^{1} |y - x| |u(x)| |x - Ax - b|^{p-2} dx$$

$$A_2 = \int_{B_r(y+r|\omega|)}^{1} |y - x| |u(x)|^{p-1} dx$$

$$A_3 = \int_{B_r(y+r|\omega|)}^{1} |u(x)| |y - x|^{p-1} dx.$$

If $x \in B_r(y + r|\omega|)$, then $|x - y| \leq 2r$. Since $|Ax - Ay| \leq ||A|| |x - y| \leq 2r||A||$, we get

$$A_1 \leq 2r \int_{B_r(y+r|\omega|)}^{1} |u(x)| |x - Ax - b|^{p-2} dx \leq 2r (2r(1 + ||A||) + |y - Ay - b|)^{p-2} \int_{B_r(y+r|\omega|)}^{1} |u(x)| dx$$

$$A_2 \leq 2r \int_{B_r(y+r|\omega|)}^{1} |u(x)|^{p-1} dx$$

$$A_3 \leq (2r)^{p-1} \int_{B_r(y+r|\omega|)}^{1} |u(x)| dx.$$ 

Consequently,

$$I' \leq C_p \Lambda \left( r (1 + ||A||) + |y - Ay - b|^{p-2} \int_{B_r(y+r|\omega|)}^{1} |u(x)| dx + r \int_{B_r(y+r|\omega|)}^{1} |u(x)|^{p-1} dx + r^{p-1} \int_{B_r(y+r|\omega|)}^{1} |u(x)| dx \right).$$
Combining the estimates (4.6), (4.8), (4.10), and (4.9) it follows that

\[
\delta \lambda \mathcal{C}_p |u(y)|^2 g(y)^{p-2} \\
\leq C_{n,p} |u(y)|^2 \delta^2 g(y)^{p-2} \\
+ C_p \Lambda \left( r \left( r(1 + ||A||) + g(y) \right)^{p-2} + \mathcal{R}_{B_r(0,r)} |u(x)|^p dx + r \int_{B_r(0,r)} |u(x)|^{p-1} dx + r^{p-1} \int_{B_r(0,r)} |u(x)| dx \right) \\
+ \Lambda ||A||\mathcal{C}_p \left( g(y)^{p-2} r^2 + (||A|| + 1)^{p-2} r^p \right)
\]

for \(0 < \delta < 1/2, \omega = u(y)/|u(y)|, \) and \(r = \delta|u(y)|.\) Dividing the last inequality by \(\delta\) yields

\[
\lambda \mathcal{C}_p |u(y)|^2 g(y)^{p-2} \\
\leq C_{n,p} |u(y)|^2 \delta g(y)^{p-2} \\
+ \frac{C_p \Lambda}{\delta} \left( r \left( r(1 + ||A||) + g(y) \right)^{p-2} + \mathcal{R}_{B_r(0,r)} |u(x)|^p dx + r \int_{B_r(0,r)} |u(x)|^{p-1} dx + r^{p-1} \int_{B_r(0,r)} |u(x)| dx \right) \\
+ \frac{\Lambda ||A||\mathcal{C}_p}{\delta} \left( g(y)^{p-2} \delta^2 |u(y)|^2 + (||A|| + 1)^{p-2} \delta^p |u(y)|^p \right)
\]

\[
= C_{n,p} |u(y)|^2 \delta \left( g(y)^{p-2} \right)
\]

\[
+ C_p \Lambda \left( |u(y)| \left( \delta(1 + ||A||) g(y) + g(y) \right)^{p-2} + \mathcal{R}_{B_r(0,r)} |u(x)|^p dx + \delta |u(y)| \int_{B_r(0,r)} |u(x)|^{p-1} dx + \delta g(y)^{p-2} |u(y)| \int_{B_r(0,r)} |u(x)| dx \right) \\
+ \Lambda ||A||\mathcal{C}_p \left( g(y)^{p-2} \delta |u(y)|^2 + (||A|| + 1)^{p-2} \delta^p |u(y)|^p \right)
\]

\[
\leq C_{n,p} |u(y)|^2 g(y)^{p-2} \delta \\
+ C_p \Lambda \left( |u(y)| g(y)^{p-2} \left( \delta(1 + ||A||) + 1 \right)^{p-2} + \mathcal{R}_{B_r(0,r)} |u(x)|^p dx + \delta^{p-2} |u(y)| g(y)^{p-2} \int_{B_r(0,r)} |u(x)| dx \right) \\
+ \Lambda ||A||\mathcal{C}_p \left( g(y)^{p-2} \delta |u(y)|^2 + (||A|| + 1)^{p-2} \delta^p |u(y)|^p \right)
\]
where we have used $\frac{g(y)}{|u(y)|} \geq 1$.

Choosing $\delta_0 = C(p, \Lambda, \|A\|) > 0$ sufficiently small, we then get for all $\delta \leq \delta_0$ that

$$(\Lambda/2) C_p |u(y)|^p \leq C_p \Lambda \left( (\delta(1 + \|A\|) + 1)^{p-2} + \delta^{p-2} \right) |u(y)|^{p-2} \int_{B_r(y+r\omega)} |u(x)| \, dx + |u(y)| \left( \frac{g(y)}{|u(y)|} \right)^p \int_{B_r(y+r\omega)} |u(x)|^{p-1} \, dx$$

+ $\Lambda \|A\| C_p (\|A\| + 1)^{p-2} \delta^{p-1} |u(y)|^p$.

Multiplying this inequality by $\frac{|u(y)|^{p-3}}{g(y)^{p-2}}$ yields

$$|u(y)|^{p-1} \leq C(p, \Lambda, \|A\|) \left( |u(y)|^{p-2} \int_{B_r(y+r\omega)} |u(x)| \, dx + \int_{B_r(y+r\omega)} |u(x)|^{p-1} \, dx + \delta^{p-2} |u(y)|^{p-2} \int_{B_r(y+r\omega)} |u(x)| \, dx \right)$$

+ $C(p, \Lambda, \|A\|) \delta^{p-1} |u(y)|^{p-2} |u(y)|^{p-2} \int_{B_r(y+r\omega)} |u(x)| \, dx$.

since $|u(y)| \leq g(y)$. Hence, since $p > 1$, we can choose $\delta_0$ even smaller than before, depending only on $\Lambda, p$ and $\|A\|$, so that the following estimate holds for $0 < \delta < \delta_0$

$$|u(y)|^{p-1} \leq C(p, \Lambda, \|A\|) \left( |u(y)|^{p-2} \int_{B_r(y+r\omega)} |u(x)| \, dx + \int_{B_r(y+r\omega)} |u(x)|^{p-1} \, dx + \delta^{p-2} |u(y)|^{p-2} \int_{B_r(y+r\omega)} |u(x)| \, dx \right),$$

with $r = \delta |u(y)|$. If $p = 2$ we then get the inequality

$$|u(y)| \leq C(p, \Lambda, \|A\|) \int_{B_r(y+r\omega)} |u(x)| \, dx.$$

When $p > 2$ we write

$$|u(y)|^{p-2} \int_{B_r(y+r\omega)} |u(x)| \, dx = \frac{|u(y)|^{p-2}}{\epsilon} \int_{B_r(y+r\omega)} |u(x)| \, dx$$

$$\leq \frac{p-2}{p-1} \left( \frac{|u(y)|^{p-2}}{\epsilon} \right)^{(p-1)/(p-2)} + \frac{1}{p-1} \left( \epsilon \int_{B_r(y+r\omega)} |u(x)| \, dx \right)^{p-1}$$

$$= \frac{p-2}{p-1} \left( \frac{1}{\epsilon} \right)^{(p-1)/(p-2)} |u(y)|^{p-1} + \frac{1}{p-1} \left( \epsilon \int_{B_r(y+r\omega)} |u(x)| \, dx \right)^{p-1}.$$
we then obtain the inequality
\[
|u(y)|^{p-1} \leq C \left( \left( \int_{B_r(y+\tau \omega)} |u(x)|^p dx \right)^{1/p-1} + \int_{B_r(y+\tau \omega)} |u(x)|^{p-1} dx + \delta^{p-2} \int_{B_r(y+\tau \omega)} |u(x)|^{p-2} dx \right),
\]
from Hölder’s inequality, \( p - 1 \geq 1 \). Since \( r = \delta |u(y)| \) we get
\[
|u(y)|^{p-1} \leq C \left( \int_{B_r(y+\tau \omega)} |u(x)|^{p-1} dx + \delta^{p-2} \left( \int_{B_r(y+\tau \omega)} |u(x)|^{p-1} dx \right)^{1/(p-1)} \right),
\]
By Cauchy’s inequality
\[
\delta^{p-2} \left( \int_{B_r(y+\tau \omega)} |u(x)|^{p-1} dx \right)^{1/(p-1)} \leq \frac{1}{p-1} \left( \int_{B_r(y+\tau \omega)} |u(x)|^{p-1} dx \right)^{1/(p-1)} \left( \int_{B_r(y+\tau \omega)} |u(x)|^{p-2} dx \right)^{p-2} \left( \int_{B_r(y+\tau \omega)} |u(x)|^{p-2} dx \right)^{1/(p-2)},
\]
and so we obtain the estimate
\[
|u(y)|^{p-1} \leq C \left( \int_{B_r(y+\tau \omega)} |u(x)|^{p-1} dx + \delta^{p-1} \right),
\]
with a constant \( C \) depending only on \( n, p, \lambda, \Lambda \) and \( \|A\| \).

Let us now fix a ball \( B_R(x_0) \), and suppose \( y \in B_{\beta R}(x_0) \) with \( 0 < \beta < 1, R > 0 \). Then \( B_r(y + \tau \omega) \subset B_R(x_0) \) for \( r \leq \frac{1-\beta}{2} R \) and so
\[
\int_{B_r(y+\tau \omega)} |u(x)|^{p-1} dx \leq \frac{C_n}{r^{n-1}} \int_{B_{\beta R}(x_0)} |u(x)|^{p-1} dx.
\]
Given any \( 0 < r \leq \frac{1-\beta}{2} R \), let \( \delta = \frac{r}{|u(y)|} \). If \( \delta \leq \delta_0 \), we then obtain
\[
(4.11) \quad |u(y)|^{p-1} \leq C \left( \frac{1}{r^{n-1}} \int_{B_{\beta R}(x_0)} |u(x)|^{p-1} dx + r^{p-1} \right) := H(r).
\]
If \( \delta \geq \delta_0 \), then
\[
|u(y)| \leq \frac{r}{\delta_0}.
\]
We then obtain
\[
|u(y)| \leq \max \left\{ H(r)^{1/(p-1)}, \frac{r}{\delta_0} \right\} := \wbar{H}(r)
\]
for any \( 0 < r \leq (1-\beta)R/2 \), \( y \in B_{\beta R}(x_0) \). This means that
\[
|u(y)| \leq \min_{0<r \leq (1-\beta)R/2} \wbar{H}(r).
\]
Since the constant \( C'' \) in the definition of \( H(r) \) can be enlarged with the estimate \( (4.11) \) remaining to hold, we can take \( C'' \) so that \( C'' \geq 1/\delta_0^{p-1} \) and in this way \( H(r)^{1/(p-1)} \geq \frac{r}{\delta_0} \).
and so $\max \left( H(r)^{1/(p-1)}, \frac{r}{\delta_0} \right) = H(r)^{1/(p-1)}$. Therefore we obtain the estimate

\[(4.12) \quad |u(y)| \leq \min_{0 < r \leq (1-\beta)R/2} H(r)^{1/(p-1)} \quad \text{for } y \in B_{\beta R}(x_0).\]

Next we estimate the minimum on the right hand side of (4.12). Set

$$\Delta = \int_{B_R(x_0)} |u(x)|^{p-1} \, dx,$$

so $H(r) = C'' \left( \Delta r^{-n} + r^{p-1} \right)$. The minimum of $H$ over $(0, \infty)$ is attained at

$$r_0 = \left( \frac{n \Delta}{p-1} \right)^{1/(n+p-1)},$$

$H$ is decreasing in $(0, r_0)$ and increasing in $(r_0, \infty)$, and

$$\min_{[0, \infty)} H(r) = H(r_0) = C'' \left( \left( \frac{n}{p-1} \right)^{-n/(n+p-1)} + \left( \frac{n}{p-1} \right)^{(p-1)/(n+p-1)} \right) \Delta^{(p-1)/(n+p-1)}.$$

If $r_0 < (1-\beta)R/2$, then $\min_{0 < r \leq (1-\beta)R/2} H(r) = H(r_0)$. On the other hand, if $r_0 \geq (1-\beta)R/2$, that is, $\Delta \geq \left( \frac{1-\beta}{2} \right)^{n+p-1} \frac{p-1}{n} := \Delta_0$, then we have

$$\min_{0 < r \leq (1-\beta)R/2} H(r) = H \left( \frac{1-\beta}{2} R \right) = C'' \left( \Delta \left( \frac{1-\beta}{2} R \right)^{-n} + \left( \frac{1-\beta}{2} \right)^{p-1} \right)$$

$$= C'' \left( \Delta \left( \frac{1-\beta}{2} R \right)^{-n} + \Delta \frac{1}{\Delta} \left( \frac{1-\beta}{2} R \right)^{p-1} \right)$$

$$\leq C'' \left( \Delta \left( \frac{1-\beta}{2} R \right)^{-n} + \frac{n-1}{p-1} \Delta \left( \frac{1-\beta}{2} \right)^{p-1} \right)$$

$$= C'' \left( \frac{p + n - 1}{p-1} \left( \frac{1-\beta}{2} \right)^{-n} \right) \Delta := K_2 R^{-n} \Delta.$$

We then obtain the following estimate valid for all $0 < \beta < 1$ for $y \in B_{\beta R}(x_0)$

\[(4.13) \quad |u(y)|^{p-1} \leq \begin{cases} K_1 \Delta^{(p-1)/(n+p-1)} & \text{if } \Delta \leq \Delta_0 \\ K_2 R^{-n} \Delta & \text{if } \Delta \geq \Delta_0, \end{cases}\]

with $K_1 = C'' \left( \left( \frac{n}{p-1} \right)^{-n/(n+p-1)} + \left( \frac{n}{p-1} \right)^{(p-1)/(n+p-1)} \right)$, $K_2 = C'' \left( \frac{p + n - 1}{p-1} \left( \frac{1-\beta}{2} \right)^{-n} \right)$, $\Delta = \int_{B_R(x_0)} |u(x)|^{p-1} \, dx$, and $\left( \frac{1-\beta}{2} \right)^{n+p-1} \frac{p-1}{n} := \Delta_0$. 


Proof. Let \( u \in \mathcal{X} \), that is, \( T = 0 \) when \( p = 1 \). Let \( T \) be \( h \)-monotone. If \( T \) is \( h \)-monotone, then the inequality (4.14) can be rewritten in the form (4.1) which resembles [GM22, inequality (2.5)]. This completes the proof of Theorem 6. \( \Box \)

5. Application to the differentiability of \( h \)-monotone maps

Following Calderón and Zygmund [CZ61], see also [Zi89 Sect. 3.5], we recall the notion of differentiability in \( L^p \)-sense.

**Definition 8.** Let \( 1 \leq p \leq \infty \), \( k \) is a positive integer and \( f \in L^p(\Omega) \), with \( \Omega \subset \mathbb{R}^n \) open, and let \( x_0 \in \Omega \). We say that \( f \in T^k(p)(x_0)(f \in t^k(p)(x_0)) \) if there exists a polynomial \( \phi \) of degree \( \leq k - 1 \) \( \phi(x) \) such that

\[
\left( \int_{B_r(x_0)} |f(x) - \phi(x)|^p dx \right)^{1/p} = O(r^k) \quad \text{as } r \to 0
\]

or

\[
\left( \int_{B_r(x_0)} |f(x) - \phi(x)|^p dx \right)^{1/p} = o(r^k) \quad \text{as } r \to 0;
\]

when \( p = \infty \) the averages are replaced by ess sup \( x \in B_r(x_0) \) \( |f(x) - \phi(x)| = \|f - \phi\|_{L^\infty(B_r(x_0))} \).

From Theorem 6, we then get the following two corollaries.

**Corollary 9.** Let \( T \) be \( h \)-monotone. If \( T \in L^{p-1}(B_1(x_0)) \) and there exists a matrix \( A = A_{x_0} \) and a vector \( b = b_{x_0} \) such that

\[
\left( \int_{B_R(x_0)} |Tx - Ax - b|^p dx \right)^{1/p - 1} = o(R) \quad \text{as } R \to 0,
\]

that is, \( T \in t^{1,p-1}(x_0) \), then \( T \) is differentiable in the ordinary sense at \( x_0 \).

**Proof.** Let \( u(x) = Tx - Ax - b \), and \( \phi(R) = \left( \int_{B_R(x_0)} |u(x)|^p dx \right)^{1/p - 1} \). From the assumption \( \phi(R)/R \to 0 \) as \( R \to 0 \), and so from (4.14)

\[
\sup_{B_R(x_0)} |u(x)| \leq K_1 R^{n/(n+p-1)} \phi(R)^{(p-1)/(n+p-1)} = C R \left( \frac{\phi(R)}{R} \right)^{(p-1)/(n+p-1)},
\]

that is, \( T \) is differentiable at \( x_0 \). \( \Box \)
Corollary 10. Let $T$ be an $h$-monotone map in $L^{p-1}_{loc}(\mathbb{R}^n)$, $p \geq 2$, and such that

$$
\left( \int_{B_R(x_0)} |Tx - b|^{p-1} \right)^{1/(p-1)} = O(R) \quad \text{as } R \to 0
$$

for some vector $b = b_{x_0}$, i.e., $Tx \in T^{1,p-1}(x_0)$ for all $x_0$ in a measurable set $E$. Then

$$
||Tx - A(x - x_0) - Tx_0||_{L^\infty(B_R(x_0))} = o(R) \quad \text{as } R \to 0
$$

for almost all $x_0 \in E$ and some $A = A_{x_0} \in \mathbb{R}^{n \times n}$, i.e., $Tx \in t^{1,\infty}(x_0)$ for a.e. $x_0 \in E$.

Proof. For each $x_0 \in E$ there exist constants $M(x_0) \geq 0$, $R_0 > 0$ and $b \in \mathbb{R}^n$ such that

$$
\left( \int_{B_R(x_0)} |Tx - b|^{p-1} dx \right)^{1/(p-1)} \leq M(x_0) R
$$

for all $0 < R < R_0$, i.e., $Tx \in T^{1,p-1}(x_0)$. Given $0 < R < R_0$ we have either

$$
\begin{align*}
\sup_{B_R(x_0)} |Tx - b| &\leq K_1 R^{n/(n+p-1)} \left( \int_{B_R(x_0)} |Tx - b|^{p-1} dx \right)^{1/(n+p-1)} \\
&\leq C(n,p,\beta) R
\end{align*}
$$

In the first case, from (4.14)

$$
\begin{align*}
\sup_{B_R(x_0)} |Tx - b| &\leq K_2 \left( \int_{B_R(x_0)} |Tx - b|^{p-1} dx \right)^{1/(p-1)} \\
&\leq K_2 M(x_0) R.
\end{align*}
$$

This means $\sup_{B_R(x_0)} |Tx - b| = O(R)$ as $R \to 0$ for all $x_0 \in E$, i.e., $Tx \in T^{1,\infty}(x_0)$. By Stepanov’s theorem [St70, Chap. VIII, Thm. 3, p. 250] this implies that $Tx$ is differentiable for a.e. $x_0 \in E$, i.e., $Tx \in t^{1,\infty}(x_0)$ for a.e. $x_0 \in E$. \hfill \square

Remark 11. In the particular case when $T$ is the optimal transport map, from [AGS05, Theorem 6.2.7] $T$ is approximately differentiable.

5.1. $h$-monotone maps and bounded deformation. We use the formulas

$$
\begin{align*}
h(a) - h(b) &= \int_0^1 Dh(b + s(a - b)) \cdot (a - b) \, ds \\
Dh(a) - Dh(b) &= \int_0^1 D^2 h(b + \tau(a - b)) \cdot (a - b) \, d\tau.
\end{align*}
$$

From the monotonicity with $y = x + t\xi$

$$
0 \leq h(x + t\xi - Tx) - h(x - Tx) + h(x - T(x + t\xi)) - h(x + t\xi - T(x + t\xi)).
$$
Multiplying the last inequality by $\phi \in C_c^\infty$, $\phi \geq 0$, and assuming that $Tx \in L^p_\text{loc}(\mathbb{R}^n)$ since $h$ is homogeneous of degree $p$ it follows that the function $h(x - Tx)$ is locally integrable and we get

\[
0 \leq \int_{\mathbb{R}^n} [h(x + t \xi - Tx) - h(x - Tx)] \phi(x) \, dx + \int_{\mathbb{R}^n} [h(x - T(x + t \xi)) - h(x + t \xi - T(x + t \xi))] \phi(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} [h(x + t \xi - Tx) - h(x - Tx)] \phi(x) \, dx + \int_{\mathbb{R}^n} [h(z - t \xi - Tz) - h(z - Tz)] \phi(z - t \xi) \, dz
\]

\[
= \int_{\mathbb{R}^n} \left[ \int_0^1 Dh(x - Tx + st \xi) \cdot (t \xi) \, ds \right] \phi(x) \, dx + \int_{\mathbb{R}^n} \left[ \int_0^1 Dh(z - Tz - st \xi) \cdot (-t \xi) \, ds \right] \phi(z - t \xi) \, dz
\]

\[
= \int_{\mathbb{R}^n} \left[ \int_0^1 Dh(x - Tx + st \xi) \cdot (t \xi) \, ds \right] \phi(x) \, dx
\]

\[
+ \int_{\mathbb{R}^n} \left[ \int_0^1 Dh(x - Tx - st \xi) \cdot (-t \xi) \, ds \right] \left( \phi(x - t \xi) - \phi(x) + \phi(x) \right) \, dx
\]

\[
= \int_{\mathbb{R}^n} \left[ \int_0^1 Dh(x - Tx + st \xi) - Dh(x - Tx - st \xi) \right] \cdot (t \xi) \, ds \phi(x) \, dx
\]

\[
- \int_{\mathbb{R}^n} \left[ \int_0^1 Dh(x - Tx - st \xi) \cdot (t \xi) \, ds \right] \left( \phi(x - t \xi) - \phi(x) \right) \, dx
\]

\[
= t^2 \left( 2 \int_{\mathbb{R}^n} \left[ \int_0^1 \int_0^1 D^2 h(x - Tx - st \xi + \tau(2st \xi)) (2st \xi) \, d\tau \right] \cdot (s \xi) \, ds \phi(x) \, dx
\]

\[
- \int_{\mathbb{R}^n} \left[ \int_0^1 Dh(x - Tx - st \xi) \cdot (s \xi) \, ds \right] \phi(x - t \xi) - \phi(x) \right) \, dx
\]

\[
= t^2 (2A - B),
\]

with

\[
A = \int_{\mathbb{R}^n} \left[ \int_0^1 \int_0^1 D^2 h(x - Tx - st \xi + \tau(2st \xi)) (s \xi) \, d\tau \right] \cdot (s \xi) \, ds \phi(x) \, dx
\]

\[
B = \int_{\mathbb{R}^n} \left[ \int_0^1 Dh(x - Tx - st \xi) \cdot (s \xi) \, ds \right] \phi(x - t \xi) - \phi(x) \right) \, dx.
\]

\[\text{Notice that from [GM22 Theorem 2.1] the local } L^p \text{ integrability of } Tx \text{ implies that } Tx \text{ is locally bounded.}\]
We calculate the limits of $A$, $B$ when $t \to 0$:

$$
\int_0^1 \int_0^1 D^2 h (x - Tx - st\xi + \tau(2st\xi)) s \, d\tau \, ds
\rightarrow \int_0^1 \int_0^1 D^2 h (x - Tx) s \, d\tau \, ds = \frac{1}{2} D^2 h (x - Tx)
$$

as $t \to 0$ on compact subsets of $\mathbb{R}^n$ under the assumption that $Tx \in L^p_{\text{loc}}(\mathbb{R}^n)$ (and so $Tx$ is locally bounded by [GM22, Theorem 2.1]) since $D^2 h$ is uniformly continuous on compact sets. Hence

$$
A \rightarrow \frac{1}{2} \int_{\mathbb{R}^n} \left< D^2 h (x - Tx) \xi, \xi \right> \phi(x) \, dx.
$$

And also

$$
B \rightarrow -\int_{\mathbb{R}^n} (Dh(x - Tx) \cdot \xi) \partial_\xi \phi(x) \, dx.
$$

We then obtain the inequality

\begin{align}
(5.1) \quad & \int_{\mathbb{R}^n} \left< D^2 h (x - Tx) \xi, \xi \right> \phi(x) \, dx + \int_{\mathbb{R}^n} (Dh(x - Tx) \cdot \xi) \partial_\xi \phi(x) \, dx \geq 0 \\
& \text{for all } \phi \in C^\infty_0, \phi \geq 0, \text{ and each unit vector } \xi \text{ under the assumption that } Tx \in L^p_{\text{loc}}. \text{ Now}
\end{align}

$$
\int_{\mathbb{R}^n} (Dh(x - Tx) \cdot \xi) \partial_\xi \phi(x) \, dx \\
= \sum_{i,j=1}^n \int_{\mathbb{R}^n} h_{x_i}(x - Tx) \phi_{x_j}(x) \xi_i \xi_j \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^n} \left( h_{x_i}(x - Tx) \phi_{x_j}(x) + h_{x_j}(x - Tx) \phi_{x_i}(x) \right) \xi_i \xi_j \, dx \\
= -\sum_{i,j=1}^n \left< \frac{1}{2} \left( \frac{\partial h_{x_i}(x - Tx)}{\partial x_j} + \frac{\partial h_{x_j}(x - Tx)}{\partial x_i} \right), \phi \right> \xi_i \xi_j
$$

with derivatives in the sense of distributions, and

$$
\int_{\mathbb{R}^n} \left< D^2 h (x - Tx) \xi, \xi \right> \phi(x) \, dx \\
= \sum_{i,j=1}^n \int_{\mathbb{R}^n} h_{x_i x_j}(x - Tx) \phi(x) \, dx \xi_i \xi_j = \sum_{i,j=1}^n \left< h_{x_i x_j}(x - Tx), \phi \right> \xi_i \xi_j.
$$

Therefore, if $a_{ij}$ is the scalar distribution defined by

\begin{align}
(5.2) \quad & a_{ij} = h_{x_i x_j}(x - Tx) - \frac{1}{2} \left( \frac{\partial h_{x_i}(x - Tx)}{\partial x_j} + \frac{\partial h_{x_j}(x - Tx)}{\partial x_i} \right),
\end{align}
then the matrix valued distribution $a = (a_{ij})$ is non-negative and so it is equals a matrix-valued Radon measure, see [LH90, Thm. 2.1.7]. It is assumed for this that $Tx \in L^p_{\text{loc}}(\mathbb{R}^n)$ and consequently $T$ is locally bounded. Setting

$$Dh(x - Tx) = (h_{x_1}(x - Tx), \ldots, h_{x_n}(x - Tx))$$

and

$$\frac{\partial Dh(x - Tx)}{\partial x} = \begin{pmatrix} \frac{\partial h_{x_1}(x - Tx)}{\partial x_1} & \cdots & \frac{\partial h_{x_1}(x - Tx)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_{x_n}(x - Tx)}{\partial x_1} & \cdots & \frac{\partial h_{x_n}(x - Tx)}{\partial x_n} \end{pmatrix}$$

with derivatives in the sense of distributions, we can then write the equation

$$(5.3) \quad a = D^2h(x - Tx) - \frac{1}{2} \left( \frac{\partial Dh(x - Tx)}{\partial x} + \left( \frac{\partial Dh(x - Tx)}{\partial x} \right)^T \right)$$

in the sense of distributions; and (5.1) is equivalent that the matrix $a \geq 0$ in the sense of distributions.

Summarizing, we have then observed that if $T$ is an $h$-monotone map that is locally in $L^p(\mathbb{R}^n)$ (which from the $L^\infty$ estimate implies that $T$ is locally bounded), then the matrix $a$ in (5.3) is positive semidefinite in the sense of distributions and therefore it can be represented with a matrix valued Radon measure.

When $h(x) = |x|^2/2$, we have $h_{x_i} = \delta_{ij}$ and $h_{x_i} = x_i$ and

$$a_{ij} = \delta_{ij} - \frac{1}{2} \left( \frac{\partial(x_i - T_jx)}{\partial x_j} + \frac{\partial(x_j - T_i x)}{\partial x_i} \right) = \delta_{ij} - \frac{1}{2} \left( \delta_{ij} - \frac{\partial T_{ix}}{\partial x_j} + \delta_{ij} - \frac{\partial T_{jx}}{\partial x_i} \right) = \frac{1}{2} \left( \frac{\partial T_{ix}}{\partial x_j} + \frac{\partial T_{jx}}{\partial x_i} \right)$$

is the symmetrized gradient of $T$ in the sense of distributions; in this case $a \geq 0$ means $T$ is of bounded deformation. Therefore, for a general cost function $h$, that $a \geq 0$ can be viewed as a generalization of the notion of bounded deformation.

When $Tx = (T_1 x, \ldots, T_n x)$ is $C^1$, we will obtain a more compact expression for (5.1). In

---

4As usual, a measure is a nonnegative (in the sense needed) $\sigma$-additive set function, and a signed measure is a $\sigma$-additive set function that can take positive and negative values.
fact, we have
\[
\left\langle \frac{1}{2} \left( \frac{\partial h_{x_i}(x - Tx)}{\partial x_j} + \frac{\partial h_{x_j}(x - Tx)}{\partial x_i} \right), \phi \right\rangle 
\]
\[
= -\frac{1}{2} \left( \int_{\mathbb{R}^n} h_{x_i}(x - Tx) \phi_{x_j}(x) \, dx + \int_{\mathbb{R}^n} h_{x_j}(x - Tx) \phi_{x_i}(x) \, dx \right) 
\]
\[
= \frac{1}{2} \left( \int_{\mathbb{R}^n} \left( h_{x_i}(x - Tx) \phi(x) \right)_{x_j} - (h_{x_i}(x - Tx))_{x_j} \phi(x) \right) \, dx + \int_{\mathbb{R}^n} \left( (h_{x_j}(x - Tx) \phi(x))_{x_j} - (h_{x_j}(x - Tx))_{x_i} \phi(x) \right) \, dx 
\]
\[
= -\frac{1}{2} \left( \int_{\mathbb{R}^n} (h_{x_i}(x - Tx))_{x_j} \phi(x) \, dx + \int_{\mathbb{R}^n} (h_{x_j}(x - Tx))_{x_i} \phi(x) \, dx \right) 
\]
\[
= -\frac{1}{2} \left( \int_{\mathbb{R}^n} \sum_{k=1}^n h_{x_{ik}}(x - Tx) \frac{\partial (x_k - T_k x)}{\partial x_j} \right) \phi(x) \, dx + \int_{\mathbb{R}^n} \left( \sum_{k=1}^n h_{x_{ki}}(x - Tx) \frac{\partial (x_k - T_k x)}{\partial x_i} \right) \phi(x) \, dx 
\]
\[
= -\frac{1}{2} \left( \int_{\mathbb{R}^n} \left( \sum_{k=1}^n h_{x_{ik}}(x - Tx) \delta_{kj} - \frac{\partial T_k x}{\partial x_j} \right) \phi(x) \, dx 
\right.
\]
\[
+ \int_{\mathbb{R}^n} \left( h_{x_{j}}(x - Tx) + \sum_{k=1}^n h_{x_{k}}(x - Tx) \left( -\frac{\partial T_i x}{\partial x_j} \right) \phi(x) \, dx \right) 
\]
\[
= -\int_{\mathbb{R}^n} h_{x_{ik}}(x - Tx) \phi(x) \, dx + \frac{1}{2} \int_{\mathbb{R}^n} \sum_{k=1}^n \left( h_{x_{ik}}(x - Tx) \frac{\partial T_k x}{\partial x_j} + h_{x_{ki}}(x - Tx) \frac{\partial T_k x}{\partial x_i} \right) \phi(x) \, dx. 
\]
So
\[
a_{ij}(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{k=1}^n \left( h_{x_{ik}}(x - Tx) \frac{\partial T_k x}{\partial x_j} + h_{x_{ki}}(x - Tx) \frac{\partial T_k x}{\partial x_i} \right) \phi(x) \, dx,
\]
for \( T \in C^1 \). If
\[
\frac{\partial T}{\partial x}(x) = \begin{pmatrix} \frac{\partial T_1 x}{\partial x_1} & \cdots & \frac{\partial T_n x}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial T_1 x}{\partial x_n} & \cdots & \frac{\partial T_n x}{\partial x_n} \end{pmatrix},
\]
then recalling \( a = (a_{ij}) \) in (5.3) yields the following expression
\[
a = \frac{1}{2} \left( D^2 h(x - Tx) \frac{\partial T}{\partial x}(x) + \left( D^2 h(x - Tx) \frac{\partial T}{\partial x}(x) \right)^t \right).
\]
To conclude the analysis in this section we summarize/recall the following.

**Definition 12.** \( a \) is a matrix valued distribution if \( a : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}^{n \times n} \) is linear and if \( \phi_n \to 0 \) in \( \mathcal{D}(\mathbb{R}^n) \), then \( a(\phi_n) \to 0 \) in \( \mathbb{R}^{n \times n} \). Here \( \mathcal{D}(\mathbb{R}^n) \) denotes the class of real valued functions infinitely differentiable with compact support.
Definition 13. If $a$ is a matrix valued distribution, we say that $a \geq 0$ if $\langle a(\phi) \xi, \xi \rangle \geq 0$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\phi \geq 0$, and for all $\xi \in \mathbb{R}^n$.

Theorem 14. If $a$ is a matrix valued distribution with $a \geq 0$, then there exists a unique matrix valued Radon measure $\mu_a$ defined on the Borel sets of $\mathbb{R}^n$ such that $a(\phi) = \int_{\mathbb{R}^n} \phi(x) \, d\mu_a(x)$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$, i.e., $a$ can be represented by a matrix valued Radon measure. This means that for each Borel set $E$, the matrix $\mu_a(E) \geq 0$, i.e., is positive semidefinite, $\mu_a$ is $\sigma$-additive and $\mu_a(K)$ is finite for each compact $K \subset \mathbb{R}^n$.

Definition 15. A mapping $u : \mathbb{R}^n \to \mathbb{R}^n$ is of bounded deformation, denoted $u \in \text{BD}(\mathbb{R}^n)$, if $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the symmetrized gradient $\frac{1}{2} \left( \frac{\partial u}{\partial x} + \left( \frac{\partial u}{\partial x} \right)^T \right)$, as a matrix valued distribution, can be represented by a matrix valued signed Radon measure.

From (5.1), the matrix $a$ in (5.3) is positive semidefinite in the sense of distributions and then by Theorem 14, $a$ can be represented by a matrix valued measure. Since $T$ is locally bounded and $h \in C^2$, then matrix $D^2 h(\mathbf{x} - \mathbf{T} \mathbf{x})$ defines a matrix valued distribution of order zero. Therefore the matrix

$$
\frac{1}{2} \left( \frac{\partial D h(\mathbf{x} - \mathbf{T} \mathbf{x})}{\partial \mathbf{x}} + \left( \frac{\partial D h(\mathbf{x} - \mathbf{T} \mathbf{x})}{\partial \mathbf{x}} \right)^T \right)
$$

can be represented in the sense of distributions with a matrix valued signed Radon measure. This means that the mapping $D h(\mathbf{x} - \mathbf{T} \mathbf{x})$ is of bounded deformation. Consequently, by [ACDM97, Theorem 7.4] (see also [H96]) the mapping $D h(\mathbf{x} - \mathbf{T} \mathbf{x}) \in t^{1,1}(x_0)$ for a.e. $x_0 \in \mathbb{R}^n$. We remark that to show this we have used the estimate [GM22, Theorem 2.1] only to conclude the qualitative fact that $T \mathbf{x}$ is locally bounded but have not used the explicit form of the estimate. Perhaps a quantitative use of such estimate may lead to stronger differentiability properties of $h$-monotone maps $T$ such us [GM22, Theorem 4.5] established for $T$ standard monotone.

6. Appendix

Recall that $\Gamma(x) = \frac{1}{n \omega_n (2-n)} |x|^{2-n}$, with $n > 2$ where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$, and the Green’s representation formula

$$
v(\mathbf{y}) = \int_{\partial \Omega} \left( v(\mathbf{x}) \frac{\partial \Gamma}{\partial v}(\mathbf{x} - \mathbf{y}) - \Gamma(\mathbf{x} - \mathbf{y}) \frac{\partial v}{\partial v}(\mathbf{x}) \right) \, d\sigma(\mathbf{x}) + \int_{\Omega} \Gamma(\mathbf{x} - \mathbf{y}) \Delta v(\mathbf{x}) \, d\mathbf{x}
$$
where $v$ is the outer unit normal and $y \in \Omega$. If $\Omega = B_{\rho}(y)$, then $\frac{\partial \Gamma}{\partial v}(x-y) = \frac{1}{n \omega_n} |x-y|^{1-n}$ and so the representation formula reads

$$v(y) = \int_{|x-y| = \rho} v(x) \, d\sigma(x) - \Gamma(\rho) \int_{|x-y| = \rho} \frac{\partial v}{\partial v}(x) \, d\sigma(x) + \int_{|x-y| \leq \rho} \Gamma(x-y) \, \Delta v(x) \, dx$$

from the divergence theorem. Multiplying the last identity by $\rho^{n-1}$ and integrating over $0 \leq \rho \leq r$ yields

$$v(y) = \int_{|x-y| \leq r} v(x) \, dx + \frac{n}{r^n} \int_0^r \rho^{n-1} \int_{|x-y| \leq \rho} (\Gamma(x-y) - \Gamma(\rho)) \, \Delta v(x) \, dx \, d\rho. \tag{6.1}$$

6.1. Density points. The following lemma yields the differentiability property we use in the proof of Theorem 1.

**Lemma 16.** Let $S \subset \mathbb{R}^n$ be a set not necessarily Lebesgue measurable and consider

$$f(x) := \limsup_{r \to 0} \frac{|S \cap B_r(x)|}{|B_r(x)|},$$

where $| \cdot |$, and $| \cdot |$ denote the Lebesgue outer measure and Lebesgue measure respectively. If

$$M = \{x \in S : f(x) < 1\},$$

then $|M| = 0$. Here $B_r(x)$ is the Euclidean ball centered at $x$ with radius $r$.

Moreover, if $B_r(x)$ is a ball in a metric space $X$ and $\mu^*$ is a Carathéodory outer measure on $X$, then a similar result holds true for all $S \subset X$.

**Proof.** Fix $x \in M$. There exists a positive integer $m$ such that $f(x) < 1 - \frac{1}{m}$, and let $m_x$ be the smallest integer with this property. So for each $\eta > 0$ sufficiently small we have

$$\sup_{0 < \delta \leq \eta} \frac{|S \cap B_r(x)|}{|B_r(x)|} < 1 - \frac{1}{m_x}, \quad \text{for all } 0 < \delta \leq \eta. \tag{6.2}$$

Given a positive integer $k$, let $M_k = \{x \in M : m_x = k\}$. We have $M = \bigcup_{k=1}^{\infty} M_k$. We shall prove by contradiction that $|M_k| > 0$ for some $k$, we may assume also that $|M_k| < \infty$. Let us consider the family of balls $\mathcal{F} = \{B_r(x) : x \in M_k\}$ with $B_r(x)$ satisfying (6.2). Then we have that the family $\mathcal{F}$ covers $M_k$ in the Vitali sense, i.e., for every $x \in M_k$ and for every $\eta > 0$ there is ball in $\mathcal{F}$ containing $x$ whose diameter is less than $\eta$. Therefore from [WZ15, Corollary (7.18) and equation (7.19)] we have that given

\[\text{From [WZ15] Theorem (11.5)] every Borel subset of X is Carathéodory measurable.}\]
\( \epsilon > 0 \) there exists a family of disjoint balls \( B_1, \ldots, B_N \) in \( F \) such that

\[
|M_k|_* - \epsilon < |M_k \cap \bigcup_{i=1}^N B_i|_* , \quad \text{and} \quad \sum_{i=1}^N |B_i| < (1 + \epsilon)|M_k|_* .
\]

Since \( M_k \subset S \), then from (6.2) we get

\[
|M_k|_* - \epsilon < |S \cap \bigcup_{i=1}^N B_i|_* \leq \sum_{i=1}^N |S \cap B_i|_* \leq \left( 1 - \frac{1}{k} \right) \sum_{i=1}^N |B_i| < (1 + \epsilon) \left( 1 - \frac{1}{k} \right) |M_k|_* ,
\]

then letting \( \epsilon \to 0 \) we obtain a contradiction.

\[ \square \]

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**References**


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