Uniform refraction in negative refractive index materials

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We study the problem of constructing an optical surface separating two homogeneous, isotropic media, one of which has a negative refractive index. In doing so, we develop a vector form of Snell's law, which is used to study surfaces possessing a certain uniform refraction property, in both the near- and far-field cases. In the near-field problem, unlike the case when both materials have positive refractive indices, we show that the resulting surfaces can be neither convex nor concave.

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1. INTRODUCTION

Given a point \( O \) inside medium I and a point \( P \) inside medium II, the question we consider is whether we can find an interface surface \( \Gamma \) separating medium I and medium II that refracts all rays emanating from the point \( O \) into the point \( P \). Suppose that medium I has an index of refraction \( n_1 \) and medium II has an index of refraction \( n_2 \). In the case in which both \( n_1, n_2 > 0 \), this question has been studied extensively (see, in particular, [1] and the references therein). This is the so-called near-field problem. The far-field problem is as follows: given a direction \( m \) fixed in the unit sphere in \( \mathbb{R}^3 \), if rays of light emanate from the origin inside medium I, what is the surface \( \Gamma \), interface between media I and II, that refracts all these rays into rays parallel to \( m \)? In the case in which both \( n_1, n_2 > 0 \), this problem has also been studied extensively [2].

A natural question in this direction is what happens in the case when one of the media has a negative index of refraction? These media, as of now still unknown to exist naturally, constitute the so-called negative refractive index materials (NIMs) or left-handed materials. The theory behind these materials was developed by V. G. Veselago in the late 1960s [3], yet was put on hold for more than 30 years until the existence of such materials was shown by a group at the University of California at San Diego [4]. For a more detailed description, including results on near- and far-field imaging with NIMs, see [5, Chaps. 6 and 8]. Refraction laws for NIMs have been previously established, for instance, at the interface between an isotropic and anisotropic medium where negative refraction occurs [6].

This note is devoted to studying surfaces having the uniform refraction property (as discussed above) in the case when \( \kappa := n_2/n_1 < 0 \), in both the far-field and near-field cases. We begin discussing in Subsection 2.A the notion of negative refraction, and in Subsection 2.B, we formulate Snell’s law in vector form for NIMs. Next, in Section 3 we find the surfaces having the uniform refraction property in the setting of NIMs. In contrast with standard materials, i.e., those having positive refractive indices, the surfaces for NIMs in the near-field case can be neither convex nor concave; this is analyzed in detail in Subsection 3.D.

Finally, in Section 4, using the Snell law described in Subsection 4.B, we study the Fresnel formulas of geometric optics, which give the amount of radiation transmitted and reflected through a surface separating two homogeneous and isotropic media, one of them being a NIMs medium. We expect that the study in this paper will be useful in the design of surfaces refracting energy in a prescribed manner for NIMs as it is done for standard materials in [1,2,7].

2. GEOMETRIC OPTICS IN NEGATIVE REFRACTIVE INDEX MATERIALS

A. Introduction to Negative Refractive Index Materials

The notion of negative refraction goes back to the work of Veselago in [3]. In the Maxwell system of electromagnetism, the material parameters \( \epsilon, \mu \) are characteristic quantities which determine the propagation of light in matter. If \( \mathbf{E} = A \cos(\mathbf{r} \cdot \mathbf{k} + \omega t) \), then in the case of isotropic material, the dispersion relation [8, Formula (7.9)] is given by
\[ |k|^2 = \left(\frac{\omega}{c}\right)^2 n^2, \]

with \( n^2 \) being the square of the index of refraction of the material, which takes the form

\[ n^2 = \varepsilon \mu. \]  

(1)

Veselago showed that, in the case when both \( \varepsilon, \mu < 0 \), we are forced by the Maxwell system to take the negative square root for the refractive index, i.e.,

\[ n = -\sqrt{-\varepsilon \mu}. \]  

(2)

In particular, he observed that a slab of material with \( n = -1 \) and behave like a lens. A remarkable property of this material was shown in [9]: that the focusing is perfect provided the condition (3) is exactly met. The theory behind NIMs opened the door to study the so-called metamaterials; see, e.g., [10].

We end this section by making some remarks on some important differences in geometric optics in the case of NIMs. One must generalize the classical Fermat principle of optics (see, e.g., [11, Subsection 3.3.2]) to handle materials with negative refractive indices. This was done not too long ago by Veselago in [12]. In particular, in the setting of NIMs, the optical length of a light ray propagating from one NIM to another is negative because the wave vector is opposite the direction of travel of the ray. However, in this case, it is not possible a priori to determine that the path of light is a maximum or minimum of the optical length, as this depends heavily on the geometry of the problem. Finally, the Fresnel formulas of geometric optics must also be appropriately adjusted for negative refraction; we analyze this in detail at the end.

### B. Snell’s Law in Vector Form for \( \kappa < 0 \)

To explain the Snell law of refraction for media with \( \kappa < 0 \), we first review this for media with positive refractive indices.

Suppose \( \Gamma \) is a surface in \( \mathbb{R}^3 \) that separates two media I and II that are homogeneous and isotropic, with refractive indices \( n_1 \) and \( n_2 \), respectively. If a ray of light (assuming that light rays are monochromatic, since the refraction angle depends on the frequency of the radiation) having direction \( x \in S^2 \), the unit sphere in \( \mathbb{R}^3 \), and traveling through medium I strikes \( \Gamma \) at the point \( P \), then this ray is refracted in the direction \( m \in S^2 \) through medium II according to the Snell law in vector form:

\[ n_1 (x \cdot \nu) = n_2 (m \cdot \nu), \]

(4)

where \( \nu \) is the unit normal to the surface to \( \Gamma \) at \( P \) pointing toward medium II; see [13, Subsection 4.1]. It is assumed here that \( x \cdot \nu \geq 0 \).

This has several consequences:

(a) the vectors \( x, m, \nu \) are all on the same plane (called the plane of incidence);
(b) the well-known Snell’s law in scalar form holds:

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2, \]

where \( \theta_1 \) is the angle between \( x \) and \( \nu \) (the angle of incidence) and \( \theta_2 \) is the angle between \( m \) and \( \nu \) (the angle of refraction).

Equation (4) is equivalent to \( (n_1 x - n_2 m) \times \nu = 0 \), which means that the vector \( n_1 x - n_2 m \) is parallel to the normal vector \( \nu \). If we set \( \kappa = n_2/n_1 \), then

\[ x - \kappa m = \lambda \nu, \]

(5)

for some \( \lambda \in \mathbb{R} \). Notice that (5) univocally determines \( \lambda \). Taking dot products with \( x \) and \( m \) in (5), we get

\[ \lambda = \cos \theta_1 - \kappa \cos \theta_2, \cos \theta_1 = x \cdot \nu > 0, \text{ and } \cos \theta_2 = m \cdot \nu = \sqrt{1 - \kappa^2} \left(1 - (x \cdot \nu)^2\right) \].

In fact, there holds

\[ \lambda = x \cdot \nu - \kappa \sqrt{1 - \kappa^2} \left(1 - (x \cdot \nu)^2\right). \]

(6)

It turns out that refraction behaves differently for \( 0 < \kappa < 1 \) and for \( \kappa > 1 \).

1. If \( 0 < \kappa < 1 \), then for refraction to occur, we need \( x \cdot \nu \geq \sqrt{1 - \kappa^2} \) and, in (6), we have \( \lambda > 0 \) and the refracted vector \( m \) is so that \( x \cdot m \geq \kappa \).

2. If \( \kappa > 1 \), then refraction always occurs, and we have \( x \cdot m \geq 1/\kappa \), and in (6), we have \( \lambda < 0 \); see [2, Subsection 2.1].

We now consider the case when either medium I or medium II has a negative refractive index so that \( \kappa = n_2/n_1 < 0 \). (The calculation below holds, regardless of the sign of \( \kappa \).) Let us take the formulation of the Snell law as in (5). This is again equivalent to

\[ x - \kappa m = \lambda \nu, \]

(7)

with \( x, m \) being unit vectors and \( \nu \) being the unit normal to the interface pointing toward medium II. We will show that

\[ \lambda = x \cdot \nu + \sqrt{(x \cdot \nu)^2 - (1 - \kappa^2)}. \]

(8)

In fact, taking dot products in (7), first with \( x \), and then with \( m \), yields

\[ 1 - \kappa x \cdot m = \lambda x \cdot \nu, \text{ and } x \cdot m - \kappa = \lambda m \cdot \nu. \]

This implies that \( \frac{1}{\kappa} - \frac{1}{\kappa} \lambda x \cdot \nu = x \cdot m = \kappa + \lambda m \cdot \nu \). We seek to get rid of the term \( m \cdot \nu \). To do this, from (7), we have that

\[ m = \frac{x \nu}{\kappa}, \text{ so } m \cdot \nu = \frac{x \cdot \nu}{\kappa} - \frac{1}{\kappa} \lambda. \]

Therefore, by substitution, we obtain the following quadratic equation in \( \lambda \):

\[ \lambda^2 - 2\lambda x \cdot \nu + (1 - \kappa^2) = 0. \]

Solving this equation yields

\[ \lambda = x \cdot \nu \pm \sqrt{(x \cdot \nu)^2 - (1 - \kappa^2)}. \]

If \( |\kappa| \geq 1 \), then \( (x \cdot \nu)^2 - (1 - \kappa^2) \geq 0 \) and the square root is real. On the other hand, if \( |\kappa| < 1 \), then the square root is real only if \( x \cdot \nu \geq \sqrt{1 - \kappa^2} \), which means that the incident direction \( x \) must satisfy this condition. It remains to check which sign (±) to take for \( \lambda \). Recalling that

\[ x \cdot \nu - \kappa m \cdot \nu = \lambda = x \cdot \nu \pm \sqrt{(x \cdot \nu)^2 - (1 - \kappa^2)}, \]

we see that, since \( \kappa < 0 \) and \( m \cdot \nu \geq 0 \), we must take the positive square root. Hence, we conclude (8). Notice that, in contrast with the case when \( \kappa > 0 \), the value of \( \lambda \) given in (8) is always positive when \( \kappa < 0 \), and it can be also written in the following form:

\[ \lambda = x \cdot \nu + |\kappa| \sqrt{1 - \kappa^2 (1 - (x \cdot \nu)^2)}, \]

which is (6) when \( \kappa < 0 \). In the “reflection” case, \( \kappa = -1 \), since \( x \cdot \nu > 0 \), this formula yields
\[ \lambda = 2x \cdot \nu, \]
and, hence, the “reflected” vector \( m \) is given by
\[ m = 2(x \cdot \nu)\nu - x. \]

That is, after striking the interface \( \Gamma \), the wave with direction \( m \) travels in the material with a refractive index \( n_2 \).

Note that the vector formulation (7) is compatible with the Snell law for a negative refractive index [14]; see Fig. 1. Indeed, taking the cross product in (7) with the normal \( \nu \) yields \( x \times \nu = km \times \nu \); then taking absolute values yields
\[ \sin \theta_1 = -k \sin \theta_2, \]
where \( \theta_1 \) is the angle between \( x \) and \( \nu \) and \( \theta_2 \) is the angle between \( m \) and \( \nu \).

C. Physical Condition for Refraction When \( k < 0 \)

We analyze here conditions under which an electromagnetic wave is transmitted from medium I to II and there is no total internal reflection.

1. Case When \( -1 \leq k < 0 \)

The maximum angle of refraction is \( \theta_2 = \pi/2 \) which is attained when \( \sin \theta_1 = -k \); that is, the critical angle is \( \theta_c = \arcsin(-k) \). If \( \theta_1 > \theta_c \), there is no refraction. Thus, \( \theta_2 = \arcsin(-\frac{1}{k} \sin \theta_1) \) for \( 0 \leq \theta_1 \leq \theta_c \). The dot product \( x \cdot m = \cos(\theta_1 + \theta_2) \). Let \( b(\theta_1) = \theta_1 + \theta_2 = \theta_1 + \arcsin(-\frac{1}{k} \sin \theta_1) \). We have
\[ b(\theta_1) = 1 - \frac{1}{k} \cos \theta_1 \left( \frac{1}{\sqrt{1 - \sin^2 \theta_1/k^2}} \right)^2 \geq 0, \]
for \( 0 \leq \theta_1 \leq \theta_c \); thus, \( b \) is increasing on \([0, \theta_c]\), and, therefore,
\[ \theta_1 + \theta_2 \leq \theta_c + \pi/2, \quad \text{for } 0 \leq \theta_1 \leq \theta_c. \]

Then the physical constraint for refraction is
\[ x \cdot m = \cos(\theta_1 + \theta_2) \geq \cos(\pi/2 + \arcsin(-1/k)) = -\sin \theta_c = k. \quad (9) \]

2. Case When \( k < -1 \)

In this case, \( 0 \leq \theta_1 \leq \pi/2 \), and since \( \theta_2 = \arcsin(-\frac{1}{k} \sin \theta_1) \), the maximum angle of refraction is when \( \theta_1 = \pi/2 \); that is, the maximum angle of refraction is \( \theta_2 = \arcsin(-1/k) \). Now \( b(\theta_1) \) is increasing on \([0, \pi/2]\), so \( \theta_1 + \theta_2 \leq \pi/2 + \arcsin(-1/k) \), and
\[ x \cdot m = \cos(\theta_1 + \theta_2) \geq \cos(\pi/2 + \arcsin(-1/k)) = \frac{1}{k}. \quad (10) \]

given by \( r(t) = \rho(x(t))x(t) \) for \( x(t) \in S^2 \). According to (7), the tangent vector \( r'(t) \) to \( \Gamma \) satisfies \( r'(t) \cdot (x(t) - km) = 0 \). That is, \( [\rho(x(t))](x'(t) + \rho(x(t))x'(t)) \cdot (x(t) - km) = 0 \), which yields \( (\rho(x(t))(1 - km \cdot x(t))) = 0 \). Therefore,
\[ \rho(x) = \frac{b}{1 - km \cdot x}, \quad (11) \]
for \( x \in S^2 \) and for some \( b \in \mathbb{R} \). To see the surface described by (11), we assume first that \( -1 < k < 0 \). Since \( m \cdot x \geq 1 \), we have \( 1 - km \cdot x \geq 1 + k > 0 \); thus, \( b > 0 \).

Suppose, for simplicity, that \( m = e_3 \), the third-coordinate vector. If \( y = (y, y_3) \in \mathbb{R}^3 \) is a point on \( \Gamma \), then \( y = \rho(x) x \) with \( x = y/|y| \). From (11), \( |y| = ky_3 = b_3 \) is that is, \( |y|^2 + y_3^2 = (ky_3 + b)^2 \), which yields \( |y|^2 + (1 - k^2)y_3^2 - 2ky_3b = b^2 \). This equation can be written in the form
\[ \left( \frac{|y|^2}{b^2} \right)^2 + \left( \frac{y_3 - \frac{kb}{1-k^2}}{b} \right)^2 = 1, \quad (12) \]
which is an ellipsoid of revolution about the \( y_3 \) axis with upper focus \((0,0,0)\) and lower focus \((0,2kb/(1-k^2))\). Since \( |y| = ky_3 + b \) and the physical constraint for refraction (9), \( \frac{b}{m} \cdot e_3 \geq k \) is equivalent to \( y_3 \geq \frac{kb}{1-k^2} \). That is, for refraction to occur, \( y \) must be in the upper half of the ellipsoid (12); we denote this semi-ellipsoid by \( E(e_3, b) \). To verify that \( E(e_3, b) \) has the uniform refracting property, that is, that it refracts any ray emanating from the origin in the direction \( e_3 \) to the upper half of the ellipsoid (12); we denote this semi-ellipsoid by \( E(e_3, b) \). To verify that \( E(e_3, b) \) has the uniform refracting property, that is, that it refracts any ray emanating from the origin in the direction \( e_3 \), we check that (7) holds at each point. Indeed, if \( y \in E(e_3, b) \), then \( \frac{y}{\sqrt{b^2 - 1}} = 1 - \kappa e_3 \cdot \frac{y}{|y|} \geq 1 + k > 0 \) and \( \frac{y}{\sqrt{b^2 - 1}} \cdot e_3 \geq 0 \); thus, \( \frac{y}{\sqrt{b^2 - 1}} \cdot e_3 \) is an outward normal to \( E(e_3, b) \) by \( y \).

Rotating the coordinates, it is easy to see that the surface given by (11) with \(-1 < k < 0\) is an ellipsoid of revolution about the axis of direction \( m \) with upper focus \((0,0,0)\) and lower focus \( \frac{2kb}{1-k^2} \). Moreover, the semi-ellipsoid \( E(m, b) \) given by
\[ E(m, b) = \left\{ \rho(x)x : \rho(x) = \frac{b}{1 - km \cdot x}, x \in S^2, x \cdot m \geq k \right\} \quad (13) \]
has the uniform refracting property: any ray emanating from the origin \( O \) is refracted in the direction \( m \), where \(-1 < k < 0 \). See Figs. 2 and 3 comparing refraction when \( k < 0 \) versus \( k > 0 \), where \( P = (0,2kb/(1-k^2)) \) and \( m = e_3 \).

![Fig. 1. Snell's law in the case of normal refraction and negative refraction (dotted arrow).](image)
Let us now assume that \( \kappa = -1 \). From (11), we have \( |y| + y_3 = b; \) thus, \( |y'|^2 + y_3^2 = b^2 - 2by_3 + y_3^2 \), and, therefore,
\[
y_3 = \frac{1}{2b} (k^2 - |y'|^2) .
\]

This is a paraboloid with axis \( e_3 \) and focus at \( O \). Since \( \kappa = -1 \), from the physical constraint for refraction (9), we get that any ray with direction \( x \in S^2 \) emanating from the origin is refracted by the paraboloid.

Now we turn to the case \( \kappa < -1 \). Due to the physical constraint of refraction (10), we must have \( b > 0 \) in (11); we determine the type of surface. Define for \( b > 0 \)
\[
H(m, b) = \left\{ \rho(x) : \rho(x) = \frac{b}{1 - \kappa m \cdot x}, x \in S^2, x \cdot m \geq 1/\kappa \right\}.
\]

We claim that \( H(m, b) \) is the sheet with an opening in direction \( m \) of a hyperboloid of revolution of two sheets about the axis of direction \( m \). To prove the claim, set for simplicity \( m = e_3 \). If \( y = (y', y_3) \in H(e_3, b) \), then \( y = \rho(x)x \) with \( x = y'/|y| \). From (14), \(-ky_3 + |y| = b\), and, therefore, \( |y'|^2 + y_3^2 = (ky_3 + b)^2 \), which yields \( |y'|^2 - (k^2 - 1)(y_3 + \frac{kb}{k^2 - 1})^2 = b^2 \). Thus, any point \( y \) on \( H(e_3, b) \) satisfies the equation
\[
\left( y_3 + \frac{kb}{k^2 - 1} \right)^2 - \left( \frac{|y'|}{\sqrt{k^2 - 1}} \right)^2 = 1 ,
\]
which represents a hyperboloid of revolution of two sheets about the \( y_3 \) axis with foci \((0,0)\) and \((0, -2kb/(k^2 - 1))\).

The sheets of this hyperboloid of revolution are given by
\[
y_3 = -\frac{kb}{k^2 - 1} \pm \frac{b}{k^2 - 1} \sqrt{1 + \frac{|y'|^2}{(b/\sqrt{k^2 - 1})^2}} .
\]

We decide which one satisfies the refracting property. The sheet with the minus sign in front of the square root satisfies \( ky_3 + b \geq \frac{kb}{1 - \kappa e_3 \cdot x} \); therefore, this is the sheet to consider. For a general \( m \), by a rotation, we obtain that \( H(m, b) \) is the sheet with an opening in the direction opposite \( m \) of a hyperboloid of revolution of two sheets about the axis of direction \( m \) with foci \((0,0)\) and \((-2kb/(k^2 - 1), m)\).

Notice that the focus \((0,0)\) is inside the region enclosed by \( H(m, b) \) and the focus \((2kb/(k^2 - 1), m)\) is outside that region. The vector \( \frac{y}{|y|} - \kappa m \) is an outer normal to \( H(m, b) \) at \( y \), since by (14)
\[
\left( \frac{y}{|y|} - \kappa m \right) \cdot \left( \frac{-2kb}{k^2 - 1} m - y \right) \geq \frac{-2b}{k^2 - 1} + \frac{2k^2 b}{k^2 - 1} + \kappa m \cdot y - |y|
\]
\[
= \frac{-2b}{k^2 - 1} + \frac{2k^2 b}{k^2 - 1} - b = b > 0 .
\]

Clearly, \( (\frac{y}{|y|} - \kappa m) \cdot m \geq \frac{1}{2} - \kappa > 0 \) and \( (\frac{y}{|y|} - \kappa m) \cdot \frac{m}{|m|} > 0 \).

Therefore, \( H(m, b) \) satisfies the uniform refraction property. We summarize the uniform refraction property below.

**Theorem 3.1.** Let \( n_1, n_2 \) be two indices of refraction for media I and II, respectively, and set \( \kappa = n_2/n_1 \). Assume the origin is in medium I, and \( E(m, b) \), \( H(m, b) \) defined by (11) and (14), respectively.

Then:

1. If \(-1 < \kappa < 0\) and \( E(m, b) \) is the interface between medium I and medium II, then \( E(m, b) \) refracts all rays from the origin \( O \) into rays in medium II with direction \( m \).
2. If \( \kappa < -1 \) and \( H(m, b) \) is the interface between medium I and medium II, then \( H(m, b) \) refracts all rays from the origin \( O \) into rays in medium II with direction \( m \).

**B. Near-Field Problem**

Given a point \( O \) inside medium I and a point \( P \) inside medium II, we construct a surface \( \Gamma \) separating medium I and medium II such that \( \Gamma \) refracts all rays emanating from \( O \) into the point \( P \). Suppose \( O \) is the origin and let \( X(e) \) be a curve on \( \Gamma \). Recall that Snell’s law says that
\[
x - \kappa m = \lambda u ,
\]
longer convex, and its level sets are convex sets only for a range

that

follows by the triangle inequality; since

However, in the case when

where \( \nu \) is the normal to the surface \( \Gamma \), and we are considering

the case when \( \kappa < 0 \). Then, via Snell’s law, we see that the
tangent vector \( x'(t) \) must satisfy

That is to say,

Hence, \( \Gamma \) is the surface

\( |X| + \kappa |X - P| = b. \tag{16} \)

Compare this surface to the Cartesian ovals which occur in

the near-field case when \( \kappa > 0 \). In this case, since the function

\( F(X) = |X| + \kappa |X - P| \) is convex and the ovals are the level

sets of \( F \), then the ovals are convex; see [1, Section 4].

However, in the case when \( \kappa < 0 \), the function \( F(X) \) is no

longer convex, and its level sets are convex only for a range of

values in the parameter; see Fig. 4. That is, the surface (16)
is, in general, not a convex surface; see Theorem 3.4.

Case \(-1 < \kappa < 0\). We will show that the surface (16) is

nonempty if and only if \( b \) is bounded from below. In fact, this

follows by the triangle inequality; since \( 1 + \kappa > 0 \), we see that

\( |X| + \kappa |X - P| \geq |P| \) for all \( X \). Therefore, if the surface (16)
is nonempty, then

\[ b \geq \kappa |P|. \tag{17} \]

Vice versa, if \( b \) satisfies (17), then the surface (16) is nonempty.

Let \( X_0 \) be of the form \( X_0 = \lambda P \). We will find \( \lambda > 0 \) so that

\( X_0 \) is on the surface defined by (16). This happens if \( \lambda |P| + \kappa |\lambda - 1| |P| = b \). Suppose \( b > |P| \). Then we must have \( \lambda > 1 \). Thus, \( b = \lambda |P| + \kappa (\lambda - 1)|P| \), and solving for \( \lambda \) gives

\[ \lambda = \frac{b + \kappa |P|}{(1 + \kappa)|P|}. \]

Now suppose \( \kappa |P| \leq b \leq |P| \), and let

\[ \lambda = \frac{b - \kappa |P|}{(1 - \kappa)|P|}. \]

We then have \( 0 < \lambda \leq 1 \), and \( X_0 = \lambda P \) is on the surface (16).

(Notice that the set \( \{ X : |X| + \kappa |X - P| \leq \kappa |P| + \delta \}, \delta > 0 \), is contained in the ball with center zero and radius \( \delta/(1 + \kappa) \).

Case \( \kappa = -1 \). In this case, to have a nonempty surface, we

show that \( b \) must be bounded above and below. If the surface is

nonempty, then by application of the triangle inequality we get that

\[ -|P| \leq b \leq |P|. \tag{18} \]

Vice versa, if (18) holds, then the surface is nonempty. In fact, the

point \( X_0 = \lambda P \) with \( \lambda = \frac{b + \kappa |P|}{|P|} \) belongs to the surface.

Case \( \kappa < -1 \). We will show that the surface (16) is nonempty if and only if \( b \) is bounded above. First, notice that, by the triangle inequality and the fact that \( \kappa < -1 \), we have that \( |X| + \kappa |X - P| \leq |P| \) for all \( X \), and, therefore, if (16) is nonempty, then

\[ b \leq |P|. \tag{19} \]

On the other hand, if (19) holds, then we will find \( X_0 = \lambda P \in \Gamma \) for some \( \lambda > 0 \). We want \( \lambda |P| + \kappa |P| |\lambda - 1| = b \). If we let \( \lambda = \frac{b + \kappa |P|}{|P|} \) then \( \lambda \geq 1 \) and the point \( \lambda P \) is on the curve.

(Notice that the set \( E = \{ X : |X| + \kappa |X - P| \geq |P| - \delta \}, \delta > 0 \), is contained in the ball with center 0 and radius \( \delta/(1 + \kappa) \).

C. Polar Equation of (16)

Next we will find the polar equation of the refracting surface (16) when \( \kappa < 0 \). We show that only a piece of this surface does the refracting job; otherwise, total internal reflection occurs. This follows from the physical conditions described in Subsection 3.C.1.

Let \( X = \rho(x)x \) with \( x \in S^2 \). Then we write

\[ \kappa |\rho(x)x - P| = b - \rho(x). \]

Squaring both sides yields the following quadratic equation:

\[ (1 - \kappa^2)\rho(x)^2 + 2(\kappa^2 x \cdot P - b)\rho(x) + b^2 - \kappa^2 |P|^2 = 0. \tag{20} \]

Solving for \( \rho \), we obtain

\[ \rho(x) = \frac{(b - \kappa^2 x \cdot P) \pm \sqrt{(b - \kappa^2 x \cdot P)^2 - (1 - \kappa^2)(b^2 - \kappa^2 |P|^2)}}{1 - \kappa^2}. \]

Define an auxiliary quantity

\[ \Delta(t) := (b - \kappa^2 t)^2 - (1 - \kappa^2)(b^2 - \kappa^2 |P|^2), \]

so that
\[ \rho_\pm(x) = \frac{(b - \kappa^2 x \cdot P) \pm \sqrt{\Delta(x \cdot P)}}{1 - \kappa^2} . \]

1. Case \(-1 < \kappa < 0\)

We begin with a proposition.

**Proposition 3.2** Assume \(-1 < \kappa < 0\). If \(x \in S^2\); then, \(\Delta(x \cdot P) \geq \kappa^2(x \cdot P - b)^2\), \(\text{(21)}\) with equality if and only if \(|x \cdot P| = |P|\). Hence, the quantity \(\sqrt{\Delta(x \cdot P)}\) is well defined.

**Proof.** First notice that \(|x \cdot P| \leq |P|\) comes for free since \(|x| = 1\). We have, since \(0 < \kappa^2 < 1\) and \(x \cdot P \leq |P|\),

\[ \Delta(x \cdot P) - \kappa^2(x \cdot P - b)^2 = -2\kappa^2 x^2 \cdot P + \kappa^4(x \cdot P)^2 + \kappa^2 b^2 \]

\[ - \kappa^4|P|^2 + \kappa^2|P|^2 - \kappa^2(x \cdot P)^2 \]

\[ + 2\kappa^2 x \cdot b \cdot P - \kappa^2 b^2 \]

\[ = \kappa^2(x \cdot P)^2 - \kappa^2|P|^2 + \kappa^2|P|^2 \]

\[ - \kappa^2(x \cdot P)^2 = -\kappa^4(|P|^2 - (x \cdot P)^2) \]

\[ + \kappa^2(|P|^2 - (x \cdot P)^2) \]

\[ = \kappa^2(1 - \kappa^2)(|P|^2 - (x \cdot P)^2) \geq 0 , \quad \text{(22)} \]

as desired.

Now, notice that in this case, to make sense of the physical problem, we must have \(P\) lying outside the oval, that is, outside the region \(|X| + \kappa|X - P| \leq b\). In addition to the requirement that the surface be nonempty, we see that the condition on \(b\) so that the ovals are meaningful is

\[ \kappa|P| < b < |P|. \]

Recalling that

\[ \kappa|x(x - P) = b - \rho(x), \]

we see that, since \(\kappa < 0\), we must have \(\rho(x) > b\). We now choose which sign (\(\pm\)) to take in the definition of \(\rho_\pm\) so that \(\rho > b\). In fact, by Proposition 3.2,

\[ \rho_+(x) = \frac{(b - \kappa^2 x \cdot P) + \sqrt{\Delta(x \cdot P)}}{1 - \kappa^2} \]

\[ \geq \frac{(b - \kappa^2 x \cdot P) + |x| |b - x \cdot P|}{1 - \kappa^2} \geq b, \]

where the last inequality holds since \(|x| < 1\). Therefore, the equality \(\rho_+(x) = b\) holds only if \(|x \cdot P| = |P|\) and \(b = x \cdot P\). In addition,

\[ \frac{(b - \kappa^2 x \cdot P) + |x| |b - x \cdot P|}{1 - \kappa^2} \times \begin{cases} (b - \kappa^2 x \cdot P) + |x| |b - x \cdot P| & \text{for } b > x \cdot P \\ (b - \kappa^2 x \cdot P) + |x| (x \cdot P - b) & \text{for } b < x \cdot P \end{cases} > b. \]

Hence, \(\rho_+(x) > b\) for all \(x\) and \(b\) such that \(b \neq x \cdot P\). From (17), the oval is not degenerate only for \(|\kappa| |P| < b\). Reversing the above inequalities, one can show that

\[ \rho_-(x) \leq b. \]

Thus, the polar equation of the surface \(\Gamma\) is given by

\[ b(x, P, b) = \rho_+(x) = \frac{(b - \kappa^2 x \cdot P) + \sqrt{\Delta(x \cdot P)}}{1 - \kappa^2}, \quad \text{(23)} \]

provided that \(|\kappa| |P| < b\). Furthermore, from the physical constraint for refraction (9), in this case, with \(m = \frac{P \cdot (x - x')}{|P - P_0| (x - x')^2}\), so \(x \cdot m \geq \kappa\), and using Eq. (16), we must have

\[ x \cdot P \geq b. \]

Ending with the case when \(-1 < \kappa < 0\), given a point \(P \in \mathbb{R}^3\) and \(|\kappa| |P| \leq b\), a refracting oval is the set

\[ O(P, b) = \{ b(x, P, b) : x \in S^2, x \cdot P \geq b \}, \quad \text{(24)} \]

where

\[ b(x, P, b) = \frac{(b - \kappa^2 x \cdot P) + \sqrt{(b - \kappa^2 x \cdot P)^2 - (\kappa^2 b^2 - (x \cdot P)^2)}}{1 - \kappa^2} . \]

Figure 5 illustrates an example of such a refracting oval.

2. Case \( \kappa < -1 \)

In this case, we must have \(O\) not on the oval; that is, \(O\) must lie outside the set \(|X| + \kappa|X - P| \geq b\). Hence, in combination with the requirement that the surface be nonempty, we have

\[ \kappa|P| < b < |P| . \]

Notice that this is the same condition on \(b\) as in the case \(-1 < \kappa < 0\).

Multiplying (20) by \(-1\) and solving for \(\rho_\pm\) yields

\[ \rho_\pm(x) = \frac{\kappa^2 x \cdot P - b \pm \sqrt{\frac{(b - \kappa^2 x \cdot P)^2 - (\kappa^2 b^2 - (x \cdot P)^2)}}{1 - \kappa^2}} . \]

To have the square root defined, we need

\[ \Delta(x \cdot P) = (b - \kappa^2 x \cdot P)^2 - (\kappa^2 b^2 - (x \cdot P)^2) \geq 0 . \]

From (19), to have a nonempty oval, we must have \(b \leq |P|\). To find the directions \(x\) for which \(\rho\) is well defined, we need to find the values of \(t\) for which

\[ \Delta(t) = (b - \kappa^2 t)^2 - (\kappa^2 b^2 - b^2) \geq 0 . \]

Let \(z = b - \kappa^2 t\); then we want \(t\) such that

\[ z^2 \geq (\kappa^2 - 1)(\kappa^2 b^2 - b^2) . \]

If \(\kappa^2 |P|^2 - b^2 \leq 0\), then this is true for any \(t\). On the other hand, \(\kappa^2 |P|^2 - b^2 \geq 0\) if and only if \(|b| \leq \kappa|P|\). Thus, for \(b\) with \(|b| \leq \kappa|P|\), we have to choose \(t\) such that \(z = b - \kappa^2 t\) satisfies (Notice that the requirement \(|b| \leq \kappa|P|\) is satisfied, since we are assuming that \(b \leq |P|\) so, in particular, \(|b| \leq |P| \leq \kappa|P|\).)

\[ |z| \geq \sqrt{(\kappa^2 - 1)(\kappa^2 |P|^2 - b^2)} . \]

If \(\kappa^2 |P|^2 - b^2 \geq 0\), i.e., \(t < b / \kappa^2\), then \(b - \kappa^2 t \geq \sqrt{(\kappa^2 - 1)(\kappa^2 |P|^2 - b^2)}\); that is,

\[ t \leq \frac{b - \sqrt{(\kappa^2 - 1)(\kappa^2 |P|^2 - b^2})}{\kappa^2} . \]

If \(\kappa^2 |P|^2 - b^2 \leq 0\), i.e., \(t \geq b / \kappa^2\), then \(\kappa^2 t - b \geq \sqrt{(\kappa^2 - 1)(\kappa^2 |P|^2 - b^2)}\); that is,

\[ t \geq \frac{b + \sqrt{(\kappa^2 - 1)(\kappa^2 |P|^2 - b^2})}{\kappa^2} . \quad \text{(25)} \]

From (10), to have refraction in this case, we need to have

\[ \rho_\pm(x) = \frac{\kappa^2 x \cdot P - b \pm \sqrt{\frac{(b - \kappa^2 x \cdot P)^2 - (\kappa^2 b^2 - (x \cdot P)^2)}}{1 - \kappa^2}} . \]
A picture of the refracting piece of this oval can be obtained with the refracting piece of the oval is given by

\[ \rho(x) = \frac{\kappa^2 x \cdot P - b}{\kappa^2 - 1}. \]

That is, we need

\[ \rho(x) \leq \frac{\kappa^2 x \cdot P - b}{\kappa^2 - 1}. \]

Note that, since \( \kappa^2 > 1 \), we see that \( \rho(x) < 0 \) for \( \kappa^2 x \cdot P - b < 0 \). Thus, we must have \( \kappa^2 x \cdot P \geq b \) if we are to have \( \rho(x) \geq 0 \). From (25), we must have

\[ x \cdot P \geq \frac{b + \sqrt{(\kappa^2 - 1)(\kappa^2|P|^2 - b^2)}}{\kappa^2}, \]

so that \( \Delta(x \cdot P) \geq 0 \). We are left with choosing either \( \rho_+ (x) \) or \( \rho_-(x) \) now. However, by the physical restraint for refraction, we see that only \( \rho_-(x) \) will do the job since

\[ \rho_+(x) = \frac{\kappa^2 x \cdot P - b - \sqrt{\Delta(x \cdot P)}}{\kappa^2 - 1} \leq \frac{\kappa^2 x \cdot P - b}{\kappa^2 - 1}, \]

provided that

\[ x \cdot P \geq \frac{b + \sqrt{(\kappa^2 - 1)(\kappa^2|P|^2 - b^2)}}{\kappa^2}. \]

Hence, for \( \kappa < -1 \), refraction only occurs when \( b \leq |P| \) and the refracting piece of the oval is given by

\[ \mathcal{O}(P, b) = \left\{ h(x; P, b) : x \cdot P \geq \frac{b + \sqrt{(\kappa^2 - 1)(\kappa^2|P|^2 - b^2)}}{\kappa^2} \right\}, \tag{26} \]

with

\[ h(x; P, b) = \rho_-(x) \]

\[ = \frac{(\kappa^2 x \cdot P - b) - \sqrt{(\kappa^2 x \cdot P - b)^2 - (\kappa^2 - 1)(\kappa^2|P|^2 - b^2)}}{\kappa^2 - 1}. \]

A picture of the refracting piece of this oval can be obtained from Fig. 5 by reversing the roles of \( P \) and \( O \), and changing the direction of the rays; see also the explanation at the end of the proof of Theorem 3.4.

3. Case \( \kappa = -1 \)

In this case, we have \( |\rho(x) x - P| = \rho(x) - b \), so squaring and solving for \( \rho \) yields

\[ \rho(x) = \frac{b^2 - |P|^2}{2(b - x \cdot P)}. \]

To have a nonempty surface, \( b \) must satisfy (18). Since \( \rho \geq 0 \) and the numerator in \( \rho \) is nonpositive, we must have \( b \leq x \cdot P \).

The last inequality follows from the physical condition for refraction (9) with \( \kappa = -1 \) and \( m = \frac{P - p(x)}{|P - p(x)|} \). Notice that \( \rho(x) \geq b \); that is, \( \frac{\kappa^2 x \cdot P}{2(b - x \cdot P)} \geq b \), since this is equivalent to \( (b - x \cdot P)^2 + |P|^2 - (x \cdot P)^2 \geq 0 \).

Therefore, when \( \kappa = -1 \), the surface refracting the origin \( O \) into \( P \) is given by

\[ \mathcal{E}(P, b) = \left\{ \frac{b^2 - |P|^2}{2(b - x \cdot P)} : x \in S^2, x \cdot P > b \right\}. \tag{27} \]

with \( |b| \leq |P| \).

The uniform refraction property is summarized below.

**Theorem 3.3.** Let \( n_1, n_2 \) be two indices of refraction of two media, I and II, respectively, such that \( \kappa := n_2 / n_1 < 0 \). Assume that \( O \) is a point inside medium I, \( P \) is a point inside medium II, and \( b \in (\kappa|P|, |P|) \). Then,

1. If \( -1 < \kappa < 0 \) and \( \Gamma := \mathcal{E}(P, b) \) is given by (24), then \( \Gamma \) refracts all rays emitted from \( O \) into \( P \).
2. If \( \kappa < -1 \) and \( \Gamma := \mathcal{E}(P, b) \) is given by (26), then \( \Gamma \) refracts all rays emitted from \( O \) into \( P \).
3. If \( \kappa = -1 \) and \( \Gamma := \mathcal{E}(P, b) \) is given by (27), then \( \Gamma \) refracts all rays emitted from \( O \) into \( P \).

**D. Analysis of the Convexity of the Surfaces**

We assume \( \kappa < 0 \) and analyze the convexity of the surface:

\[ |X| + \kappa|X - P| = b. \]

We will show the surface is convex for a range of \( b \)’s, and neither convex nor concave for the remaining range of \( b \)’s. (To be precise, by convexity we mean that the set that is enclosed by \( F(X) = |X| + \kappa|X - P| = b \) is a convex set.) We begin by letting \( X = (x_1, \ldots, x_n) \) and \( P = (p_1, \ldots, p_n) \), and analyzing the function

\[ F(X) = |X| + \kappa|X - P|. \]

We have the gradient

\[ DF(X) = \frac{X}{|X|} + \kappa \frac{X - P}{|X - P|}, \]

and

\[ F_{x_i x_j} = \delta_{ij} \frac{1}{|X|^2} \sum_{k=1}^n x_k x_j \frac{x_k}{|X|^2} \frac{x_j}{|X - P|^2} \]

\[ - \kappa \frac{1}{|X - P|^2} \sum_{k=1}^n (x_k - p_k)(x_j - p_j) \frac{x_k}{|X|^2} \]

\[ = \frac{|X|^2}{|X|} \frac{1}{|X - P|^2} (\kappa - 1)(X \cdot \xi)^2 + \kappa \frac{|\xi|^2}{|X - P|^2} \]

\[ = \kappa \frac{1}{|X - P|^2} \left( |\xi|^2 |X - P|^2 - (X \cdot \xi)^2 \right) \]

\[ + \frac{1}{|X|^2} |\xi|^2 |X - P|^2 - (X \cdot \xi)^2 \].

By Cauchy–Schwarz’s inequality \( |v \cdot w| \leq |v||w| \) with equality if and only if \( v \) is a multiple of \( w \). If we set \( \xi = X \) in the quadratic form, we get \( Q(X,X) = \kappa \frac{1}{|X|^2} |X|^2 |X - P|^2 - (X \cdot X)^2 \). Since \( P \neq 0 \), \( X \) is not a multiple
of $X - P$ if and only if $X$ is not a multiple of $P$ and, in this case, we get $Q(X, X) < 0$ since $\kappa < 0$. On the other hand, if we set $\xi = X - P$, then $Q(X, X - P) = \frac{1}{2}\left(\frac{1}{|X|^2} - (X - P)^2\right)$ and, again, if $X$ is not a multiple of $P$, we obtain $Q(X, X - P) > 0$. Therefore, the quadratic form $Q(X, \xi)$ is indefinite, and, therefore, the function $F$ is neither concave nor convex at each $X$ which is not a multiple of $P$.

Now let $X$ have the form $X = \lambda P$, $0 < \lambda < \infty$, and see for what values of $\lambda$ there holds that $Q(\lambda P, \xi) \geq 0$ for all $\xi$. In other words, we study which values of $\lambda$ make the Hessian $F_{xx}(\lambda P) \geq 0$. In particular, we wish to find the range of $b$ that gives a convex surface. The range we obtain will be proven rigorously in Theorem 3.4. First, we have

$$Q(\lambda P, \xi) = \kappa \frac{1}{|1 - \lambda|} \left( (1 - \lambda)^2 |\xi|^2 |P|^2 - (1 - \lambda)^2 (P \cdot \xi)^2 \right)$$

$$+ \left( \frac{1}{|\lambda|^2} \left[ |\lambda|^2 |\xi|^2 |P|^2 - \lambda^2 (P \cdot \xi)^2 \right] \right)$$

$$= \left( \frac{1}{\lambda^2} \kappa \frac{1}{|1 - \lambda|} \right) \left( |\xi|^2 |P|^2 - (P \cdot \xi)^2 \right) \geq 0,$$

if and only if

$$\phi(\lambda) := \frac{1}{\lambda} + \kappa \frac{1}{|1 - \lambda|} \geq 0.$$

Let us assume $-1 < \kappa < 0$. Then $\phi(\lambda) > 0$ on $0, 1/(1 - \kappa)$, $\phi(\lambda) < 0$ on $(1/(1 - \kappa), 1)$, $\phi(\lambda) < 0$ on $(1, 1/(1 + \kappa))$, and $\phi(\lambda) > 0$ on $(1/(1 + \kappa), +\infty)$. Therefore,

$$Q(\lambda P, \xi) \begin{cases} > 0 & \text{for } \lambda \in (0, 1/(1 - \kappa)) \cup (1/(1 + \kappa), +\infty) \\ < 0 & \text{for } \lambda \in (1/(1 - \kappa), 1) \cup (1/(1 + \kappa), +\infty). \end{cases}$$

Suppose $b \geq \kappa|P|$; see (17). If $\lambda \in (0, 1/(1 - \kappa))$, then, since $\kappa < 0$, we have $\lambda < 1$. If the point $X = \lambda P$ is on the surface $16)$, then $\lambda |P| + \kappa(1 - \lambda) |P| = b$ and, thus, $\lambda = \frac{b - \kappa |P|}{(1 - \kappa) |P|}$. Therefore, when $0 < \lambda < 1/(1 - \kappa)$, to have $F_{xx}(\lambda P) \geq 0$, we must have

$$\frac{b - \kappa |P|}{(1 - \kappa) |P|} \leq \frac{1}{1 - \kappa}.$$

That is,

$$b \leq (1 + \kappa) |P|.$$

Thus, $F_{xx}(\lambda P) \geq 0$, $0 < \lambda < 1/(1 - \kappa)$, with $\lambda P$ on the surface $\lambda |P| + \kappa(1 - \lambda) |P| = b$ if and only if $b \in [\kappa |P|, (1 + \kappa) |P|]$. $1/(1 + \kappa) |P|$.

Now suppose $\lambda \in (1/(1 - \kappa), 1)$. Let us see for which $b$ we have $F_{xx}(\lambda P) \leq 0$. Since $\lambda = \frac{b - \kappa |P|}{(1 - \kappa) |P|}$, we see that, to have $F_{xx}(\lambda P) \leq 0$, we need

$$\frac{1}{1 - \kappa} \leq \frac{b - \kappa |P|}{(1 - \kappa) |P|} < 1,$$

so that

$$0 < \lambda < 1/(1 + \kappa).$$

That is, $F_{xx}(\lambda P) \leq 0$ precisely when $b \in [(1 + \kappa) |P|, |P|]$. $1/(1 + \kappa)$, then $\lambda > 1$ since $-1 < \kappa < 0$. If $\lambda P$ is on the surface, which means $\lambda |P| + \kappa(\lambda - 1) |P| = b$, we have

$$\lambda = \frac{b + \kappa |P|}{(1 + \kappa) |P|}.$$

Thus, $\lambda \geq 1/(1 + \kappa) |P|$.

However, $(1 - \kappa) |P| > |P|$ and, by the physical restriction on $b$, we must have $\kappa |P| < b < |P|$. Thus, we see that

$$|P| < (1 - \kappa) |P| \leq b < |P|,$$

which is impossible.

Now let $\lambda \in (1, 1/(1 + \kappa))$. Then since $\lambda = \frac{b + \kappa |P|}{(1 + \kappa) |P|}$, we see that

$$1 < \frac{b + \kappa |P|}{(1 + \kappa) |P|} < \frac{1}{1 + \kappa},$$

so that

$$|P| < b < (1 - \kappa) |P|.$$
and

\[ C = \kappa |X|(|X|^2 - X \cdot P)(2|X - P|^2 + |X|^2 - X \cdot P) \]

\[ = 2\kappa |X||X - P|^2(|X|^2 - X \cdot P) + \kappa |X|(|X|^2 - X \cdot P)^2 \]

\[ = \kappa |X||X - P|^2(|X|^2 + |X - P|^2 - |P|^2) \]

\[ + \frac{1}{4} \kappa |X|(|X|^2 + |X - P|^2 - |P|^2)^2. \]

The numerator of \( Q(X, (-F_{\rho}, F_{\chi})) \) then equals

\[ |X|^2 - X \cdot P\left((\kappa^2 |X| + |X - P|) \right) \]

\[ + \frac{1}{4} \kappa |X|^2(|X|^2 + |X - P|^2 - |P|^2)^2 \]

\[ = H(|X|, |X - P|, \kappa, |P|), \]

and we want to optimize this quantity when \( X \) is on the curve \( F(X) = |X| + \kappa |X - P| = b \). If \( X \) is on the curve, there holds

\[ |X - P| = \frac{b - |X|}{\kappa} \geq 0. \]

Hence, \( |X| \geq b \) and \(|P| - |X| \leq |X - P| = \frac{b - |X|}{\kappa}\). Hence,

\[ -b - \frac{|X|}{\kappa} \leq |P| - |X| \leq \frac{b - |X|}{\kappa}. \]

Multiplying by \( \kappa < 0 \) yields

\[ -b + |X| \geq \kappa |P| - \kappa |X| \geq b - |X|, \]

so

\[ (1 - \kappa)|X| \geq b - \kappa |P|, \]

and

\[ (1 + \kappa)|X| \geq b + \kappa |P|. \]

Then the last two inequalities, together with \( |X| \geq b \), imply that \( X \) must satisfy

\[ |X| \geq \max \left\{ \frac{b - \kappa |P|}{1 - \kappa}, \frac{b + \kappa |P|}{1 + \kappa}, b \right\}. \]

In addition, from \( |X - P| = \frac{b - |X|}{\kappa} \), we get that

\[ |X| + |P| \geq |X - P| = \frac{b - |X|}{\kappa}, \]

which implies \( |X| \leq \frac{b - |P|}{1 + \kappa} \) since \(-1 < \kappa < 0\). (This means that the ball centered at zero with radius \( \frac{b - |P|}{1 + \kappa} \) contains the oval. In fact, this is the smallest ball centered at zero that contains the oval because the point \( X = \lambda P \) with \( \lambda = \frac{b - |P|}{1 + \kappa} \) satisfies the equation \( F(X) = b \) [notice that \( \lambda \leq 0 \) and \( \lambda - 1 = \frac{b - |P|}{1 + |P|} < 0 \) since \(-b \leq -|P| < |P| \) since \(-1 < \kappa < 0\).]

Substituting \( |X - P| = \frac{b - |X|}{\kappa} \) in the expression for \( H \) above, we obtain the following expression in terms of \( \kappa, b, |X| \) and \( |P| \):

\[ \max \left\{ \frac{b - \kappa |P|}{1 - \kappa}, \frac{b + \kappa |P|}{1 + \kappa}, b \right\}. \]

Note that this minimum is positive since \(-1 < \kappa < 0\) and the maximum is positive as well since \( b \geq |P| \). Hence, the maximum and minimum have the same sign and, as such, the curve is convex for this range of \( b \).

**Case 0 < b \leq (1 + \kappa)|P|**

By calculation, we have that \( L_b > 0 \). Let us analyze the sign of \( t_b \). Since \( b > 0 \) and \(-1 < \kappa < 0\), we have \( t_b > 0 \) if and only if \( b > -|P| \) and \( t_b < 0 \) if and only if \( b < -|P| \). Let us compare \(-|P| \) with \((1 + \kappa)|P| \). We have \(-|P| \leq (1 + \kappa)|P| \) if and only if \( |X| \leq 1/2, \) and \(-|P| > (1 + \kappa)|P| \) if and only if \( \kappa > -1/2 \). Therefore, if \( \kappa > -1/2, \) then \( t_b < 0 \). On the other hand, if \( \kappa \leq -1/2, \) then \( b \in (0, -|P|) \cup (|P|, (1 + \kappa)|P|) \). If \( b \in (0, -|P|) \), then \( t_b < 0 \). If \( b \in [-|P|, (1 + \kappa)|P|) \), then \( t_b > 0 \). Notice also that \( t_b < b \) since \( b < (1 + \kappa)|P| \). If \( \kappa > -1/2, \) then \( t_b < 0 \) and \( \partial_b G \leq 0 \) on \((0, b)\), while \( \partial_b G > 0 \) on \((b, +\infty)\). Since \( \frac{b - |P|}{1 + \kappa} > b \), we have that the interval \( I \subset (b, +\infty) \) and
\( \min_{t \in I} G(t, b, \kappa) = G\left(\frac{b - \kappa|P|}{1 - \kappa}, b, \kappa\right) \)
\[
= \frac{(b - \kappa|P|)^2(b - (1 + \kappa)|P|)}{(-1 + \kappa)^2}.
\]

If \( \kappa \leq -1/2 \), then \( 0 < t_b < b \); thus, \( \partial_t G > 0 \) on \( (0, t_b) \), \( \partial_t G < 0 \) on \( (t_b, b) \), and \( \partial_t G > 0 \) on \( (b, +\infty) \). Once again, \( I \subset (b, +\infty) \), and we obtain
\[
\min_{t \in I} G(t, b, \kappa) = G\left(\frac{b - \kappa|P|}{1 - \kappa}, b, \kappa\right) \]
\[
= \frac{(b - \kappa|P|)^2(b - (1 + \kappa)|P|)}{(-1 + \kappa)^2}.
\]

Note that this minimum is positive since \( b < 0 \). Furthermore, we have
\[
\max_{t \in I} G(t, b, \kappa) = G\left(\frac{b - \kappa|P|}{1 + \kappa}, b, \kappa\right),
\]
which is positive because \( b \geq |k|P| \). Hence, the maximum and minimum have the same sign and, for this range of \( b \)'s, the curve is also convex.

**Case** \((1 + \kappa)|P| < b < 0\).

In this case, we have that \( t_b < b < \frac{b - \kappa|P|}{1 - \kappa}, \frac{b - |P|}{1 - \kappa} \) so that the interval \( I = (\frac{b - |P|}{1 - \kappa}, \frac{b - \kappa|P|}{1 - \kappa}) \subset (b, \infty) \). Furthermore, we have that \( t_b > 0 \) provided that \(-1 < \kappa < -1/2 \) and \(-|k|P| < b < |P| \) or, if \(-1/2 < \kappa < 0 \) and \((1 + \kappa)|P| < b < |P| \). Moreover, we have that \( t_b < 0 \) if \(-1 < \kappa < -1/2 \) and \((1 + \kappa)|P| < b < -|k|P| \).

In addition, on the interval \( I \), we have that \( \partial_t G(t, b, \kappa) > 0 \) so that
\[
\min_{t \in I} G(t, b, \kappa) = G\left(\frac{b - \kappa|P|}{1 - \kappa}, b, \kappa\right) \]
\[
= \frac{(b - \kappa|P|)^2(b - (1 + \kappa)|P|)}{(-1 + \kappa)^2}.
\]

which is negative since \( b > (1 + \kappa)|P| \). Furthermore, we have that
\[
\max_{t \in I} G(t, b, \kappa) = G\left(\frac{b - \kappa|P|}{1 + \kappa}, b, \kappa\right) \]
\[
= \frac{(b + |P|)^2(b - |P|)^2(b + (1 - \kappa)|P|)}{(1 + \kappa)^2}.
\]

which is positive since \(-1 < \kappa < 0 \). Thus, the minimum and maximum of \( G \) on the interval have opposite signs, which implies that the curve is neither convex nor concave for this range of \( b \).

In dimension three, the above result still holds true because the oval is radially symmetric with respect to the axis \( OP \). Finally, suppose that \( \kappa < -1 \).

We have \( X \in \{X; |X| + k|X - P| = b\} \) if and only if \( X - P \in \{Z; |Z| + k|Z + P| = \frac{b}{k}\} \) := \( O' \). The ovals \( O \) and \( O' \) have the same curvature and, by the \(-1 < \kappa < 0 \) case, \( O' \) is convex provided \( \frac{1}{k}|P| < \frac{1}{k} < (1 + \frac{1}{k})|P| \), and neither convex nor concave provided \((1 + \frac{1}{k})|P| < \frac{1}{k} < |P| \). Multiplying both inequalities by \( \kappa \), we obtain the desired range of \( b \).

4. FRESNEL FORMULAS FOR NIMS

To obtain the Fresnel formulas for NIMs, we briefly review the calculations leading to the Fresnel formulas for standard materials; see [11, Subsection 1.5.2].

The electric field is denoted by \( \mathbf{E} \), and the magnetic field is denoted by \( \mathbf{H} \). These are three-dimensional vector fields, \( \mathbf{E} = \mathbf{E}(\mathbf{r}, \tau) \) and \( \mathbf{H} = \mathbf{H}(\mathbf{r}, \tau) \), where \( \mathbf{r} \) represents a point in three-dimensional space \( \mathbf{r} = (x, y, z) \). The way in which \( \mathbf{E} \) and \( \mathbf{H} \) interact is described by Maxwell’s equations [11] and [16, Subsection 4.8]:
\[
\nabla \times \mathbf{E} = -\frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial \tau}, \tag{28}
\]
\[
\nabla \times \mathbf{H} = \frac{2\pi}{c} \sigma \mathbf{E} + \frac{\epsilon}{c} \frac{\partial \mathbf{E}}{\partial \tau}, \tag{29}
\]
\[
\nabla \cdot (\sigma \mathbf{E}) = 4\pi \rho, \tag{30}
\]
\[
\nabla \cdot (\mu \mathbf{H}) = 0, \tag{31}
\]

\( c \) being the speed of light in vacuum.

We consider plane wave solutions to the Maxwell equations (28)–(31), with \( \rho = 0 \) and \( \sigma = 0 \), and having components of the form
\[
a \cos \left(\omega \left(t - \frac{\mathbf{r} \cdot \mathbf{s}}{v}\right) + \delta\right) = a \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta),
\]
with \( \mathbf{k} = \omega \mathbf{s} \) and \( a, \delta \) being real numbers, and \( \mathbf{s} \) being a unit vector. If \( \mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{k} \cdot \mathbf{r} - \omega t) \) and \( \mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{k} \cdot \mathbf{r} - \omega t) \) solve the Maxwell equations (28)–(30), one can show that
\[
\mathbf{H} = \frac{c}{\mu \omega} (\mathbf{k} \times \mathbf{E}), \quad \text{and} \quad \mathbf{E} = -\frac{\epsilon}{c \omega} (\mathbf{k} \times \mathbf{H}). \tag{32}
\]

We assume that the incident vector \( \mathbf{s} = \mathbf{s}' \) with
\[
\mathbf{s}' = \sin \theta \hat{i} + \cos \theta \hat{k}.
\]

That is, \( \mathbf{s}' \) lives on the \( x-z \) plane and, thus, the direction of propagation is perpendicular to the \( y \) axis. In addition, the boundary between the two media is the \( x-y \) plane, \( \nu \) denotes the normal vector to the boundary at the point \( P \); that is, \( \nu \) is on the \( z \) axis, and \( \theta \) is the angle between the normal vector \( \nu \) and the incident direction \( \mathbf{s}' \) (as usual, \( \hat{i}, \hat{j}, \hat{k} \) denote the unit coordinate vectors). Let \( I_{\parallel} \) and \( I_{\perp} \) denote the parallel and perpendicular components, respectively, of the incident field.

The electric field corresponding to this incident field is
\[
\mathbf{E}(\mathbf{r}, t) = (-I_{\parallel} \cos \theta, I_{\perp} \sin \theta) \cos \left(\omega \left(t - \frac{\mathbf{r} \cdot \mathbf{s}'}{v_1}\right)\right) \]
\[
= \mathbf{E}_0 \cos \left(\omega \left(t - \frac{\mathbf{r} \cdot \mathbf{s}'}{v_1}\right)\right), \tag{33}
\]
with
\[
v_1 = \frac{c}{\sqrt{\epsilon_1 \mu_1}}.
\]

From (32), the magnetic field is then
We have that

\[ R_\parallel = \frac{z_2 \cos \theta_i - z_1 \cos \theta_i}{z_2 \cos \theta_i + z_1 \cos \theta_i} I_\parallel \]

\[ R_\perp = \frac{z_1 \cos \theta_i - z_2 \cos \theta_i}{z_1 \cos \theta_i + z_2 \cos \theta_i} I_\perp. \]

These are the Fresnel equations expressing the amplitudes of the reflected and transmitted waves in terms of the amplitude of the incident wave.

We now apply this calculation to deal with NIMs. We will replace \( s' \) by \( x \) and \( s'' \) by \( m \), and we also set

\[ \kappa = -\frac{\sqrt{\varepsilon_2 \mu_2}}{\sqrt{\varepsilon_1 \mu_1}}. \]

Recall \( \nu \) is the normal to the interface. We have \( \cos \theta_i = x \cdot \nu \) and \( \cos \theta_i = m \cdot \nu \). Suppose medium I has \( \varepsilon_1 > 0 \) and \( \mu_1 > 0 \); and medium II has \( \varepsilon_2 < 0 \) and \( \mu_2 < 0 \). In other words, medium I is “right-handed” and medium II is “left-handed.”

This means that, in medium II, the refracted ray stays on the same side of the incident ray \( x \).

From the Snell law formulated in (7) (notice that \( \theta < 0 \)), we have \( x = \kappa \cdot m \), so the Fresnel equations (34) and (35) take the form

\[ T_\parallel = \frac{2z_1 x \cdot \nu}{(z_2 x + z_1 m) \cdot \nu} I_\parallel = \frac{2z_1 x \cdot (x - \kappa m)}{(z_2 x + z_1 m) \cdot (x - \kappa m)} I_\parallel, \]

\[ T_\perp = \frac{2z_1 x \cdot \nu}{(z_2 x + z_1 m) \cdot \nu} I_\perp = \frac{2z_1 x \cdot (x - \kappa m)}{(z_2 x + z_1 m) \cdot (x - \kappa m)} I_\perp, \]

\[ R_\parallel = \frac{(z_2 x - z_1 m) \cdot \nu}{(z_2 x + z_1 m) \cdot \nu} I_\parallel = \frac{(z_2 x - z_1 m) \cdot (x - \kappa m)}{(z_2 x + z_1 m) \cdot (x - \kappa m)} I_\parallel, \]

\[ R_\perp = \frac{(z_1 x - z_2 m) \cdot \nu}{(z_1 x + z_2 m) \cdot \nu} I_\perp = \frac{(z_1 x - z_2 m) \cdot (x - \kappa m)}{(z_1 x + z_2 m) \cdot (x - \kappa m)} I_\perp. \]

Notice that the denominators of the perpendicular components are the same and, likewise, for the parallel components.

As an example, suppose that \( \varepsilon_1, \mu_1 \) are both positive, and \( \varepsilon_2 = -\varepsilon_1 \) and \( \mu_2 = -\mu_1 \). This is the so-called mirror-like material. Then \( z_1 = z_2 \) and \( \kappa = -1 \); thus, \( R_\parallel = R_\perp = 0 \) for all incident rays. This means that all the energy is transmitted and nothing is reflected internally. Notice also that, if \( \kappa \neq -1 \), then there is a reflected wave; that is, \( R_\perp \) and \( R_\parallel \) may be different from zero.

The wave impedance, like the index of refraction, is a unique characteristic of the medium in consideration. However, unlike the refractive index \( n \), the wave impedance \( z \) remains positive for negative values of \( \varepsilon, \mu \). In addition, compare Eq. (35) with the table in [14, p. 765] giving \( r_\perp \). There the Fresnel formula for \( r_\perp \) in the exact Fresnel formula column contains a misprint. We believe that our calculations above are correct and consistent with what is expected in the case of negative refraction.

**A. Brewster Angle**

This is the case when \( R_\parallel = 0 \); this means when

\[ z_2 \cos \theta_i - z_1 \cos \theta_i = 0, \]

which, together with the Snell law (sin \( \theta_i = \kappa \sin \theta_i \)), yields

\[ \left( \frac{z_2}{z_1} \right)^2 \cos^2 \theta_i + \frac{1}{\kappa} \sin^2 \theta_i = 1 = \cos^2 \theta_i + \sin^2 \theta_i. \]
Thus,
\[ \tan^2 \theta_i = \left( \frac{\varepsilon}{\mu} \right)^2 - 1 = \frac{\varepsilon_1 \mu_2 - \varepsilon_2 \mu_1}{\mu_1 (\varepsilon_2 \mu_1 - \varepsilon_1 \mu_2)}, \]
if \(1/k^2 \neq 1\), and the Brewster angle is \( \theta_i \) with
\[ \tan \theta_i = \sqrt{\frac{\mu_2}{\mu_1} \left( \frac{\varepsilon_1 \mu_2 - \varepsilon_2 \mu_1}{\varepsilon_1 \mu_2 - \varepsilon_2 \mu_1} \right)}. \]

Let us compare this value of \( \theta_i \) with the case when \( \kappa > 0 \). As above, the Brewster angle occurs when \( R_\parallel = 0 \). In this situation, the reflected and transmitted rays are orthogonal to each other and, via Snell’s law, it follows that
\[ \tan \theta_i = n_2/n_1, \]
which is consistent with what we would expect. For related results, see [17].

**B. Fresnel Coefficients for NIMs**

To calculate these coefficients we follow the calculations [11] from Subsection 1.5,3. The Poynting vector is given by \( \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \), where \( c \) is the speed of light in free space. From (32), we get that
\[ \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \left( \frac{c}{\mu_0} \mathbf{k} \times \mathbf{E} \right) = \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} (\mathbf{E} \times \mathbf{s} \times \mathbf{E}), \]
where \( \mathbf{k} = \# \mathbf{s}, \mathbf{s} \) is a unit vector, and \( \mathbf{v} = \frac{c}{\sqrt{\varepsilon_0}} \). Using the form of the incident wave (33), the amount of energy \( J_i \) of the incident wave \( \mathbf{E}_i \) flowing through a unit area of the boundary per second is then
\[ J_i = |\mathbf{S}| \cos \theta_i = \frac{c}{4\pi} \sqrt{\frac{\varepsilon_1}{\mu_1}} |\mathbf{E}_0| \cos \theta_i. \]
Similarly, the amount of energy in the reflected and transmitted waves (also given in the previous section) leaving a unit area of the boundary per second is given by
\[ J' = |\mathbf{S}'| \cos \theta_i = \frac{c}{4\pi} \sqrt{\frac{\varepsilon_2}{\mu_2}} |\mathbf{E}_0| \cos \theta_i, \]
\[ J'' = |\mathbf{S}''| \cos \theta_i = \frac{c}{4\pi} \sqrt{\frac{\varepsilon_3}{\mu_3}} |\mathbf{E}_0| \cos \theta_i. \]

The reflection and transmission coefficients are defined by
\[ \mathcal{R} = \frac{J'}{J_i} = \left( \frac{|\mathbf{E}_0'|}{|\mathbf{E}_0|} \right)^2, \quad \text{and} \quad \mathcal{T} = \frac{J''}{J_i} = \sqrt{\frac{\varepsilon_3 \mu_1}{\varepsilon_1 \mu_2}} \cos \theta_i \left( \frac{|\mathbf{E}_0''|}{|\mathbf{E}_0|} \right)^2. \]
By conservation of energy or by direct verification, we have \( \mathcal{R} + \mathcal{T} = 1 \).

In the case when no polarization is assumed, we have from Fresnel’s Eq. (36) that
\[ |\mathbf{E}_0'|^2 = R_\parallel^2 + R_\bot^2, \]
\[ = \left( \frac{(z_2 x - z_1 m) \cdot (x - km)}{(z_2 x + z_1 m) \cdot (x - km)} \right)^2 I_\parallel + \left( \frac{(z_1 x - z_2 m) \cdot (x - km)}{(z_1 x + z_2 m) \cdot (x - km)} \right)^2 I_\bot, \]
and, thus,
\[ \mathcal{R} = \left( \frac{|\mathbf{E}_0'|}{|\mathbf{E}_0|} \right)^2 = \frac{R_\parallel^2 + R_\bot^2}{I_\parallel + I_\bot}, \]
\[ = \left( \frac{(z_2 x - z_1 m) \cdot (x - km)}{(z_2 x + z_1 m) \cdot (x - km)} \right)^2 \frac{I_\parallel}{I_\parallel + I_\bot} + \left( \frac{(z_1 x - z_2 m) \cdot (x - km)}{(z_1 x + z_2 m) \cdot (x - km)} \right)^2 \frac{I_\bot}{I_\parallel + I_\bot}. \]
which is a function only of \( x \cdot m \). We assume in these formulas that the product \( e_i \mu_i > 0 \), \( i = 1, 2 \). In principle, the coefficients \( I_\parallel \) and \( I_\bot \) might depend on the direction \( x \); in other words, for each direction \( x \) we would have a wave that changes its amplitude with the direction of propagation. The energy of the incident wave would be \( f(x) = |\mathbf{E}_0|^2 = I_\parallel (x^2) + I_\bot (x^2) \).

Notice that, if the incidence is normal, that is, \( x = m \), then \( \mathcal{R} = \left( \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2} \right)^2 \), which shows that even for radiation normal to the interface we may lose energy by reflection.

The Fresnel coefficients \( \mathcal{R} \) and \( \mathcal{T} \) being written in terms of \( x \) and \( m \) as above are useful for studying some refraction problems in geometric optics. In fact, when \( \kappa > 0 \), it was shown in [7] that it is possible to construct a surface interface between media I and II in such a way that both the incident radiation and transmitted energy are prescribed, and that loss of energy due to internal reflection across the interface is taken into account.

**5. CONCLUSION**

We have given a vector formulation of the Snell law for waves passing between two homogeneous and isotropic materials when one of them has a negative refractive index, that is, is left-handed. This formulation was used to find surfaces having the uniform refraction property in both the far-field and near-field cases. In the near-field case, in contrast with the case when both materials are standard, these surfaces can be neither convex nor concave and can wrap around the target. A quantitative analysis in terms of the parameters defining the surfaces has been carried out. We have used the vector formulation of Snell’s law to find expressions for the Fresnel formulas and coefficients for NIMs. We expect these formulas to be useful in the design of surfaces separating materials, one of them a NIM, that refract radiation with prescribed amounts of energy given in advance. This will be done in future work and requires more sophisticated mathematical tools.

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