1. Let $u, v$ smooth convex functions in $\Omega$ both vanishing on $\partial \Omega$. Suppose that $|1 - \det D^2u(x)| \leq \epsilon$ for all $x \in \Omega$, and $\det D^2v = 1$ in $\Omega$. Prove that $|u(x) - v(x)| \leq c_n \epsilon$ for all $x \in \Omega$.

HINT: comparison principle.

2. If $u, v$ are convex in $\Omega$, then $M(u+v)(E) \geq Mu(E)+Mv(E)$ on all Borel sets $E \subset \Omega$. Here $Mu$ is the Monge-Ampère measure associated with $u$, $Mu(E) = |\partial u(E)|$.

HINT: prove it first for smooth functions and then approximate. Actually the inequality holds for all $E \subset \Omega$ since the measures are regular.

3. An ellipsoid in $\mathbb{R}^n$ with center at $x_0$ is a set of the form $E = \{x \in \mathbb{R}^n : \langle A(x-x_0), x-x_0 \rangle \leq 1\}$, where $A$ is an $n \times n$ positive definite symmetric matrix and $\langle , \rangle$ denotes the Euclidean inner product. Prove that $|E| = \frac{\omega_n}{(\det A)^{1/2}}$, where $\omega_n$ is the volume of the unit ball.

Hint: formula of change of variables.

4. If $A, B$ are $n \times n$ symmetric positive definite matrices, then prove that $\det \left( \frac{A + B}{2} \right) \geq \sqrt{\det A \det B}$, with equality iff $A = B$.

HINT: if $O$ is an orthogonal matrix and $A' = O^tAO$, $B' = O^tBO$, then the inequality holds for $A, B$ iff it holds for $A', B'$. So we may assume $A$ is diagonal with diagonal $\lambda_1, \cdots, \lambda_n$. Let $T$ be diagonal with diagonal $\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}$. Show that the inequality holds for $A, B$ iff it holds for $Id, T^{-1}BT^{-1}$. Then assume $A$ is the identity and $B$ is diagonal.

The inequality also follows from Minkowski’s matrix inequality writing $\det \left( \frac{A + B}{2} \right) \geq \left( (\det(A/2))^{1/n} + (\det(B/2))^{1/n} \right)^n = \frac{1}{2^n} \left( (\det A)^{1/n} + (\det B)^{1/n} \right)^n$, and using that $x^{1/n} + y^{1/n} \geq 2 \sqrt[x^{1/n}y^{1/n}].$

5. Let $K$ be an open bounded convex set in $\mathbb{R}^n$ (a convex body) with center of mass $x_0$. Consider all ellipsoids centered at $x_0$ and containing $K$, among these there exists an ellipsoid having minimum volume. Prove that this ellipsoid is unique.
HINT: suppose \( E_1 \) and \( E_2 \) are two different ellipsoids of minimum volume with corresponding defining matrices \( A_1 \) and \( A_2 \); we have \( |E_1| = |E_2| \) and \( A_1 \neq A_2 \). Therefore, from Problem 3 \( \det A_1 = \det A_2 \). Consider the ellipsoid \( E \) with corresponding matrix \( A = \frac{A_1 + A_2}{2} \). Prove that \( E \) contains \( K \), and use Problems 3 and 4 to show that \( |E| < |E_1| \).

6. We recall the estimate

\[
\max_{\Omega} u \leq \max_{\partial \Omega} u + c_n \text{diam}(\Omega) \left\| \frac{L u}{(\det A)^{1/n}} \right\|_{L^n(\Omega)}
\]  

(1)

valid for all \( u \in C^2(\Omega) \cap C(\bar{\Omega}) \), where \( Lu(x) = \text{trace} \left( A(x)D^2 u(x) \right) \) with \( A \) symmetric and positive definite in \( \bar{\Omega} \).

Use the John lemma to prove that

\[
\max_{\Omega} u \leq \max_{\partial \Omega} u + c_n |\Omega|^{1/n} \left\| \frac{L u}{(\det A)^{1/n}} \right\|_{L^n(\Omega)}
\]

whenever \( \Omega \) is a bounded convex domain.

HINT: let \( E \) be the ellipsoid of minimum volume of \( \Omega \), then by John’s lemma \( \alpha_n E \subset \Omega \subset E \) with \( \alpha_n \) constant depending only on \( n \). Let \( T \) be an affine transformation such that \( T(E) = B_1(0) \), the unit ball, and let \( v(z) = u(T^{-1}z) \) defined for \( z \in T(\Omega) \). Apply (1) to the function \( v \) in \( T(\Omega) \).

7. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and \( \mu \) is Borel measure over \( \Omega \) such that \( \mu(\Omega) < \infty \). Prove that

(a) for each \( k \) there exist a disjoint family \( \{\Omega_j^k\}_{j=1}^{N_k} \) of Borel subsets of \( \Omega \), such that \( \text{diam}(\Omega_j^k) < 1/k \) and \( \Omega = \bigcup_{j=1}^{N_k} \Omega_j^k \).

(b) Pick \( x_j^k \in \Omega_j^k \) and let \( \mu_k = \sum_{j=1}^{N_k} \mu(\Omega_j^k) \delta_{x_j^k} \) with \( \delta_{x_j^k} \) the Dirac delta concentrated at \( x_j^k \). Prove that \( \mu_k \to \mu \) weakly, that is, \( \int f \, d\mu_k \to \int f \, d\mu \) as \( k \to \infty \) for each \( f \in C(\bar{\Omega}) \).

8. Let \( \mu_n \) and \( \mu \) be Borel measures in \( \Omega \subset \mathbb{R}^n \) that are finite on compact sets (and therefore regular). Suppose that

(a) \( \limsup_{k \to \infty} \mu_k(F) \leq \mu(F) \) for each compact \( F \subset \Omega \); and

(b) \( \liminf_{k \to \infty} \mu_k(G) \geq \mu(G) \) for each open \( G \subset \Omega \).

Prove that \( \mu_k \to \mu \) weakly, that is, \( \int f(x) \, d\mu_k \to \int f(x) \, d\mu \) for all \( f \) continuous with compact support in \( \Omega \) (or for all \( f \) continuous and bounded in \( \Omega \) if \( \mu_k(\Omega) \) and \( \mu(\Omega) \) are finite).
9. In $\mathbb{R}^3$ consider the function

$$u(x_1, x_2, x_3) = (1 + x_1^2)(x_2^2 + x_3^2)^{2/3}.$$ 

The objective here is to show that $u$ is a convex Aleksandrov solution to $Mu = \phi$ with $\phi \in C^\infty$, $\lambda \leq \phi \leq \Lambda$ ($\lambda, \Lambda$ some positive constants) in a sufficiently small ball around the origin $B_\epsilon(0)$, $u$ on $\partial B_\epsilon(0)$ is continuous, and $u \in C^1(B_\epsilon(0))$ but $u \notin C^2(B_\epsilon(0))$.

(a) calculate $D^2u(x_1, x_2, x_3)$ when $(x_1, x_2, x_3) \neq 0$;

(b) show that $\det D^2u(x_1, x_2, x_3) = 32/9 (1 + x_1^2)^2 ((1/3) - (7/3)x_1^2) := \phi(x_1, x_2, x_3)$ for $((x_1, x_2, x_3) \neq 0$;

(c) the function $u$ is not convex in a domain sufficiently far from zero.

(d) calculate the principal minors of $D^2u$ when $(x_1, x_2, x_3) \neq 0$;

(e) prove that for $x_1^2 + x_2^2 + x_3^3 = \epsilon$ with $\epsilon$ sufficiently small, the determinants of the principal minors are positive and therefore $D^2u(x_1, x_2, x_3)$ is positive definite for all $(x_1, x_2, x_3) \neq 0$ in the ball $B_{\sqrt{\epsilon}}(0) = \{x_1^2 + x_2^2 + x_3^3 \leq \epsilon\}$.

HINT: for the principal minor of order two use Lagrange multipliers.

(f) show that the function $u$ is strictly convex in any convex domain not intersecting the line $\ell \equiv \{x_2 = 0, x_3 = 0\}$ and contained in $B_{\sqrt{\epsilon}}(0)$, and it is $C^\infty$ away from the line $\ell$.

(g) the function $u$ is convex in $B_{\sqrt{\epsilon}}(0)$. HINT: use that if a one variable function $f$ is continuous non negative in $[-a, a]$, $f(0) = 0$, $f$ is convex in $(0, a)$ and in $(-a, 0)$, then $f$ is convex in $[-a, a]$. Use this as follows. If
$P_1, P_2$ are in $B_{\sqrt{\epsilon}}(0)$, then we need to show the function $u$ restricted to the segment $\overline{P_1P_2}$ is convex as a function of one variable. There are two cases: (a) when the segment $\overline{P_1P_2}$ does not intersect $\ell$, then the convexity follows from (f); if $\overline{P_1P_2}$ intersects $\ell$, then use the convexity result in one variable.

(h) the graph of $u$ contains the line $x_2 = 0, x_3 = 0$;

(i) $\partial u((x_1, 0, 0))$ has measure zero. HINT: use Aleksandrov’s lemma (the set of supporting hyperplanes containing a segment in the graph of $u$ has measure zero).

(j) if $E \subset B_{\sqrt{\epsilon}}(0)$ is a Borel set, then $|\partial u(E)| = \int_E \phi(x) \, dx$. HINT: $\partial u(E) = \partial u(E \cap \ell) \cup \partial u(E \cap \ell^c)$; and from (i) $|\partial u(E \cap \ell)| = 0$; next write $E$ as a disjoint union over the eight octants and since $u$ is $C^2$ and convex away from $\ell$ by adding over each piece of $E$ we can represent $|\partial u(E \cap \ell^c)|$ as the integral of $\phi$ over $E \cap \ell^c$.

(k) $u \in C^{1,1/3}(\partial B_{\sqrt{\epsilon}}(0))$. 