WAVE AND MAXWELL'S EQUATIONS

CRISTIAN E. GUTIÉRREZ
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1. Model for the vibrant string

Suppose that a perfectly elastic string is attached at two end points on the x-axis, $x = 0$ and $x = \ell$, and its equilibrium position is the segment $[0, \ell]$. A tension T_0 acts at the end points of the string to maintain the equilibrium position. Let us pluck the string at $t = 0$ so that the initial shape of the string is given by a function $f(x)$ and we give at the same time at the point $(x, f(x))$ an initial velocity $g(x)$. We let the time run and then the shape of the string changes with the time t. Let us then denote by $y(x, t)$ the y-coordinate of the string at x and time t. We will show that

$y(x,t)$ satisfies a second order pde. Suppose the density of the string is uniform, that is, the mass of a section Δx of the string is $\mu\,\Delta x$, with $\mu > 0$ a constant. We will deduce the equation of the string by using Newton's second law. Consider a small interval $[x, x+\Delta x]$ on the x-axis and the corresponding piece of string on that interval. Let T be the tension force acting at the extreme $(x, y(x,t))$ of the string, and let T' be the tension acting at the extreme $(x+\Delta x, y(x+\Delta x, t))$. Let θ be the angle that the tension force T makes with the positive horizontal direction, and let θ' be the angle that T' makes with the positive horizontal direction. We then have that $T = (|T|\cos\theta, |T|\sin\theta)$, and $T' = (|T'|\cos\theta', |T'|\sin\theta')$. We assume that the tensions are tangential at the end points of the segment $x, x+\Delta x$. Since we assume no motion of the string in the horizontal direction, the tensions T, T' have constant opposite components in the horizontal direction. That is, we have $-|T|\cos\theta = |T'|\cos\theta' = T_0$ =constant. Notice that this implies that the tension at the end points of the string is then T_0. Since Δx is very small then $T \approx -T'$. The net vertical force acting on the piece of string from x to $x+\Delta x$ is then $|T|\,\sin\theta + |T'|\,\sin\theta'$. From Newton's second law, the sum of the acting forces in the vertical direction equals mass times the acceleration of the string at $y(x,t)$ in the vertical direction. Since the mass of the string from x to $x+\Delta x$ is $\mu\,\Delta x$, we get

$$|T'|\,\sin\theta' + |T|\,\sin\theta \approx \mu\,\Delta x\,\frac{\partial^2 y}{\partial t^2}(x,t).$$

Dividing by T_0 yields

$$(\tan\theta' - \tan\theta) \approx \frac{\mu}{T_0}\,\Delta x\,\frac{\partial^2 y}{\partial t^2}(x,t).$$

On the other hand, since the tensions are tangential to $y(x,t)$ (t fixed), we have $\tan\theta \approx \dfrac{\partial y}{\partial x}(x,t)$, and $\tan\theta' \approx \dfrac{\partial y}{\partial x}(x+\Delta x, t)$. Therefore

$$\frac{\partial y}{\partial x}(x+\Delta x, t) - \frac{\partial y}{\partial x}(x,t) \approx \Delta x\,\frac{\partial^2 y}{\partial x^2}(x,t).$$

Therefore

$$\Delta x\,\frac{\partial^2 y}{\partial x^2}(x,t) = \frac{\mu}{T_0}\,\Delta x\,\frac{\partial^2 y}{\partial t^2}(x,t);$$

that is, we obtain the equation

$$\frac{\partial^2 y}{\partial t^2}(x,t) = v^2\,\frac{\partial^2 y}{\partial x^2}(x,t), \quad \text{with } v = \sqrt{\frac{T_0}{\mu}}; \tag{1.1}$$

v denotes the speed of propagation in the string and the formula for v was discovered by Vincenzo Galilei in the 1500's, talented musician and father of Galileo Galilei, see `https://en.wikipedia.org/wiki/Vincenzo_Galilei`.

2. Solution of the wave equation in dimension one

The wave operator is given by

$$\Box u = u_{tt} - c^2 \Delta_x u$$

here $u = u(x,t)$, $x \in \mathbb{R}^n$, $t > 0$, and c is a constant. A problem to solve is

$$\begin{aligned} \Box u &= 0, \qquad \text{in } \mathbb{R}^n \times (0, +\infty) \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x). \end{aligned}$$

Suppose $n = 1$ and write

$$\partial_{tt} - c^2 \partial_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x).$$

Then solve

$$\begin{aligned} (\partial_t + c\partial_x) u &= v \\ (\partial_t - c\partial_x) v &= 0. \end{aligned}$$

Using for example the method of solving 1st order pdes, we get that

$$v(x,t) = \phi(x + ct),$$

with ϕ an arbitrary function. Next solve $(\partial_t + c\partial_x) u = \phi(x + ct)$. Let us replace t by y, so we solve the first order pde

$$cu_x + u_y = \phi(x + cy),$$

with the initial condition $u(x,0) = f(x)$. So the characteristic equations of this pde are

$$\dot{x} = c, \qquad \dot{y} = 1, \qquad \dot{z} = \phi(x + cy),$$

with $x(0,s) = s; y(0,s) = 0; z(0,s) = f(s)$. So we obtain

$$x(t,s) = ct + s, \qquad y(t,s) = t, \qquad z(t,s) = \int_0^t \phi(s + 2c\eta)\, d\eta + f(s).$$

Inverting, we get $t = y$ and $s = x - cy$ and so

$$u(x,y) = z(y, x - cy) = f(x - cy) + \int_0^y \phi(x - cy + 2c\eta)\, d\eta.$$

Returning to the original variables, i.e., replacing y by t we get

$$u(x,t) = f(x-ct) + \int_0^t \phi(x-ct+2c\eta)\,d\eta = f(x-ct) + \frac{1}{2}\int_{-t}^{t} \phi(x+cz)\,dz.$$

We now impose the boundary condition $u_t(x,0) = g(x)$. We have

$$u_t(x,t) = -c\,f'(x-ct) + \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right),$$

so $u_t(x,0) = -c\,f'(x) + \phi(x) = g(x)$, and so $\phi(x) = g(x) + cf'(x)$ and we therefore obtain the d'Lambert formula

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz, \tag{2.1}$$

valid for $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. Notice that since x represents position and t time, then c represents position/time, that is, c is a velocity; the velocity of propagation. From (1.1), we see that if the tension T_0 is very large, then the velocity v increases, as well as if the density μ of the string is small, then v also is large.

3. Method of the spherical averages

Suppose h is a continuous function in $\mathbb{R}^n$, for $r \in \mathbb{R}$ let

$$M_h(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} h(x+r\xi)\,d\sigma(\xi), \tag{3.2}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. The function M_h is defined in $\mathbb{R}^n \times \mathbb{R}$, and $M_h(x,-r) = M_h(x,r)$. If $r > 0$, then changing variables we get

$$M_h(x,r) = \fint_{|x-y|=r} h(y)\,d\sigma(y).$$

If $h \in C^k(\mathbb{R}^n)$, then $M_h \in C^k(\mathbb{R}^{n+1})$. Let us calculate $\partial_r M_h(x,r)$:

$$\begin{aligned}
\partial_r M_h(x,r) &= \frac{1}{\omega_n}\partial_r\left(\int_{|\xi|=1} h(x+r\xi)\,d\sigma(\xi)\right)\\
&= \frac{1}{\omega_n}\int_{|\xi|=1}\sum_{j=1}^n h_{x_j}(x+r\xi)\xi_i\,d\sigma(\xi) = \frac{1}{\omega_n}\int_{|\xi|=1} Dh(x+r\xi)\cdot\xi\,d\sigma(\xi)\\
&= \frac{r^{1-n}}{\omega_n}\int_{|\eta-x|=r} Dh(\eta)\cdot\left(\frac{\eta-x}{r}\right)d\sigma(\eta)\\
&= \frac{r^{1-n}}{\omega_n}\int_{|\eta-x|\le r}\Delta h(\eta)\,d\eta \qquad \text{by the divergence theorem}\\
&= \frac{r^{1-n}}{\omega_n}\int_{|\xi|\le r}\Delta h(x+\xi)\,d\xi = \frac{r^{1-n}}{\omega_n}\int_{|\xi|\le r}\Delta_x(h(x+\xi))\,d\xi\\
&= \frac{r^{1-n}}{\omega_n}\Delta_x\left(\int_{|\xi|\le r} h(x+\xi)\,d\xi\right)\\
&= \frac{r^{1-n}}{\omega_n}\Delta_x\left(\int_0^r \rho^{n-1}\int_{|\xi|=1} h(x+\rho\xi)\,d\sigma(\xi)\,d\rho\right)\\
&= r^{1-n}\Delta_x\left(\int_0^r \rho^{n-1}M_h(x,\rho)\,d\rho\right)\\
&= r^{1-n}\int_0^r \rho^{n-1}\Delta_x M_h(x,\rho)\,d\rho.
\end{aligned}$$

Hence

$$r^{n-1}\partial_r M_h(x,r) = \int_0^r \rho^{n-1}\Delta_x M_h(x,\rho)\,d\rho,$$

and differentiating with respect to r yields

$$\partial_r\left(r^{n-1}\partial_r M_h(x,r)\right) = r^{n-1}\Delta_x M_h(x,r),$$

which can be re-written as

$$\left(\partial_{rr}^2 + \frac{n-1}{r}\partial_r\right)M_h(x,r) = \Delta_x M_h(x,r). \tag{3.3}$$

This is called Darboux's equation, and we showed it is satisfied for $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$ and for any $h \in C^2(\mathbb{R}^n)$. In addition

$$M_h(x,0) = h(x), \qquad \text{and } \partial_r M_h(x,0) = 0$$

where the last identity follows since $M_h(x,r)$ is even in r.

4. Solution of the wave equation with the method of the spherical averages

This method is due to Poisson. Suppose $u = u(x,t)$ is a C^2 solution to $\square u = 0$ in $\mathbb{R}^n \times (0,+\infty)$ with the boundary conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Let us take the spherical averages of u, that is, let

$$M_u(x,r,t) = \frac{1}{\omega_n} \int_{|\xi|=1} u(x + r\xi, t)\, d\sigma(\xi).$$

Then $M_u(x,r,t)$ satisfies (3.3) in x and r for each t. Let us calculate $\Delta_x M_u(x,r,t)$:

$$\begin{aligned}\Delta_x M_u(x,r,t) &= \frac{1}{\omega_n}\Delta_x \int_{|\xi|=1} u(x + r\xi, t)\, d\sigma(\xi) = \frac{1}{\omega_n} \int_{|\xi|=1} \Delta_x u(x + r\xi, t)\, d\sigma(\xi)\\ &= \frac{c^{-2}}{\omega_n} \int_{|\xi|=1} u_{tt}(x + r\xi, t)\, d\sigma(\xi) = c^{-2}\partial_{tt} M_u(x,r,t).\end{aligned}$$

Then from (3.3) we get that

$$\partial_{tt} M_u(x,r,t) = c^2 \left(\partial_{rr} + \frac{n-1}{r}\partial_r\right) M_u(x,r,t), \tag{4.4}$$

for each $x \in \mathbb{R}^n$ and for all $r \in \mathbb{R}$ and $t > 0$. In addition, from the boundary conditions on u we get

$$M_u(x,r,0) = M_f(x,r), \qquad \partial_t M_u(x,r,0) = M_g(x,r).$$

4.1. Solution when $n = 3$.

In this case equation (4.4) becomes

$$\partial_{tt} M_u(x,r,t) = c^2 \left(\partial_{rr} + \frac{2}{r}\partial_r\right) M_u(x,r,t),$$

and multiplying this equation by r we get that

$$\partial_{tt}\left(rM_u\right) = c^2\left(r\partial_{rr}M_u + 2\partial_r M_u\right) = c^2 \partial_{rr}\left(rM_u\right),$$

that is, rM_u satisfies the one dimensional wave equation in the variables $-\infty < r < \infty$ and $t > 0$ for each $x \in \mathbb{R}^3$. In addition,

$$rM_r(x,r,0) = rM_f(x,r), \qquad \partial_t\left(rM_u(x,r,0)\right) = rM_g(x,r).$$

Therefore from d'Lambert formula

$$rM_u(x,r,t) = \frac{(r+ct)M_f(x,r+ct) + (r-ct)M_f(x,r-ct)}{2} + \frac{1}{2c}\int_{r-ct}^{r+ct} \xi M_g(x,\xi)\, d\xi.$$

Now the function $M_f(x,\xi)$ is even in ξ and the function $\xi M_g(x,\xi)$ is odd in ξ, so

$$\int_{-r+ct}^{r+ct} \xi M_g(x,\xi)\, d\xi = \int_{-r+ct}^{r-ct} + \int_{r-ct}^{r+ct} = \int_{r-ct}^{r+ct},$$

and we obtain

$$M_u(x,r,t) = \frac{(r+ct)M_f(x,r+ct) - (ct-r)M_f(x,ct-r)}{2r} + \frac{1}{2cr}\int_{ct-r}^{ct+r} \xi M_g(x,\xi)\,d\xi := A+B.$$

We have

$$\lim_{r\to 0} B = tM_g(x,ct).$$

If we set $w(s) = (ct+s)M_f(x,ct+s)$, then

$$A = \frac{w(r)-w(-r)}{2r} = \frac{w(r)-w(0)}{2r} + \frac{w(-r)-w(0)}{-2r} \to w'(0) = \partial_t\left(t\,M_f(x,ct)\right).$$

Therefore we obtain the formula

$$\begin{aligned} (4.5)\qquad u(x,t) &= tM_g(x,ct) + \partial_t\left(t\,M_f(x,ct)\right) \\ &= \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} g(y)\,d\sigma(y) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{|y-x|=ct} f(y)\,d\sigma(y)\right). \end{aligned}$$

If on the other hand $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$ then it follows by direct calculation that the function defined by (4.5) satisfies $\Box u = 0$ in $\mathbb{R}^3 \times (0,+\infty)$ and the boundary conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Notice also that from Darboux equation (3.3) when $n = 3$, both functions $tM_g(x,ct)$ and $\partial_t\left(t\,M_f(x,ct)\right)$ that appear in (4.5) satisfy the 3-d wave equation taking $r = ct$ (they satisfy the wave equation for all $x \in \mathbb{R}^n$ and $-\infty < t < \infty$). In fact, let us verify that $tM_g(x,ct)$ is a solution. We only need to notice that $\partial_{tt}\left(t\,M_g(x,ct)\right) = 2\,c\,(\partial_t M_g)(x,ct) + c^2\,t\,(\partial_{tt}M_g)(x,ct)$ is equal to $\partial_{rr}\left(c\,r\,M_g(x,r)\right)$ when $r = ct$. Now from (3.3) for $n = 3$ we obtain $\partial_{rr}\left(c\,r\,M_g(x,r)\right) = cr\Delta_x M_g(x,r)$ and the claim follows.[1]

Since $\omega_3 = 4\pi$, from equation (4.5) it follows differentiating under the integral sign the following formula due to Kirchhoff:

$$(4.6)\qquad u(x,t) = \frac{1}{4\pi c^2 t^2}\int_{|x-y|=ct} \left[tg(y) + f(y) + Df(y)\cdot(y-x)\right]\,d\sigma(y).$$

4.2. **Solution when** $n = 2$. Let $\bar{f}(x_1,x_2,x_3) = f(x_1,x_2)$ and $\bar{g}(x_1,x_2,x_3) = g(x_1,x_2)$. Applying formula (4.5) we get

$$(4.7)\qquad \bar{u}(x_1,x_2,x_3,t) = \frac{1}{4\pi c^2 t}\int_{|x-y|=ct} \bar{g}(y)\,d\sigma(y) + \partial_t\left(\frac{1}{4\pi c^2 t}\int_{|x-y|=ct} \bar{f}(y)\,d\sigma(y)\right).$$

[1]We remark that the function u in the first indentity in (4.5) is a solution to the wave equation for all $x \in \mathbb{R}^n$ and $-\infty < t < \infty$ (this will be used later on to solve Maxwell equations). As it was indicated before for $r < 0$, the average $M_g(x,r) = M_g(x,-r)$ extended as an even function to the whole line.

Now

$$
\begin{aligned}
\int_{|x-y|=ct} \bar{g}(y)\,d\sigma(y) &= c^2t^2 \int_{|z|=1} \bar{g}(x+ctz)\,d\sigma(z)\\
&= c^2t^2 \int_{z_1^2+z_2^2+z_3^2=1} g(x_1+ctz_1, x_2+ctz_2)\,d\sigma(z)\\
&= 2c^2t^2 \int_{z_1^2+z_2^2+z_3^2=1; z_3\geq 0} g(x_1+ctz_1, x_2+ctz_2)\,d\sigma(z)\\
&= 2c^2t^2 \int_{z_1^2+z_2^2\leq 1} g(x_1+ctz_1, x_2+ctz_2)\,\frac{1}{\sqrt{1-z_1^2-z_2^2}}\,dz_1dz_2, \quad \text{using }^2\\
&= 2ct \int_{(\xi_1-x_1)^2+(\xi_2-x_2)^2\leq c^2t^2} g(\xi_1,\xi_2)\,\frac{1}{\sqrt{c^2t^2-(\xi_1-x_1)^2-(\xi_2-x_2)^2}}\,d\xi_1 d\xi_2
\end{aligned}
$$

and a similar expression for f. Since the right hand side of (4.7) does not depend of x_3, we then get that the solution to $\Box u = 0$ in $\mathbb{R}^2 \times (0,+\infty)$ is given by

$$
\begin{aligned}
u(x_1,x_2,t) &= \frac{1}{2\pi c}\int_{(\xi_1-x_1)^2+(\xi_2-x_2)^2\leq c^2t^2} g(\xi_1,\xi_2)\,\frac{1}{\sqrt{c^2t^2-(\xi_1-x_1)^2-(\xi_2-x_2)^2}}\,d\xi_1 d\xi_2\\
&\quad + \partial_t\left(\frac{1}{2\pi c}\int_{(\xi_1-x_1)^2+(\xi_2-x_2)^2\leq c^2t^2} f(\xi_1,\xi_2)\,\frac{1}{\sqrt{c^2t^2-(\xi_1-x_1)^2-(\xi_2-x_2)^2}}\,d\xi_1 d\xi_2\right).
\end{aligned}
$$

4.3. **Solution of the non homogeneous equation.** We solve $\Box u = f$ in $\mathbb{R}^n \times (-\infty,+\infty)$ with $u(x,0)=0$ and $u_t(x,0)=0$. For each $s \in \mathbb{R}$ fixed, let $U(x,t,s)$ be the solution to $\Box U = 0$ for $x \in \mathbb{R}^n, -\infty < t < \infty$ with $U(x,s,s)=0$ and $U_t(x,s,s) = f(x,s)$ for $x \in \mathbb{R}^n$. Notice this is possible from the footnote on page 7. Define

$$u(x,t) = \int_0^t U(x,t,s)\,ds.$$

Let us show u is the desired solution. We have

$$u_t(x,t) = U(x,t,t) + \int_0^t U_t(x,t,s)\,ds = \int_0^t U_t(x,t,s)\,ds.$$

[2] If $x_3 = \phi(x_1,x_2)$ and a surface $S = \{(x_1,x_2,\phi(x_1,x_2)) : (x_1,x_2) \in D\}$, then $\int_S f(x_1,x_2,x_3)\,d\sigma(x_1,x_2,x_3) = \int_D f(x_1,x_2,\phi(x_1,x_2))\,\sqrt{1+(\partial_{x_1}\phi(x_1,x_2))^2+(\partial_{x_1}\phi(x_1,x_2))^2}\,dx_1dx_2$.

Also

$$\begin{aligned} u_{tt}(x,t) &= U_t(x,t,t) + \int_0^t U_{tt}(x,t,s)\,ds \\ &= f(x,t) + c^2 \int_0^t \Delta_x U(x,t,s)\,ds \\ &= f(x,t) + c^2 \Delta_x \left(\int_0^t U(x,t,s)\,ds \right) = f(x,t) + c^2 \Delta_x u(x,t). \end{aligned}$$

Therefore we have proved that the equation $u_{tt} - c^2 \Delta_x u = F$ in $\mathbb{R}^n \times (-\infty, \infty)$, with $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ has a solution that is unique.

We can write also an integral formula for the solution of the non homogenous problem when $n = 3$. Let $V(x,t,s) = U(x,t+s,s)$. Then $\Box V = 0$ for $x \in \mathbb{R}^n$ and $t > 0$ and $V(x,0,s) = U(x,s,s) = 0$ and $V_t(x,0,s) = f(x,s)$. From the Kirchhoff formula ($n = 3$) we have

$$V(x,t,s) = \frac{1}{4\pi c^2 t} \int_{|x-y|=ct} f(y,s)\,d\sigma(y).$$

So

$$U(x,t,s) = V(x,t-s,s) = \frac{1}{4\pi c^2 (t-s)} \int_{|x-y|=c(t-s)} f(y,s)\,d\sigma(y); \qquad t > s.$$

Therefore

$$\begin{aligned} u(x,t) &= \int_0^t V(x,t-s,s)\,ds = \frac{1}{4\pi c^2} \int_0^t \frac{1}{t-s} \int_{|x-y|=c(t-s)} f(y,s)\,d\sigma(y)\,ds \\ &= \frac{1}{4\pi c} \int_0^{ct} \frac{1}{r} \int_{|x-y|=r} f(y,t-(r/c))\,d\sigma(y)\,dr \\ &= \frac{1}{4\pi c} \int_{|x-y|\le ct} \frac{f\left(y, t - (|x-y|/c)\right)}{|x-y|}\,dy. \end{aligned} \tag{4.8}$$

5. Maxwell's equations

The *electromagnetic field* (EM) is a physical field produced by electrically charged objects. It extends indefinitely throughout space and describes the electromagnetic interaction. It is one of the four fundamental forces of nature (the others are gravitation, the weak interaction, and the strong interaction). The field propagates by electromagnetic radiation; in order of increasing energy (decreasing wavelength) electromagnetic radiation comprises: radio waves, microwaves, infrared, visible light, ultraviolet, X-rays, and gamma rays. The field (EM) can be viewed as the combination of an electric field **E** and a magnetic field **H**, that is, these are three-dimensional vector fields that have a value defined at every point

of space and time: $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{x}, t)$, where $\mathbf{x}$ represents a point in 3-d space $\mathbf{x} = (x, y, z)$. The electric field is produced by stationary charges, and the magnetic field by moving charges (currents); these two are often described as the sources of the field. The way in which $\mathbf{E}$ and $\mathbf{H}$ interact is described by Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \text{Gauss's law} \tag{M.1}$$

$$\nabla \cdot \mathbf{H} = 0, \quad \text{Gauss's law for magnetism} \tag{M.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \quad \text{Faraday's law} \tag{M.3}$$

$$\nabla \times \mathbf{H} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \text{Ampère-Maxwell's law.} \tag{M.4}$$

Here

$\nabla = (\partial_x, \partial_y, \partial_z)$	the gradient
$\rho = \rho(\mathbf{x}, t)$	charge density
ϵ_0	permittivity of free space
$\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$	current density vector
μ_0	permeability of free space

We have $c = 1/\sqrt{\epsilon_0 \mu_0}$, the speed of light in vacuum.

We notice that the constant ϵ_0 appears in Coulomb's law: if q_1 and q are charges located at the points P_1 and P respectively, then the force felt at P is given by

$$F(P) = \frac{1}{4\pi\epsilon_0} q_1 \, q \, \frac{P_1 - P}{|P_1 - P|^3},$$

where $\dfrac{1}{4\pi\epsilon_0} = 8.854 \times 10^9 \, N \, m^2/coulomb^2$. The value of the permeability of free space is $\mu_0 = 4\pi \times 10^{-7} N/Ampere^2$. Since 1 Amp=1 coulomb/sec, we have $c = \sqrt{8.854} \times 10^8 m/sec \approx 3 \times 10^5 km/sec.$

Assuming no currents, $\mathbf{J} = 0$, and no charges, $\rho = 0$, Maxwell's equations have the form

$$\nabla \cdot \mathbf{E} = 0, \tag{5.1}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{5.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \tag{5.3}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{5.4}$$

These are all together a system of eight partial differential equations of the first order.

5.1. **Maxwell's equations written in an equivalent way.** In general, when the electric permittivity $\epsilon = \epsilon(x, y, z)$ and the magnetic permeability $\mu = \mu(x, y, z)$, Maxwell's equations are written introducing the vectors

$$\mathbf{D} = \text{electric displacement vector} \tag{5.5}$$

$$\mathbf{B} = \text{magnetic flux density vector,} \tag{5.6}$$

in the following way

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.7}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{5.8}$$

together with the so-called constitutive relations

$$\mathbf{D} = \epsilon\, \mathbf{E} \tag{5.9}$$

$$\mathbf{B} = \mu\, \mathbf{H}. \tag{5.10}$$

In the most general case, the functions μ and ϵ are 3×3 matrices. In many cases, $\mathbf{J} = \sigma\, \mathbf{E}$, where σ is a constant called the electric conductivity. In addition, the fields $\mathbf{D}$ and $\mathbf{B}$ satisfy $\nabla \cdot \mathbf{D} = \rho$ and $\nabla \cdot \mathbf{B} = 0$; here ρ is as before the charge density.

6. Solution to the Maxwell equations

6.1. **If E and H solve Maxwell then E and H solve the wave equation.** We show that each component of the magnetic and electric field satisfy the scalar wave equation assuming the fields have second derivatives in all variables. Recall the following formula from vector analysis. For a vector $\mathbf{A} = \mathbf{A}(x, y, z) = (\mathbf{A}_1(x, y, z), \mathbf{A}_2(x, y, z), \mathbf{A}_3(x, y, z)) \in C^2$, we have:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A}. \tag{6.1}$$

Denote $\nabla \cdot \nabla = \nabla^2$, the Laplacian, and so

$$\nabla^2 \mathbf{A} =$$

$$\left(\frac{\partial^2 \mathbf{A}_1}{\partial x^2} + \frac{\partial^2 \mathbf{A}_1}{\partial y^2} + \frac{\partial^2 \mathbf{A}_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 \mathbf{A}_2}{\partial x^2} + \frac{\partial^2 \mathbf{A}_2}{\partial y^2} + \frac{\partial^2 \mathbf{A}_2}{\partial z^2}\right)\mathbf{j} + \left(\frac{\partial^2 \mathbf{A}_3}{\partial x^2} + \frac{\partial^2 \mathbf{A}_3}{\partial y^2} + \frac{\partial^2 \mathbf{A}_3}{\partial z^2}\right)\mathbf{k}.$$

Suppose the fields **E** and **H** satisfy Maxwell's equations and let us see what kind of non homogeneous wave equations these yield. From (M.3) and (M.4) we get

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Then from (6.1)

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

so from (M.1) we obtain that **E** satisfies the non homogeneous wave equation

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E} - \mu_0 \frac{\partial \mathbf{J}}{\partial t} - \nabla(\rho/\epsilon_0). \tag{6.2}$$

Similarly we get for **H** that

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla^2 \mathbf{H} + \mu_0 \nabla \times \mathbf{J}. \tag{6.3}$$

That is, we proved that if **E** and **H** are C^2 and satisfy the Maxwell equations (M.1), (M.2), (M.3), and (M.4), then **E** and **H** solve the non homogeneous wave equations (6.2) and (6.3), respectively.

6.2. **Case when ϵ is variable.** We will use here the formulation from Subsection 5.1. Suppose the magnetic permeability μ is constant and $\epsilon(x, y, z) > 0$ is a scalar function. Since $\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon\, \mathbf{E}) = \rho$, we have $(\nabla\epsilon) \cdot \mathbf{E} + \epsilon\, \nabla \cdot \mathbf{E} = \rho$, and consequently

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} - \frac{\nabla\epsilon}{\epsilon} \cdot \mathbf{E}.$$

Taking the curl of (5.7), using (5.8) and the vector identity (6.1), we obtain the equation

$$\nabla\, (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\, \epsilon \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \mu\, \frac{\partial \mathbf{J}}{\partial t}.$$

Of course, if ϵ is constant this is the non homogeneous wave equation. Otherwise, we obtain that the field **E** satisfies the equation

$$\mu\, \epsilon \frac{\partial^2 \mathbf{E}}{\partial^2 t} = \nabla^2 \mathbf{E} + \nabla \left(\frac{\nabla\epsilon}{\epsilon} \cdot \mathbf{E} \right) - \nabla \cdot \left(\frac{\rho}{\epsilon} \right) - \mu\, \frac{\partial \mathbf{J}}{\partial t}, \tag{6.4}$$

which is more complicated than the wave equation because of the term $\nabla \left(\frac{\nabla\epsilon}{\epsilon} \cdot \mathbf{E} \right)$. If ϵ changes abruptly in the neighborhood of a point then the term $\frac{\nabla\epsilon}{\epsilon}$ gets very large or infinity. This is the typical case when one has a dielectric that has two different refractive indexes in two adjacent regions. On the separating surface between the regions the gradient of ϵ becomes infinite. A very interesting

qualitative analysis of equation (6.4) can be found in the very interesting book by D. Marcuse [Mar82, pp. 9-11].

6.3. **Solution of the initial value problem for Maxwell equations.** Let us now prove the converse. We seek for $\mathbf{E}$ and $\mathbf{H}$ satisfying (M.1), (M.2), (M.3), and (M.4), with the initial conditions

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}), \qquad \text{and } \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}). \tag{6.5}$$

We first observe that from (M.1) and (M.4) (ρ independent of t) we get that

$$\nabla \cdot \mathbf{J}(\mathbf{x},t) = 0, \qquad \text{for all } \mathbf{x} \text{ and } t. \tag{6.6}$$

In addition, from (M.1) and (M.2)

$$\nabla \cdot \mathbf{E}_0(\mathbf{x}) = \rho(\mathbf{x})/\epsilon_0; \text{ and } \nabla \cdot \mathbf{H}_0(\mathbf{x}) = 0. \tag{6.7}$$

So $\mathbf{E}_0$, $\mathbf{H}_0$, and $\mathbf{J}$ must satisfy these compatibilities conditions at the outset. Also (M.4) leads to the condition

$$\frac{\partial \mathbf{E}}{\partial t}(\mathbf{x},0) = c^2\, \nabla \times \mathbf{H}_0(\mathbf{x}) - \frac{1}{\epsilon_0}\mathbf{J}(\mathbf{x},0), \tag{6.8}$$

and (M.3) leads to the condition

$$\frac{\partial \mathbf{H}}{\partial t}(\mathbf{x},0) = -\nabla \times \mathbf{E}_0(\mathbf{x}). \tag{6.9}$$

So let $\mathbf{E}$ *be the solution of the non homogenous wave equation* (6.2) *with initial conditions* (6.5) *and* (6.8)*; and let* $\mathbf{H}$ *be the solution of the non homogenous wave equation* (6.3) *with initial conditions* (6.5) *and* (6.9) *(from Section 4.3 these solutions are defined for all* $-\infty < t < \infty$*). We claim* $\mathbf{E}$ *and* $\mathbf{H}$ *solve the Maxwell equations* (M.1)*,* (M.2)*,* (M.3)*, and* (M.4)*.*

First, let us verify (M.1). Let $v(x,t) = \nabla \cdot \mathbf{E}(\mathbf{x},t) - \rho(\mathbf{x})/\epsilon_0$. We claim that $v \equiv 0$. We have from (6.7) that $v(\mathbf{x},0) = 0$. Also $\frac{\partial v}{\partial t}(\mathbf{x},0) = \nabla \cdot \frac{\partial \mathbf{E}}{\partial t}(\mathbf{x},0) = 0$ from (6.8) and (6.6). We also have $\frac{\partial^2 v}{\partial t^2}(\mathbf{x},t) - c^2\nabla^2 v(\mathbf{x},t) = \nabla \cdot \left(\frac{\partial^2 \mathbf{E}}{\partial t^2}(\mathbf{x},t) - c^2\nabla^2\mathbf{E}(\mathbf{x},t)\right) + c^2\nabla^2(\rho/\epsilon_0) = \nabla \cdot \left(\frac{1}{\epsilon_0}\frac{\partial \mathbf{J}}{\partial t} - \frac{1}{\epsilon_0^2\mu_0}\nabla\rho\right) + c^2\nabla^2(\rho/\epsilon_0) = \frac{1}{\epsilon_0}\frac{\partial(\nabla \cdot \mathbf{J})}{\partial t} = 0$ from (6.6).

Second, let us verify (M.4). Let $\mathbf{M} = \frac{\partial \mathbf{E}}{\partial t} - c^2\nabla \times \mathbf{H} + c^2\mu_0\mathbf{J}$, where $c^2 = 1/(\epsilon_0\mu_0)$. We show that $\mathbf{M}$ satisfies the homogeneous wave equation with zero boundary conditions and therefore is zero. Indeed, since $\mathbf{E}$ and $\mathbf{H}$ satisfy the non homogeneous

wave equation, we have

$$
\begin{aligned}
&\frac{\partial^2 \mathbf{M}}{\partial t^2} - c^2\nabla^2\mathbf{M}\\
&= \frac{\partial}{\partial t}\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2\nabla\times\frac{\partial^2 \mathbf{H}}{\partial t^2} + c^2\mu_0\frac{\partial^2 \mathbf{J}}{\partial t^2} - c^2\nabla^2\left(\frac{\partial \mathbf{E}}{\partial t}\right) + c^4\nabla^2\left(\nabla\times\mathbf{H}\right) - c^4\mu_0\nabla^2\mathbf{J}\\
&= c^2\nabla^2\left(\frac{\partial \mathbf{E}}{\partial t}\right) - c^2\mu_0\frac{\partial^2 \mathbf{J}}{\partial t^2} - \frac{c^2}{\epsilon_0}\frac{\partial\nabla\rho}{\partial t} - c^4\nabla\times(\nabla^2\mathbf{H}) - c^4\mu_0\nabla\times(\nabla\times\mathbf{J})\\
&\qquad + c^2\mu_0\frac{\partial^2 \mathbf{J}}{\partial t^2} - c^2\nabla^2\left(\frac{\partial \mathbf{E}}{\partial t}\right) + c^4\nabla^2\left(\nabla\times\mathbf{H}\right) - c^4\mu_0\nabla^2\mathbf{J}\\
&= -c^4\nabla\times(\nabla^2\mathbf{H}) - c^4\mu_0\nabla\times(\nabla\times\mathbf{J}) + c^4\nabla^2\left(\nabla\times\mathbf{H}\right) - c^4\mu_0\nabla^2\mathbf{J}\\
&= -c^4\nabla\times(\nabla^2\mathbf{H}) + c^4\nabla^2\left(\nabla\times\mathbf{H}\right) \text{ from (6.1) and (6.6)}\\
&= c^4\left\{\nabla^2\left(\nabla\times\mathbf{H}\right) - \nabla\times\left(\nabla^2\mathbf{H}\right)\right\} = 0
\end{aligned}
$$

where the last identity follows using formula (6.1) applied twice. In fact, from (6.1) applied with $\mathbf{A} = \nabla\times\mathbf{H}$ we get

$$\nabla\times(\nabla\times(\nabla\times\mathbf{H})) = \nabla(\nabla\cdot(\nabla\times\mathbf{H})) - \nabla^2(\nabla\times\mathbf{H}) = -\nabla^2(\nabla\times\mathbf{H})$$

since the divergence of the curl is zero. On the other hand, by (6.1)

$$\nabla\times(\nabla\times\mathbf{H}) = \nabla(\nabla\cdot\mathbf{H}) - \nabla^2\mathbf{H}$$

so

$$\nabla\times(\nabla\times(\nabla\times\mathbf{H})) = \nabla\times\left(\nabla(\nabla\cdot\mathbf{H}) - \nabla^2\mathbf{H}\right) = -\nabla\times\left(\nabla^2\mathbf{H}\right)$$

$\nabla\times\nabla\phi = 0$ for any ϕ. Let us verify the boundary conditions. We have $\mathbf{M}(\mathbf{x},0) = \frac{\partial \mathbf{E}}{\partial t}(\mathbf{x},0) - c^2\nabla\times\mathbf{H}(\mathbf{x},0) + c^2\mu_0\mathbf{J}(\mathbf{x},0) = 0$ from (6.8). Also[3]

$$
\begin{aligned}
\frac{\partial \mathbf{M}}{\partial t}(\mathbf{x},0) &= \frac{\partial^2 \mathbf{E}}{\partial t^2}(\mathbf{x},0) - c^2\nabla\times\left(\frac{\partial \mathbf{H}}{\partial t}\right)(\mathbf{x},0) + c^2\mu_0\frac{\partial \mathbf{J}}{\partial t}(\mathbf{x},0)\\
&= c^2\nabla^2\mathbf{E}(x,0) - c^2\mu_0\frac{\partial \mathbf{J}}{\partial t}(\mathbf{x},0) - c^2\nabla(\rho/\epsilon_0)(\mathbf{x})\\
&\qquad + c^2\nabla\times(\nabla\times\mathbf{E}_0(x)) + c^2\mu_0\frac{\partial \mathbf{J}}{\partial t}(\mathbf{x},0) \text{ from (6.2) and (6.7)}\\
&= c^2\nabla^2E_0(\mathbf{x}) - c^2\nabla(\rho/\epsilon_0) + c^2\left(\nabla(\nabla\cdot E_0(\mathbf{x})) - \nabla^2E_0(\mathbf{x})\right) \text{ by (6.1)}\\
&= 0 \text{ by (6.7).}
\end{aligned}
$$

[3] We are using in this calculation that the field $\mathbf{E}(x,t)$ satisfies the wave equation in $\mathbb{R}^3\times\{0\}$. This is true in view of the footnote on page 7 and Section 4.3, since the equation is satisfied in $\mathbb{R}^3\times(-\infty,\infty)$.

To verify Faraday's law (M.3), we set $\mathbf{M} = \dfrac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E}$ and proceed like in the second case.

Finally, to verify (M.4) we set $v(\mathbf{x}, t) = \nabla \cdot \mathbf{H}(x, t)$ and proceed like in the first case.

Therefore from the Kirchhoff formula (4.6) and from formula (4.8) we obtain explicit formulas for $\mathbf{E}$ and $\mathbf{H}$ in terms of $\mathbf{E}_0$ and $\mathbf{H}_0$.

6.4. **Plane waves.** Let $\mathbf{s}$ be a unit vector. Any solution to the wave equation

$$\frac{1}{v^2}\partial_t^2 V = \nabla^2 V,$$

of the form $V(\mathbf{x}, t) = F(\mathbf{x} \cdot \mathbf{s}, t)$ is called a *plane wave*, since at each time t, V is constant on each plane of the form $\mathbf{x} \cdot \mathbf{s}$=constant. That is, for each t the vector $V(\mathbf{x}, t)$ is the same on each plane $\mathbf{x} \cdot \mathbf{s}$=constant. The plane wave propagates in the direction $\mathbf{s}$. If we set $F = F(\eta, t)$, then $\dfrac{\partial^2 V}{\partial x_i^2}(\mathbf{x}, t) = F_{\eta\eta}(\mathbf{x} \cdot \mathbf{s}, t)\, s_i^2$, $i = 1, 2, 3$, and therefore $\nabla^2 V(\mathbf{x}, t) = F_{\eta\eta}(\mathbf{x} \cdot \mathbf{s}, t)$. Hence the function $F(\eta, t)$ satisfies the one dimensional wave equation

$$\frac{1}{v^2}\partial_t^2 F = F_{\eta\eta},$$

and therefore the solution V has the form

$$V(\mathbf{x}, t) = V_1(\mathbf{x} \cdot \mathbf{s} - vt) + V_2(\mathbf{x} \cdot \mathbf{s} + vt)$$

where V_1, V_2 are arbitrary functions.

Suppose that $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell equations (M.3), (M.4) with $\mathbf{J} = 0$. Since the fields $\mathbf{E}$ and $\mathbf{H}$ both satisfy the wave equation, it is then natural to consider the case when

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{e}(\mathbf{x} \cdot \mathbf{s} - vt), \quad \mathbf{H}(\mathbf{x}, t) = \mathbf{h}(\mathbf{x} \cdot \mathbf{s} - vt),$$

that is, $\mathbf{e}$ and $\mathbf{h}$ are vector valued functions of the scalar variable $\mathbf{x} \cdot \mathbf{s} - vt$ with $v = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$. We have

$$\frac{\partial \mathbf{E}}{\partial t} = -v\,\mathbf{e}', \text{ and } \nabla \times \mathbf{E} = \mathbf{s} \times \mathbf{e}';$$

and similarly for $\mathbf{H}$ under the assumption that $\mathbf{J} = 0$. Thus, from the Faraday and Ampère laws we obtain the equations

$$\mathbf{s} \times \mathbf{e}' = v\,\mathbf{h}'$$

$$\mathbf{s} \times \mathbf{h}' = -\frac{1}{v}\mathbf{e}'.$$

Since $\mathbf{s}$ is a constant vector $\mathbf{s} \times \mathbf{e}' = (\mathbf{s} \times \mathbf{e})'$, and so functions $\mathbf{e}$ and $\mathbf{h}$ satisfy the system of odes

$$(\mathbf{s} \times \mathbf{e})' = v\, \mathbf{h}'$$
$$(\mathbf{s} \times \mathbf{h})' = -\frac{1}{v}\, \mathbf{e}'.$$

By integration, we get that $\mathbf{s} \times \mathbf{h} = -\dfrac{1}{v}\, \mathbf{e} + \mathbf{c}_1$, and $\mathbf{s} \times \mathbf{e} = v\, \mathbf{h} + \mathbf{c}_2$, with $\mathbf{c}_i$ constant vectors. Using the vector identity $a \times (b \times c) = b\,(a \cdot c) - c\,(a \cdot b)$, we get that

$$\mathbf{s}\,(\mathbf{s} \cdot \mathbf{e}) = v\, \mathbf{c}_1 + \mathbf{s} \times \mathbf{c}_2; \qquad \mathbf{s}\,(\mathbf{s} \cdot \mathbf{h}) = -\frac{1}{v}\, \mathbf{c}_2 + \mathbf{s} \times \mathbf{c}_1.$$

If we choose the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ satisfying

$$\begin{cases} v\, \mathbf{c}_1 + \mathbf{s} \times \mathbf{c}_2 = 0 \\ -\dfrac{1}{v}\, \mathbf{c}_2 + \mathbf{s} \times \mathbf{c}_1 = 0 \end{cases} \tag{6.10}$$

we then get $\mathbf{s} \cdot \mathbf{e} = \mathbf{s} \cdot \mathbf{h} = 0$; obviously choosing $\mathbf{c}_1 = \mathbf{c}_2 = 0$ we have a solution to the system. Nevertheless, the system (6.10) with coefficients in terms of s and v has determinant equal to $(-1 + s_1^2 + s_2^2 + s_3^2)^2$ which is zero since $\mathbf{s}$ is a unit vector. Therefore there are other choices for $\mathbf{c}_1$ and $\mathbf{c}_2$ solving (6.10). With $\mathbf{c}_1$ and $\mathbf{c}_2$ solving (6.10) we get then that

$$\mathbf{h} = \frac{1}{v}(\mathbf{s} \times \mathbf{e}) - \mathbf{s} \times \mathbf{c}_1$$
$$\mathbf{e} = -v(\mathbf{s} \times \mathbf{h}) - \mathbf{s} \times \mathbf{c}_2.$$

Consequently, taking constants of integration $\mathbf{c}_i$ equal zero (which amounts to neglect constant fields), we obtain the very important equations relating the electric and magnetic fields

$$\mathbf{E} = -v(\mathbf{s} \times \mathbf{H}) \tag{6.11}$$

$$\mathbf{H} = \frac{1}{v}(\mathbf{s} \times \mathbf{E}). \tag{6.12}$$

This shows that $\mathbf{s} \cdot \mathbf{E} = \mathbf{s} \cdot \mathbf{H} = \mathbf{0}$, that means, the electric and magnetic field are always *perpendicular* to the direction of propagation $\mathbf{s}$. In addition, $\mathbf{E} \cdot \mathbf{H} = -v\,(\mathbf{s} \times \mathbf{H}) \cdot \mathbf{H} = 0$, that is, $\mathbf{E}$ and $\mathbf{H}$ are always perpendicular. We also obtain taking absolute values that

$$|\mathbf{E}| = v|\mathbf{H}|.$$

7. Helmholtz equation

Suppose we have the wave equation $u_{tt} = c^2 \Delta u$ in all space $\mathbb{R}^n$ and $t \in \mathbb{R}$. If $u(x,t) = a(x)\, b(t)$, then

$$c^2 \frac{\Delta a(x)}{a(x)} = \frac{b''(t)}{b(t)},$$

therefore $c^2 \dfrac{\Delta a(x)}{a(x)} = \dfrac{b''(t)}{b(t)} = \alpha$, with α a complex constant. So we get two equations $b''(t) - \alpha\, b(t) = 0$ and

$$\Delta a(x) - \frac{\alpha}{c^2}\, a(x) = 0.$$

The last equation is called the Helmholtz equation or reduced wave equation. The solutions of the equation in b have the form $b(t) = A\, e^{\alpha_1 t} + B\, e^{\alpha_1 t}$ where α_i are the roots of the polynomial $r^2 - \alpha = 0$.

A typical case consider in the applications is when $u(x,t) = a(x)\, e^{i\omega t}$, $\omega > 0$. This represents in this case an electric or magnetic monochromatic field with angular frequency ω, and amplitude $a(x)$. In this case, we obtain that a satisfies the Helmholtz equation

$$\Delta a(x) + \frac{\omega^2}{c^2}\, a(x) = 0.$$

If for example the amplitude has the form $a(x) = a_0\, e^{i k \cdot x}$, with a_0 constant and k a constant vector in $\mathbb{R}^n$, then a solves the Helmholtz equation iff $|k| = \dfrac{\omega}{c}$. All together $u(x,t) = a_0\, e^{i(k\cdot x + \omega t)}$ is a plane wave, that is, for each fixed time t, u is constant on each plane $k \cdot x + \omega\, t$ =constant.

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Department of Mathematics, Temple University, Philadelphia, PA 19122