You may not consult in any form with any other person, except your instructor, while doing this take-home test.

1. If \( u \) solves the one dimensional wave equation \( u_{tt} - c^2 u_{xx} = e^x \) and \( u(x,0) = u_t(x,0) = 0 \), find \( u(1/2, 3/2) \).

2. If \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a constant vector, prove that \( u(x,t) = \exp(\pm i(\alpha \cdot x + k c t)) \) solves the wave equation \( u_{tt} - c^2 \Delta u = 0 \) provided \( |\alpha|^2 = k^2 \).

3. If \( E \) is a plane wave of the form
   \[
   E(x,y,z) = \left(0, a \cos\left(\omega\left(t - \frac{x \sin \alpha + z \cos \alpha}{c}\right)\right), 0\right),
   \]
   find the direction of propagation of the wave. Suppose \( E \) and \( B \) solve the Maxwell equations in vacuum, i.e., \( \rho = 0 \) and \( J = 0 \). Calculate the magnetic field \( B \).

4. If \( f \) is continuous in \( \Omega \) and the weak derivative \( D^i f \) is also continuous in \( \Omega \), then the ordinary derivative \( f_{x_i} \) exists and is continuous in \( \Omega \).
   **HINT:** enough to prove it in dimension one, that is, if
   \[
   \int_a^b f(x)g'(x) \, dx = - \int_a^b h(x)g(x) \, dx \tag{1}
   \]
   with \( f \) and \( h \) continuous and for all \( g \in C_0^\infty(a,b) \), then \( f'(x) = h(x) \) in \( (a,b) \). Equation (1) is equivalent to
   \[
   \int_a^b \left(f(x) - \int_{s_0}^x h(t) \, dt\right)g'(x) \, dx = 0,
   \]
   so it is enough to show that if
   \[
   \int_a^b w(x)g'(x) \, dx = 0, \quad \forall g \in C_0^\infty(a,b) \tag{2}
   \]
   with \( w \) continuous, then \( w \) is constant in \( (a,b) \). Now, if (2) holds then prove that
   \[
   \int_a^b w(x)g'(x) \, dx = 0 \text{ for all } g \text{ absolutely continuous with compact support in } (a,b) \text{ (take } \phi \text{ smooth with compact support in } [-1,1], \phi_\varepsilon(x) = e^{-1}\phi(x/\varepsilon) \text{ and take } \phi_\varepsilon \ast g \text{).}
   \]
   Then given \( a < a' < b' < b \), define \( g(x) = \int_{a'}^x \left(w(t) - \int_{a'}^{b'} w(s) \, ds\right) \, dt \), for \( a' \leq x \leq b' \), and \( g(x) = 0 \) outside \([a', b']\). Show that \( g \) is absolutely continuous. Notice that (2) holds if and only if \( \int_a^{b'} (w(x) - c)g'(x) \, dx = 0 \) with any constant \( c \). Then from this get that
   \[
   \int_{a'}^{b'} \left(w(x) - \int_{a'}^{b'} w(s) \, ds\right)^2 \, dx = 0.
   \]

5. Let \( f(x) = |x_1| \) in the unit ball \( B_1(0) \subset \mathbb{R}^n \). Show that \( f \) has weak derivatives of order \( \alpha \) for all \( |\alpha| \leq 1 \) and \( D^i f = \text{sign } x_1 \) and \( D^i f = 0 \) for \( i = 2, \ldots, n \).
6. Let \( f \in L^1_{\text{loc}}(\Omega) \), \( \Omega \) connected and \( D^\alpha f = 0 \) for all \(|\alpha| = 1\). Then \( f \) is constant in \( \Omega \).

7. Let \( x_1, \ldots, x_{k+1} \) be numbers and consider the Vandermonde determinant

\[
V(x_1, \ldots, x_{k+1}) = \det \begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_{k+1} \\
x_1^2 & x_2^2 & \cdots & x_{k+1}^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^k & x_2^k & \cdots & x_{k+1}^k
\end{bmatrix}.
\]

We have that \( V(x_1, \ldots, x_{k+1}) = \prod_{1 \leq i < m \leq k+1} (x_m - x_i) \).

Let \( B = B_1(0) \) be the unit ball in \( \mathbb{R}^n \) and \( B^+ = \{ x \in B_1(0) : x_n > 0 \} \). Let \( f \in C^k(B^+) \), write \( x = (x', x_n) \) and define the extension

\[
\tilde{f}(x) = \begin{cases} 
  f(x), & \text{for } x_n \geq 0 \\
  \sum_{i=1}^{k+1} a_i f \left( x', -\frac{x_n}{i} \right), & \text{for } x_n < 0,
\end{cases}
\]

where \( a_i, i = 1, \ldots, k+1 \), is the unique solution of the linear system

\[
\sum_{i=1}^{k+1} \left( -\frac{1}{i} \right)^s a_i = 1, \quad s = 0, \ldots, k.
\]

Prove that \( \tilde{f} \in C^k(B) \) and

\[
\|\tilde{f}\|_{W^{k,p}(B)} \leq C \|f\|_{W^{k,p}(B^+)}.
\]