1. Let $\Omega$ be a bounded open set with $C^1$ boundary, $1 \leq p < n$; and $p^*$ such that, 
$$
\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.
$$
The Rellich-Kondrachov theorem says that the Sobolev space $W^{1,p}(\Omega)$ is compactly contained in $L^q(\Omega)$ for all $1 \leq q < p^*$.

Prove that this is not true when $q = p^*$. Proceed as follows.

(a) Construct a sequence of disjoint balls $B_{r_i}(x_i) \subset \Omega$ such that $r_i \leq 1, i = 1, 2, \cdots$

(b) Let $\eta \in C_0^\infty(B_1(0))$ such that $\eta$ is not trivial, and consider 
$$
\phi_i(x) = r_i^{1-(n/p)} \phi\left(\frac{x-x_i}{r_i}\right).
$$

(c) Show that $\phi_i \in W^{1,p}(\Omega)$ and $\|\phi_i\|_{W^{1,p}(\Omega)} \leq M$ for all $i$.

(d) Show that $\|\phi_i\|_{L^{p^*}(\Omega)} = C > 0$ for all $i$.

(e) Show that the supports of the $\phi_i$'s are all disjoint.

(f) Conclude that $\phi_i$ cannot have a Cauchy subsequence in $L^{p^*}(\Omega)$.

2. Let $u_j \in W^{1,p}$ for $1 \leq j \leq N$. Prove that the functions $\max_{1 \leq i \leq N} |u_i|$ and $\min_{1 \leq i \leq N} |u_i|$ belong to $W^{1,p}$.

3. Let $u(x) = \log(\log(1/|x|))$ for $|x| < 1/2$. Prove that 

(a) $u \in W^{1,p}(B_{1/2}(0))$ and $u$ is not bounded;

(b) $u \in BMO(B_{1/2}(0))$;

(c) look at $u$ for $|x|$ sufficiently small so that it is positive and extend this function so that it is smooth away from 0, nonnegative, and has compact support; call this extension $v$. Now consider the function 
$$
F(x) = \sum_{i=1}^\infty 2^{-i} v(x-r_i),
$$
where $r_i$ is a dense set in $\mathbb{R}^n$.

(d) show that $F \in W^{1,q}(\mathbb{R}^n)$ and $F$ is unbounded in a neighborhood of each point.

HINT: show that if $f$ is a nonnegative smooth function with compact support then 
$$
\int_{\mathbb{R}^n} f(x) \sum_{i=1}^m 2^{-i} v(x-r_i) \to 0 \text{ as } m \to \infty.
$$
This implies that 
$$
\int_{\mathbb{R}^n} \phi_{x_i}(x) F(x) dx = \sum_{i=1}^\infty 2^{-i} \int_{\mathbb{R}^n} \phi_{x_i}(x) v(x-r_i) dx = I,
$$
for all $\phi \in C_0^\infty(\mathbb{R}^n)$, and now show you can pass the derivative $x_i$ to $v$ in the integral obtaining 
$$
I = -\sum_{k=1}^\infty 2^{-k} \int_{\mathbb{R}^n} \phi(x) v_{x_i}(x-r_k) dx.
$$
Show that $\sum_{k=1}^\infty 2^{-k} v_{x_i}(x-r_k)$ belongs to $L^q(\mathbb{R}^n)$. Since $F \in L^q(\mathbb{R}^n)$ (prove it) and $\phi$ has compact support, use the Lebesgue dominated theorem to show that 
$$
\sum_{k=1}^\infty 2^{-k} \int_{\mathbb{R}^n} \phi(x) v_{x_i}(x-r_k) dx = \int_{\mathbb{R}^n} \phi(x) \sum_{k=1}^\infty 2^{-k} v_{x_i}(x-r_k) dx.
$$