1. Prove that the general solution of \( u_{xy} = 0 \) has the form \( u(x, y) = F(x) + G(y) \).

Using the change of variables \( \xi = x + ct \) and \( \eta = x - ct \), show that \( u_{tt} - c^2 u_{xx} = 0 \) if \( u_{\xi \eta} = 0 \).

Using this re derive d’Lambert formula.

2. The Darboux equation \( \left( \partial_{rr} + \frac{n-1}{r} \partial_r \right) M_h(x, r) = \Delta_x M_h(x, r) \) is satisfied for any \( h \in C^2 \) function in \( \mathbb{R}^n \), where \( M_h(x, r) = \frac{1}{\omega_n} \int_{|\xi|=1} h(x + r\xi) \, d\sigma(\xi) \). Suppose \( n = 3 \) and \( f \in C^3(\mathbb{R}^3) \). Prove that the function \( v(x, t) = \partial_t \left( t M_f(x, ct) \right) \) satisfies the wave equation \( v_{tt} = c^2 \Delta x v \).

3. Solve the initial value problem \( 3 u_{tt} - 4 u_{xx} = 0, u(x, 0) = \sin x, \) and \( u_t(x, 0) = 1 \).

4. Let \( u \in C^2(\mathbb{R}^3 \times (0, +\infty)) \) solution to \( \Box u = u_{tt} - c^2 \Delta u = 0 \) in \( \mathbb{R}^3 \times (0, +\infty) \) with \( u(x, 0) = f(x) \) and \( u_t(x, 0) = g(x) \) with \( f, g \in C^2 \).

   (a) if \( f, g \) have compact support, then prove that

   \[
   |u(x, t)| \leq \frac{C}{t}
   \]

   for all \( t > 0 \) with a constant \( C \) depending only of \( f \) and \( g \).

   (b) Suppose

   \[
   K = \int_{\mathbb{R}^3} (|g(x)| + |f(x)| + |D_f(x)| + |D_g(x)| + |\Delta f(x)|) \, dx < \infty,
   \]

   in particular, this holds if \( g, f \) have compact support. Prove that

   \[
   |u(x, t)| \leq \frac{C_1}{t} + \frac{C_2}{t^2} + \frac{C_3}{t^3}
   \]

   for all \( t > 0 \) with constants \( C_i \) depending only on \( K \) and independent of \( u \).

HINT: use the Kirchhoff formula

\[
  u(x, t) = \frac{1}{4\pi c^2 t^2} \int_{|y-x|=ct} [t \, g(y) + f(y) + D_f(y) \cdot (y-x)] \, d\sigma(y)
\]

directly to prove part (a), and for (b) using the divergence theorem convert this integral into a solid integral.