1. Fourier transform in $L^1$

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform is defined by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i x \cdot t} \, dt.$$ 

Theorem 1. We have

(a) $\|\hat{f}\|_\infty \leq \|f\|_1$

(b) $\hat{f}$ is uniformly continuous.

Proof. (a) is immediate. For (b), from the mean value theorem we have that $|e^{\alpha} - e^{\beta}| \leq \sqrt{2} |\alpha - \beta|$. Writing $\hat{f}(x) - \hat{f}(y)$ as an integral for $|t| > M$ plus an integral for $|t| \leq M$, and using the absolute continuity of the integral in the first piece and the estimate for the difference of the exponentials in the second piece, part (b) follows. 

Theorem 2 (Riemann-Lebesgue). If $f \in L^1(\mathbb{R}^n)$, then $|\hat{f}(x)| \to 0$ as $|x| \to \infty$.

Proof. If $f(x) = \chi_I(x)$ with $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$, then

$$\hat{f}(x) = \prod_{i=1}^n \int_{a_i}^{b_i} e^{-2\pi i x_i t_i} \, dt_i = \prod_{i=1}^n \frac{e^{-2\pi i x_i b_i} - e^{-2\pi i x_i a_i}}{-2\pi i x_i}.$$ 


If \(|x| \geq M\), then there exists \(k\) such that \(|x_k| \geq M/\sqrt{n}\). Write

\[
\hat{f}(x) = \frac{e^{-2\pi i x_k t_k}}{-2\pi i x_k} \prod_{i \neq k} e^{-2\pi i x_i t_i} \left| t_i = a_i \right|
\]

so

\[
|\hat{f}(x)| \leq \frac{1}{\pi |x_k|} \prod_{i \neq k} \sqrt{2} |b_i - a_i| \leq \frac{\sqrt{2n}}{\pi M} \prod_{i \neq k} \sqrt{2} |b_i - a_i| \leq \frac{C(n, I)}{M},
\]

and we are done.

For the general case, let \(\epsilon > 0\) and let \(g\) be a linear combination of characteristic functions of intervals such that \(||f - g||_1 < \epsilon\). From the linearity of the Fourier transform and the first theorem

\[
|\hat{f}(x)| \leq ||f - g||_1 + |\hat{g}(x)| \leq \epsilon + |\hat{g}(x)|,
\]

so \(\limsup_{|x| \to \infty} |\hat{f}(x)| \leq \epsilon\) for all \(\epsilon\), and the theorem follows. \(\square\)

**Theorem 3.** Let \(f, g \in L^1(\mathbb{R}^n)\). Recall that the convolution \(f \ast g\) is

\[
f \ast g(x) = \int_{\mathbb{R}^n} f(x - t)g(t) \, dt.
\]

We have

(a) \((\hat{f} \ast \hat{g})(x) = \hat{f}(x) \hat{g}(x)\)

(b) for \(h \in \mathbb{R}^n\) we set \(\tau_h f(x) = f(x - h)\), then

\[
(\hat{\tau_h f})(x) = e^{-2\pi i h \cdot x} \hat{f}(x)
\]

and

\[
(e^{2\pi i h \cdot \hat{f}}(t))(x) = (\tau_h \hat{f})(x).
\]

(c) for \(a > 0\), \((\delta_a f)(x) = f(ax)\). Then

\[
(\delta_a \hat{f})(x) = a^{-n} \hat{f}(x/a).
\]

**Theorem 4.** If \(f \in L^1(\mathbb{R}^n)\) and \(x_k f(x) \in L^1(\mathbb{R}^n)\), then \(\hat{f}\) is continuously differentiable with respect to \(x_k\) and

\[
\left( \frac{\partial \hat{f}}{\partial x_k} \right)(x) = (-2\pi i t_k f(t))(x).
\]

**Proof.** We write

\[
\frac{\hat{f}(x + h e_k) - \hat{f}(x)}{h} = \int_{\mathbb{R}^n} f(t) e^{-2\pi i x_k t} \left( \frac{e^{-2\pi i h t_k}}{h} - 1 \right) \, dt.
\]
Since by the mean value theorem the absolute value of the integrand is bounded by
\[2\pi|t_k||f(t)|,\]
then the result follows by Lebesgue’s dominated convergence theorem. \(\square\)

For \(f \in L^1(\mathbb{R}^n)\) a function \(g \in L^1(\mathbb{R}^n)\) is the derivative of \(f\) with respect to \(x_k\) in the sense of \(L^1(\mathbb{R}^n)\) if
\[
\int_{\mathbb{R}^n} \left| \frac{f(x + h e_k) - f(x)}{h} - g(x) \right| \, dx \to 0
\]
as \(h \to 0\); we write \(g = \frac{\partial f}{\partial x_k}\) in the \(L^1\)-sense. Of course if \(g\) exists is unique.

**Theorem 5.** If \(f \in L^1(\mathbb{R}^n)\) and \(g = \frac{\partial f}{\partial x_k}\) in \(L^1\)-sense, then
\[
\hat{(\frac{\partial f}{\partial x_k})}(x) = 2\pi i x_k \hat{f}(x); \quad \forall x \in \mathbb{R}^n.
\]

**Proof.** Since \(\frac{f(x + h e_k) - f(x)}{h} \to g\) in \(L^1\), by Theorem 1(a) we have
\[
\hat{g}(x) = \lim_{h \to 0} \left( \frac{f(x + h e_k) - f(x)}{h} \right)(x) = \lim_{h \to 0} \left( \frac{\tau_{-he_k}f(x) - f(x)}{h} \right)
\]
\[
= \lim_{h \to 0} \left( e^{2\pi i x \cdot he_k} - 1 \right) \hat{f}(x) = 2\pi i x_k \hat{f}(x).
\]
\(\square\)

If \(P(x) = \sum_{|\alpha| \leq k} a_\alpha x^\alpha\) is a polynomial in \(n\) variables with complex coefficients, 
\(P(D) = \sum_{|\alpha| \leq k} a_\alpha D^\alpha\), then under appropriate conditions on \(f\) we have the following formulas:
\[
P(D)\hat{f}(x) = (P(-2\pi it)f(t))(x)
\]
\[
(P(D)f)(x) = P(2\pi ix)\hat{f}(x).
\]

2. Inversion of the Fourier transform

The purpose is to show that one has a formula like
\[
f(x) = \int_{\mathbb{R}^n} \hat{f}(t)e^{2\pi i t \cdot x} \, dt
\]
but the trouble is that in general \(\hat{f}\) is not integrable, for example, if \(f(x) = \chi_{[0,1]}(x)\). The idea is to introduce a *summability* method and obtain the desired formula in that sense.
Given a function $\Phi$ continuous that tends to zero at infinity with $\Phi(0) = 1$ we say that the integral $\int_{\mathbb{R}^n} f(x) \, dx$ is $\Phi$ summable to the number $L$ if
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} f(x) \Phi(\epsilon x) \, dx = L.
\]
The case $\Phi(x) = e^{-|x|}$ corresponds to Abel summability and $\Phi(x) = e^{-|x|^2}$ corresponds to Gauss summability. We need the Fourier transforms of these functions. We have for $\alpha > 0$
\[
\begin{align*}
(1) & \quad \int_{\mathbb{R}^n} e^{-\pi \alpha |y|^2} e^{-2\pi i t \cdot y} \, dy = \alpha^{-n/2} e^{-\pi |t|^2/\alpha}; \\
(2) & \quad \int_{\mathbb{R}^n} e^{-2\pi \alpha |y|^2} e^{-2\pi i t \cdot y} \, dy = c_n \frac{\alpha}{(\alpha^2 + |t|^2)^{(n+1)/2}}.
\end{align*}
\]

**Theorem 6.** If $f, g \in L^1(\mathbb{R}^n)$, then
\[
\int_{\mathbb{R}^n} \hat{f}(x) \hat{g}(x) \, dx = \int_{\mathbb{R}^n} f(x) \hat{g}(x) \, dx.
\]

*Proof.* It follows immediately from Fubini’s theorem. \qed

Given $\epsilon > 0$ and $\varphi$ a function, we set $\varphi_\epsilon(x) = e^{-\pi \varphi(x/\epsilon)}$. If $\hat{\varphi} = \varphi$, then from the dilation formula we have that
\[
(\hat{\varphi_\epsilon}(\Phi))(x) = \varphi_\epsilon(x).
\]

**Corollary 7.** If $f, \Phi, \varphi \in L^1(\mathbb{R}^n)$, $\hat{\Phi} = \varphi$, then
\[
\int_{\mathbb{R}^n} \hat{f}(x) e^{2\pi i t \cdot x} \Phi(\epsilon x) \, dx = \int_{\mathbb{R}^n} f(x) \varphi_\epsilon(x - t) \, dx
\]
for all $t \in \mathbb{R}^n$.

**Theorem 8.** If $f, \hat{f} \in L^1(\mathbb{R}^n)$, then
\[
f(x) = \int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i t \cdot x} \, dt
\]
at each Lebesgue point $x$ of $f$.

*Proof.* Let $\alpha > 0$, and $\Phi(x) = e^{-4\pi |x|^2}$. We have $\Phi(x) = 4^{-n/2} e^{-\pi |x|^2/\alpha} := \varphi(x)$, $\int \varphi(x) \, dx = 1$, and from Corollary 7
\[
I(x, \alpha) := \int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i t \cdot x} \Phi(\sqrt{\alpha} t) \, dt = (f * \varphi_{\sqrt{\alpha}})(x) \to f(x)
\]
in $L^1(\mathbb{R}^n)$ as $\alpha \to 0$. Then there exists a sequence $\alpha_k \to 0$ such that $I(x, \alpha_k) \to f(x)$ for a.e. $x$. We have $|\hat{f}(t) e^{2\pi i t \cdot \alpha_k} - \hat{f}(t)| \leq |\hat{f}(t)|$ for all $x, t$, and so by Lebesgue dominated convergence theorem $I(x, \alpha_k) \to \int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i t \cdot \alpha_k} dt$, and the theorem follows.

\[\square\]

**Theorem 9.** If $f, \Phi, \varphi \in L^1(\mathbb{R}^n)$, $\hat{\Phi} = \varphi$, $\int \varphi(x) \, dx = 1$, then

\[\int_{\mathbb{R}^n} \hat{f}(x) \, e^{2\pi i t \cdot \varphi(x)} \, dx \to f(t)\]

in $L^1(\mathbb{R}^n)$ as $\epsilon \to 0$.

**Proof.** From Theorem 8

\[\Phi(x) = \int_{\mathbb{R}^n} \hat{\Phi}(t) e^{2\pi i t \cdot x} \, dt = \int_{\mathbb{R}^n} \varphi(t) e^{2\pi i t \cdot x} \, dt = \varphi(-x)\]

and so $\Phi(0) = \hat{\varphi}(0) = \int \varphi(x) \, dx = 0$ and by Riemann-Lebesgue, Theorem 2, $\Phi \to 0$ as $|x| \to \infty$. The theorem then follows from Corollary 7.

\[\square\]

**Corollary 10.** If $f_1, f_2 \in L^1(\mathbb{R}^n)$ and $\hat{f}_1(x) = \hat{f}_2(x)$ for all $x$, then $f_1 = f_2$ a.e.

We recall the definition of Lebesgue point: If $f$ is locally integrable, then $x_0$ is a Lebesgue point of $f$ if

\[\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |f(x) - f(x_0)| \, dx \to 0,\]

as $r \to 0$. Obviously, if $f$ is continuous at $x_0$, then $x_0$ is a Lebesgue point of $f$. We recall that if $\varphi(x)$ is radial, non-increasing, and $\varphi \in L^1(\mathbb{R}^n)$ with $\int \varphi(x) \, dx = 1$, then

\[(f * \varphi_{\epsilon})(t) \to f(t)\]

as $\epsilon \to 0$ for all $t$ Lebesgue point of $f$.

**Theorem 11.** If $f \in L^1(\mathbb{R}^n)$ is continuous at $x = 0$ and $\hat{f}(x) \geq 0$ for all $x$, then $\hat{f} \in L^1(\mathbb{R}^n)$, and so the formula in Theorem 8 holds.

**Proof.** Applying Corollary 7 with $\Phi(x) = e^{-4\pi |x|^2} \, (\hat{\Phi}(x) = 4^{-n/2} \, e^{\pi |x|^2/4} = \varphi(x)$, $\int \varphi(x) \, dx = 1$), we obtain

\[\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \hat{f}(x) \, e^{2\pi i t \cdot \varphi_{\epsilon}(x)} \, dx = \lim_{\epsilon \to 0} (f * \varphi_{\epsilon})(t) = f(t)\]

\[\text{Notice that since } \varphi \text{ radial and decreasing, the convergence } (f * \varphi_{\epsilon})(x) \to f(x) \text{ is at all Lebesgue points } x \text{ of } f.\]
at all Lebesgue points $t$ of $f$. In particular,

$$
\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \hat{f}(x) \Phi(\epsilon x) \, dx = f(0).
$$

Since $|\Phi(x)| \leq 1$, $\Phi(0) = 1$, and $\hat{f} \geq 0$, we obtain from Fatou’s Lemma that

$$
\int_{\mathbb{R}^n} \hat{f}(x) \, dx = \int_{\mathbb{R}^n} \hat{f}(x) \lim_{\epsilon \to 0} \Phi(\epsilon x) \, dx \leq \liminf_{\epsilon \to 0} \int_{\mathbb{R}^n} \hat{f}(x) \Phi(\epsilon x) \, dx = f(0),
$$

and the theorem follows. □

3. $L^2$-Theory

**Theorem 12.** If $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$ and $\|\hat{f}\|_2 = \|f\|_2$.

**Proof.** Let $g(x) = \overline{f(-x)}$. We have $g \in L^1 \cap L^2$, $h = f \ast g \in L^1$, $\hat{h} = \hat{f} \hat{g}$, and $\hat{g} = \overline{\hat{f}}$. Thus $\hat{h} = \|\hat{f}\|^2$. Since $f, g \in L^2$, we have the $h = f \ast g$ is uniformly continuous. Therefore $h$ is continuous at $x = 0$, $\hat{h} \geq 0$, and applying Theorem 11 we obtain that $\hat{h} \in L^1$ and so $\hat{f} \in L^2$. Moreover

$$
\int |\hat{f}|^2 \, dx = \int \hat{h}(x) \, dx = h(0) = \int f(x) g(0 - x) \, dx = \int f(x) \overline{f(x)} \, dx = \int |f|^2 \, dx.
$$

Therefore the mapping $f \mapsto \hat{f}$ from $L^1 \cap L^2$ to $L^2$ is a linear isometry. Since $L^1 \cap L^2$ is dense in $L^2$ there exists a unique continuous linear extension $\mathcal{F} : L^2 \to L^2$ which is by definition the Fourier transform on the space $L^2$. A way to calculate the Fourier transform of a function $f \in L^2$ is taking $h_k(x) = f(x) \chi_{|x|\leq k}(x)$. In fact, since $h_k \to f$ in $L^2$, and $h_k \in L^1 \cap L^2$ we have that

$$
\mathcal{F} f = \lim_{k \to \infty} \hat{h}_k
$$

where the limit is taken in the $L^2$ sense. We clearly have that

$$
\|\mathcal{F} f\|_2 = \|f\|_2.
$$

for all $f \in L^2$. In addition, $\mathcal{F}$ is onto. In fact, we will show first that the range of $\mathcal{F}$ is a closed subspace of $L^2$. Let $g_k = \mathcal{F} f_k$ with $f_k \in L^2$ and $g_k \to g$ in $L^2$. We prove that there is $f \in L^2$ such that $g = \mathcal{F} f$. We have $g_k - g_m = \mathcal{F} (f_k - f_m)$ and so $\|g_k - g_m\|_2 = \|\mathcal{F} (f_k - f_m)\|_2 = \|f_k - f_m\|_2$. Therefore $f_k$ is a Cauchy sequence in $L^2$ and we are done. Suppose the range($\mathcal{F}$) $\neq L^2$. Then the range has an orthogonal complement in $L^2$. That is, there exists $g \neq 0$ such that $\langle h, g \rangle = 0$ for all $h \in \text{range}(\mathcal{F})$. That is, $\langle \mathcal{F} f, g \rangle = 0$ for all $f \in L^2$. Since $L^1 \cap L^2$ is dense in $L^2$, the formula in Theorem 11 extends to $\langle \mathcal{F} f, g \rangle = \langle f, \mathcal{F} g \rangle$, and therefore $\langle f, \mathcal{F} g \rangle = 0$
for all $f \in L^2$ obtaining that $\mathcal{F} g = 0$, a contradiction. Here the inner product is complex and it is given by $\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) \overline{g(x)} \, dx$.

**Theorem 13.** The inverse operator of $\mathcal{F}$ is defined on each function $g \in L^2$, and satisfies

$$(\mathcal{F}^{-1} g)(x) = (\mathcal{F} g)(-x).$$

**Proof.** Clearly the inverse of $\mathcal{F}$ exists and is linear since $\mathcal{F}$ is an linear isometry and onto.

We first claim that

\[(1) \quad (\mathcal{F}^{-1} g)(t) = \lim_{k \to \infty} \int_{|x| \leq k} g(x) e^{2\pi i t \cdot x} \, dx,\]

for each $g \in L^2$ and where the limit is understood in the $L^2$-sense. Notice that this limit exists because if we let $G_k(t) = \int_{|x| \leq k} g(x) e^{2\pi i t \cdot x} \, dx$, then by Theorem 12 it belongs to $L^2$ and $\|G_k - G_m\|_2 = \|g \chi_{|x| \leq k} - g \chi_{|x| \leq m}\|_2 \to 0$ as $k, m \to \infty$.

Let us assume the claim holds. Then given $g \in L^2$ we have

$$\int_{|x| \leq k} g(x) e^{2\pi i t \cdot x} \, dx = (\hat{g} \chi_{|x| \leq k})(-t) = \mathcal{F}(g \chi_{|x| \leq k})(-t)$$

since $g \chi_{|x| \leq k} \in L^1 \cap L^2$. Since $g \chi_{|x| \leq k} \to g$ in $L^2$, we have $\mathcal{F}(g \chi_{|x| \leq k}) \to \mathcal{F} g$ in $L^2$, and so $\mathcal{F}(g \chi_{|x| \leq k})(-t) \to \mathcal{F} g(-t)$ in $L^2$; that is, the RHS of (1) equals $\mathcal{F} g(-t)$.

To prove the claim let

$$\hat{g}(t) = \lim_{k \to \infty} \int_{|x| \leq k} g(x) e^{2\pi i t \cdot x} \, dx,$$

with $g \in L^2$, and the limit taken in $L^2$. We shall first prove that

\[(2) \quad \langle h, \hat{g} \rangle = \langle \hat{h}, g \rangle\]

for each $h \in L^1 \cap L^2$. In fact, we write

$$\langle h, \hat{g} \rangle = \int_{\mathbb{R}^n} h(t) \overline{\hat{g}(t)} \, dt$$

$$= \int_{\mathbb{R}^n} h(t) \left( \lim_{k \to \infty} \int_{|x| \leq k} g(x) e^{2\pi i t \cdot x} \, dx \right) dt$$

$$= \lim_{k \to \infty} \int_{\mathbb{R}^n} h(t) \int_{|x| \leq k} \overline{g(x)} e^{-2\pi i t \cdot x} \, dx \, dt$$

$$\quad = \lim_{k \to \infty} \int_{\mathbb{R}^n} h(t) e^{-2\pi i t \cdot x} \, dx \, dt,$$

since $h \in L^1 \cap L^2$
Since $F$ is bijective from $L^2$ to $L^2$, given $g \in L^2$ there exists a unique $w \in L^2$ such that $Fw = g$, i.e., $w = F^{-1}g$. So we have $\langle \hat{h}, g \rangle = \langle \hat{h}, Fw \rangle$. Since $F$ is an isometry in $L^2$, we have that $F$ preserves the inner product, that is $\langle Fh, Fw \rangle = \langle h, w \rangle$. We then have from (2) that

$$\langle h, \tilde{g} \rangle = \langle \hat{h}, g \rangle = \langle \hat{h}, Fw \rangle = \langle h, w \rangle,$$

for all $h \in L^1 \cap L^2$. Therefore $\tilde{g} = w = F^{-1}g$, for each $g \in L^2$, obtaining (1). □

References

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If $H$ is a complex Hilbert space with complex inner product $\langle x, y \rangle$, then we have the formula

$$\langle x, y \rangle = \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right),$$

and the preservation of the inner product follows from this formula.