SECOND DERIVATIVES OF THE NEWTONIAN POTENTIAL

CRISTIAN E. GUTIÉRREZ

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We assume $f: \Omega \to \mathbb{R}$ is locally Hölder continuous, that is, there exists $0 < \alpha \leq 1$ such that for each $K \subset \Omega$ compact there is a constant $C_K > 0$ such that

$$|f(x) - f(y)| \leq C_K |x - y|^{\alpha}, \quad \forall x, y \in K.$$ 

We also assume $f$ is bounded in $\Omega$.

The goal is to prove the representation formula (1) below for the second derivatives of the Newtonian potential.

Assume $n \geq 3$, let $\Gamma(x) = C_n|x|^{2-n}$ be the fundamental solution, $\eta : \mathbb{R} \to \mathbb{R}$ is smooth, $\eta(t) = 0$ for $t \leq 1$, $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $t \geq 2$ and $0 \leq \eta' \leq 2$.

Let $\Omega_0$ be a domain for which the divergence theorem holds, and $\Omega \subset \Omega_0$, and let $\tilde{f}(x) = f(x)$ for $x \in \Omega$ and $\tilde{f}(x) = 0$ in $\Omega_0 \setminus \Omega$. Define for $x \in \Omega$

$$u(x) = \int_{\Omega_0} (D_{ij}\Gamma)(x - y)(\tilde{f}(y) - f(x)) \, dy - f(x) \int_{\partial \Omega_0} D_i \Gamma(x - y) \nu_j(y) \, d\sigma(y).$$

Since $|D_{ij}\Gamma(x)| \leq C_n|x|^{-n}$, $f$ is bounded and locally Hölder continuous, the function $u(x)$ is well defined for all $x \in \Omega$. Let $\epsilon > 0$ and define

$$v_\epsilon(x) = \int_{\Omega} D_i \Gamma(x - y) \eta((x - y)/\epsilon)f(y) \, dy,$$

and let $v(x) = D_i w(x)$, where

$$w(x) = \int_{\Omega} \Gamma(x - y)f(y) \, dy.$$
If $f$ is integrable in $\Omega$, then by the homework, $v_\epsilon \in C^1(\mathbb{R}^n)$, and if in addition $f$ is bounded, $w \in C^1(\mathbb{R}^n)$. We have

$$D_j v_\epsilon(x) = \int_{\Omega} D_x (D_j \Gamma(x-y) \eta((x-y)/\epsilon)) f(y) \, dy$$

$$= \int_{\Omega_0} D_x (D_j \Gamma(x-y) \eta((x-y)/\epsilon)) \tilde{f}(y) \, dy$$

$$= \int_{\Omega_0} D_x (D_j \Gamma(x-y) \eta((x-y)/\epsilon)) (\tilde{f}(y) - f(x)) \, dy$$

$$+ f(x) \int_{\Omega_0} D_x (D_j \Gamma(x-y) \eta((x-y)/\epsilon)) \, dy$$

$$= \int_{\Omega_0} D_x (D_j \Gamma(x-y) \eta((x-y)/\epsilon)) (\tilde{f}(y) - f(x)) \, dy$$

$$- f(x) \int_{\Omega_0} D_j \Gamma(x-y) \eta((x-y)/\epsilon) \, dy$$

from the divergence theorem. Since $x \in \Omega$, if we take $\epsilon \leq \text{dist}(x, \partial \Omega_0)/2$, then $|x-y| \geq 2\epsilon$ for $y \in \partial \Omega_0$ and so $\eta((x-y)/\epsilon) = 1$. So

$$D_j v_\epsilon(x) = \int_{\Omega_0} D_x (D_j \Gamma(x-y) \eta((x-y)/\epsilon)) (\tilde{f}(y) - f(x)) \, dy$$

$$- f(x) \int_{\partial \Omega_0} D_j \Gamma(x-y) \nu_j(y) \, d\sigma(y).$$

Then subtracting we get

$$u(x) - D_j v_\epsilon(x)$$

$$= \int_{\Omega_0} \left(D_j \Gamma(x-y) - D_x (D_j \Gamma(x-y) \eta((x-y)/\epsilon)) \right) (\tilde{f}(y) - f(x)) \, dy$$

$$= \int_{\Omega_0} D_x \left[(1 - \eta((x-y)/\epsilon)) \right] D_j \Gamma(x-y) \, dy$$

$$= \int_{|x-y| \leq 2\epsilon} D_x \left[(1 - \eta((x-y)/\epsilon)) \right] D_j \Gamma(x-y) \, dy$$

$$= \int_{|x-y| \leq 2\epsilon} \left[D_j \Gamma(x-y) \left(1 - \eta((x-y)/\epsilon)) - D_j \Gamma(x-y) \frac{1}{\epsilon} \eta'((x-y)/\epsilon) \frac{x_j - y_j}{x - y} \right] (\tilde{f}(y) - f(x)) \, dy.$$
Let $K \subset \Omega$ be compact, and $K' = \{ y : \text{dist}(y, K) \leq \text{dist}(K, \partial \Omega)/2 \}$. We have $K'$ is compact, $K' \subset \Omega$, and if $\epsilon < \text{dist}(K, \partial \Omega)/2$, then $B_\epsilon(x) \subset K'$ for all $x \in K$. Therefore estimating the last integral we obtain for $x \in K$

$$|u(x) - D_j v_\epsilon(x)| \leq \int_{|x-y| \leq 2\epsilon} \left[ |D_{ij} \Gamma(x-y)| + |D_i \Gamma(x-y)| \frac{C}{\epsilon} \right] |f(y) - f(x)| \, dy$$

$$\leq C_{K'} \int_{|x-y| \leq 2\epsilon} \left[ \frac{C}{|x-y|^n} + \frac{C}{|x-y|^n-1} \frac{C}{\epsilon} \right] |x-y|^\alpha \, dy \leq C_{K'} \epsilon^\alpha.$$

Therefore $D_j v_\epsilon \to u$ uniformly on compact subsets of $\Omega$ as $\epsilon \to 0$, so $u$ is continuous in $\Omega$. In addition, $v_\epsilon \to v$ uniformly in $\mathbb{R}^n$ (this follows directly by subtracting). This implies that $w \in C^2(\Omega)$ and $u(x) = D_{ij} w(x)$ for $x \in \Omega$ by the fundamental theorem of calculus, because we write

$$v_\epsilon(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n) = \int_{x_j}^{y_j} D_j v_\epsilon(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) \, dt + v_\epsilon(x),$$

and pass to the limit as $\epsilon \to 0$. So we have proved the following representation formula for the second derivatives of the Newtonian potential $w$ for $x \in \Omega$:

1. $D_{ij} w(x) = \int_{\Omega_0} (D_{ij} \Gamma)(x-y)(f(y) - f(x)) \, dy - f(x) \int_{\partial \Omega_0} D_i \Gamma(x-y)v_j(y) \, d\sigma(y),$  

under the assumption that $f$ is bounded in $\Omega$ and locally Hölder continuous for some $0 < \alpha \leq 1$.

We next prove that $\Delta w = f$ in $\Omega$. Let $x \in \Omega$, take $R$ sufficiently large such that $\Omega \subset B_R(x)$, and apply (1) with $\Omega_0 = B_R(x)$, then

$$D_{ii} w(x) = \int_{|x-y| < R} (D_{ij} \Gamma)(x-y)(f(y) - f(x)) \, dy - f(x) \int_{|x-y| = R} D_i \Gamma(x-y)v_j(y) \, d\sigma(y),$$

adding over $1 \leq i \leq n$ and using that $\Delta \Gamma(x) = 0$ for $x \neq 0$, we obtain

$$\Delta w(x) = -f(x) \int_{|x-y| = R} D \Gamma(x-y) \cdot \nu(y) \, d\sigma(y).$$

We have $D \Gamma(x) = C_n(2-n) \frac{x}{|x|^n}$ and $\nu(y) = \frac{y-x}{|x-y|}$. Then $\Delta w(x) = f(x) C_n(2-n) \omega_n$, where $\omega_n$ is the surface area of the unit sphere in $\mathbb{R}^n$. Since $C_n = \frac{1}{(2-n)\omega_n}$, we are done.

We also have that $\Delta w(x) = 0$ for $x \notin \partial \Omega$, which follows differentiating twice under the integral sign (the justification follows from the homework).

**Department of Mathematics, Temple University, Philadelphia, PA 19122**

**E-mail address:** gutierre@temple.edu