Assume \((S, \Sigma, \mu)\) is a measure space, i.e., \(\Sigma\) is a \(\sigma\)-algebra of sets in \(S\) and \(\mu\) is a measure on \(\Sigma\).

1. Prove that the following triplets \((S, \Sigma, \mu)\) are measure spaces.
   1. \(S\) is an infinite set, \(\Sigma = \mathcal{P}(S)\), and \(\mu(E) = \#(E)\) if \(E\) is finite and \(\mu(E) = +\infty\) if \(E\) is infinite.
   2. \(S = [0, 1]\), \(\Sigma = \mathcal{P}([0, 1])\), \(\mu(E) = 0\) if \(E\) is of the first category, and \(\mu(E) = +\infty\) if \(E\) is of the second category.
   3. \(S\) is an infinite set, \(\Sigma = \mathcal{P}(S)\). Let \(x_1, x_2, \ldots\) be a fix sequence of distinct points in \(S\); for \(E \subset S\) set \(\mu(E) = \sum_{x_i \in E} \frac{1}{2^i}\).

2. Let \(E, F \in \Sigma\). Prove that \(\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \cup F)\).

3. Given \(E, F \in \Sigma\) let \(\rho(E, F) = \mu(E \Delta F)\). Prove that \(\rho(E, F) \geq 0\); \(\rho(E, F) = \rho(F, E)\); and \(\rho(E, F) \leq \rho(E, G) + \rho(G, F)\).

4. (Borel-Cantelli) Let \(E_k \in \Sigma, k = 1, 2, \ldots\). Suppose that \(\sum_{k=1}^{\infty} \mu(E_k) < \infty\). Prove that \(\mu(\lim \sup_k E_k) = 0\).

5. Let \(f \in L(\mathbb{R}^n)\) and for a Lebesgue measurable set \(E \subset \mathbb{R}^n\) define \(\phi(E) = \int_E f(x) \, dx\). Show that \(\phi\) is an additive set function on the \(\sigma\)-algebra of Lebesgue measurable sets.
   
   Let \(V\) and \(\underline{V}\) be the upper and lower variation of \(\phi\). Show that
   \[
   \overline{V}(E) = \int_E f^+(x) \, dx \quad \text{and} \quad \underline{V}(E) = \int_E f^-(x) \, dx.
   \]

6. Let \((S, \Sigma, \mu)\) be a \(\sigma\)-finite measure space, that is, \(S = \bigcup_{k=1}^{\infty} E_k\) with \(\mu(E_k) < \infty\). Suppose that \(\mu(S) = +\infty\). Show that given \(t > 0\) there exists \(E \in \Sigma\) such that \(t \leq \mu(E) < +\infty\).