1. Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a continuously differentiable and monotone function with $\phi(0) = 0$. Prove
\[\int_X \phi(f(x)) \, d\mu(x) = \int_0^\infty \phi'(t) \mu(\{x \in X : f(x) > t\}) \, dt\]
where $f \geq 0$, $f \in L^1(X, \mu)$ and $(X, \mu)$ is some $\sigma$-finite measure space.

2. We say a sequence $\{f_n\}_{n=1}^\infty \subset L^1(\mu)$ is uniformly integrable if for every $\epsilon > 0$ there exists $\delta > 0$ such that
\[\mu(E) < \delta \implies \sup_n \int_E |f_n| \, d\mu < \epsilon.\]
Suppose $\mu$ is a finite measure. Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a continuous increasing function with $\lim_{t \rightarrow \infty} \frac{\phi(t)}{t} = +\infty$. Prove that
\[\sup_n \int \phi(|f_n|) \, d\mu < \infty\]
implies that $\{f_n\}$ is uniformly integrable.

3. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence in $L^1([0,1], dx)$. Show that there is a subsequence $\{f_{n_j}\}_{j=1}^\infty$ and a measure $\mu$ with $f_{n_j} \xrightarrow{\sigma^*} \mu$ provided $\sup_n \|f_n\|_1 < \infty$. Here $\sigma^*$ is the weak-star convergence of measures. Show that in general $\mu \notin L^1([0,1], dx)$. However, if we assume that, in addition, $\{f_n\}_{1}^\infty$ is uniformly integrable, then $d\mu = f dx$ for some $f \in L^1([0,1])$. Can we conclude anything about strong convergence (ie, in the $L^1$-norm) of $\{f_n\}$? Consider the analogous question on $L^p([0,1], dx)$, $p > 1$.

4. Let $m$ denote Lebesgue measure on $\mathbb{R}^d$. Fix some $f \in L^p(m)$, $1 \leq p < \infty$. Define
\[\Phi_f : \mathbb{R}^d \rightarrow L^p(m) \text{ by } \Phi_f(y)(x) = f(x+y)\]
Show that $\Phi_f$ is continuous.

5. Let $\mu$ be a finite Borel measure on $\mathbb{R}^n$. Prove that
\[\varphi(x) = \mu(B(x,r))\]
is a lower semi-continuous function in $x$. Here $B(x,r)$ is the open ball of radius $r$ and center $x$ ($r > 0$ is fixed).
6. Let $\mu$ be a finite Borel measure on $\mathbb{R}^d$. Recall

$$M\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{m(B(x,r))}$$

(a) Show that $\mu \perp m$ implies $\mu(\{x : M\mu(x) < \infty\}) = 0$

(b) Show that if $\mu \perp m$, then $\limsup_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = \infty$ $\mu$-a.e.

7. For any $f \in L^1(\mathbb{R}^d)$ and $1 \leq k \leq d$ let

$$M_kf(x) = \sup_{r>0} r^{-k} \int_{B(x,r)} |f(y)| \, dy.$$ 

Show that

$$\text{mes}(\{x \in L : M_kf(x) > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{L^1}$$

where $L$ is an arbitrary affine $k$-dimensional subspace and “mes” stands for Lebesgue measure (i.e., $k$-dimensional measure) on this space. $C$ is an absolute constant.

8. Prove the Besicovitch covering lemma on the circle: Suppose $\{I_j\}$ are finitely many arcs with $|I_j| < 1$. Then there is a sub-collection $\{I_{j_k}\}$ such that

(a) $\cup_k I_{j_k} = \cup_j I_j$

(b) No point belongs to more than $C I_{j_k}$’s where $C$ is a numerical constant. Give an explicit value for $C$, as good as you can.

9. (a) Prove that if $\mu \in \mathcal{M}(\mathbb{T}) \setminus \{0\}$ satisfies $d\mu \perp d\theta$, then $M\mu \notin L^1$ ($M$ is the Hardy-Littlewood maximal function). In fact, show that

$$\text{mes}\{\theta \in \mathbb{T} : M\mu(\theta) > \lambda\} \geq \frac{c}{\lambda} \|\mu\|$$

with an absolute constant $c > 0$.

(b) Prove that there is a numerical constant $C$ such that if $\mu \in \mathcal{M}(\mathbb{T})$ is a positive measure and $F$ the associated harmonic function, then $M\mu \leq CF^*$. Conclude that if $\mu$ is singular, then $F^* \notin L^1$.

10. (a) Let $f$ be the standard Cantor-Lebesgue function on the middle-third Cantor set on $[0,1]$. Show that $f$ is Hölder continuous with exponent $\alpha = \frac{\log 2}{\log 3}$. 

2
(b) Let $C$ be the usual middle-third Cantor set on $[0, 1]$. Show that $C + C \supset [0, 1]$. Can you find a larger interval than $[0, 1]$ with this property?

11. Let $\mu$ be a measure on $X$ with $\mu(X) = 1$. Let $f, g$ be two nonnegative measurable functions with $\int g d\mu = 1$. Prove
\[ \int fg d\mu \leq \int g \log g d\mu + \log \int e^f d\mu. \]

12. Suppose $f \in L^p(\mathbb{R}), 1 \leq p < \infty$, $f$ is absolutely continuous with $f' \in L^p(\mathbb{R})$. Prove
\[ \lim_{h \to 0} \int \left| \frac{f(x + h) - f(x)}{h} - f'(x) \right|^p dx = 0 \]

13. Suppose that $f \in L^1(\mathbb{T})$ and that $\{S_n f\}_{n=1}^\infty$ (the sequence of partial sums of the Fourier series) converges in $L^p(\mathbb{T})$ to $g$ for some $p \in [1, \infty]$ and some $g \in L^p$. Prove that $f = g$. If $p = \infty$ conclude that $f$ is continuous.

14. Let $K_n$ denote the Fejer kernel with Fourier support $[-(n - 1), n - 1]$. Show that de la Vallée Poussain’s kernel
\[ V_n(\theta) = (1 + e^{2\pi in\theta} + e^{-2\pi in\theta}) K_n(\theta) \]
satisfies
(a) $\hat{V}_n(j) = 1$ where $|j| \leq n$
(b) $\| V_n' \|_{1} \leq Cn$ with $C$ independent of $n$.

15. Prove the following result (Bohr’s inequality) which is a sort of converse to Bernstein’s inequality: Suppose that $f \in C^1(\mathbb{T})$ and that $\hat{f}(j) = 0$ for all $j$ with $|j| < n$. Then
\[ \left\| \frac{df}{d\theta} \right\|_p \geq C n \| f \|_p \]
for all $p \in [1, \infty]$, where $C$ is independent of $n \in \mathbb{Z}^+$, $f$ and $p$.

16. Show that the Hilbert transform preserves the Hölder class $C^\alpha(\mathbb{T}), 0 < \alpha < 1$.

17. If $\omega$ is an irrational number, show that
\[ \left\| \frac{1}{N} \sum_{n=1}^{N} f(\cdot + n\omega) - \int_{\mathbb{T}} f(\theta) d\theta \right\|_{L^2} \to 0 \]
for any $f \in L^2(\mathbb{T})$. In particular, if $f \in L^2$ is such that $f(x+\omega) = f(x)$ for a.e. $x$, then $f = \text{const.}$

18. Let $\alpha$ be an irrational number. Can there be a non-constant function $f \in L^2(\mathbb{T}^2)$ so that

$$f(x_1 + \alpha, x_1 + x_2) = f(x_1, x_2)$$

for a.e. $(x_1, x_2) \in \mathbb{T}^2$?

19. For a real-valued function $\varphi$ on $\mathbb{T}$ let $A_\varphi$ denote the multiplication operator $(A_\varphi f)(x) = \varphi(x)f(x)$. Let $P_N : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ be the projection onto span $\{1, e^{2\pi i \theta}, \ldots, e^{2\pi i(N-1)\theta}\}$. Let $\varphi(\theta) = \cos(2\pi \theta)$. Denote the eigenvalues of $P_N A_\varphi P_N$ by $\{\lambda_{j,N}\}_{j=1}^N$. Show that

$$\frac{1}{N} \sum_{j=1}^N \lambda_{j,N}^k = a_k + O\left(\frac{1}{N}\right) \quad k = 0, 1, 2, \ldots$$

(1)

for some constants $a_k$, which you should compute. Also show that

$$\frac{1}{N} \# \{j : \lambda_{j,N} \leq E\} \rightarrow \rho(E) \quad \text{as} \quad N \rightarrow \infty$$

(2)

uniformly in $E \in \mathbb{R}$. Find the function $\rho$.

20. Now let $\varphi \in C^\infty(\mathbb{T})$ be arbitrary and define $A_\varphi$ and $\lambda_{j,N}$ as above. Show that (3) below and (2) hold and find $\{a_k\}_{k=0}^\infty$ and $\rho$ in terms of $\varphi$. If $\varphi$ is non-degenerate of order $s$ (i.e. $\sum_{\ell=0}^s |\varphi^{(\ell)}(x)| \neq 0$ on $\mathbb{T}$), show that $\rho \in C^{1/s}(\mathbb{R})$. Here (3) means

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \lambda_{j,N}^k = a_k, \quad k = 0, 1, 2, \ldots$$

(3)

21. Let $\varphi \in C^\infty(\mathbb{T})$ and denote $A_\varphi$ as in No. 19. If $H$ is the Hilbert transform on $\mathbb{T}$, show that

$$[A_\varphi, H] = A_\varphi \circ H - H \circ A_\varphi$$

is a smoothing operator, i.e., if $\mu \in \mathcal{M}(\mathbb{T})$ is an arbitrary measure, then

$$[A_\varphi, H] \mu \in C^\infty(\mathbb{T}).$$
22. In No. 20 show that not only (3) holds, but also (1). I.e., show that for \( \varphi \in C^\infty(\mathbb{T}) \)
\[
\frac{1}{N} \sum_{j=1}^{N} \lambda_{j,N}^k - a_k = 0 \left( \frac{1}{N} \right) \quad \text{as } N \to \infty
\]
with the same \( a_k \) as in 20.

23. Let \( f \in L^1(\mathbb{T}) \). Given \( \lambda > 1 \), show that there exists \( E \subset \mathbb{T} \) (depending on \( \lambda \) and \( f \)) so that \( \text{mes}(E) < \lambda^{-1} \) and for all \( N \in \mathbb{Z}^+ \)
\[
\frac{1}{N} \int_{\mathbb{T}\setminus E} \left| \sum_{n=N}^{2N} S_n f(x) \right|^2 \, dx \leq C \lambda \| f \|_1^2.
\]
\( C \) is a constant independent of \( f, N, \lambda \).

24. Let \( \{r_j\}_{j=1}^\infty \) be a sequence of independent, identically distributed random variables with \( \mathbb{P}(r_1 = 1) = \mathbb{P}(r_1 = -1) = \frac{1}{2} \) (coin tossing sequence). Show that for \( N = 1, 2, \ldots \)
\[
\mathbb{P} \left( \left| \sum_{j=1}^{N} r_j a_j \right| > \lambda \left( \sum_{j=1}^{N} a_j^2 \right)^{\frac{1}{2}} \right) \leq 2e^{-\lambda^2/2}
\]
for any \( \{a_j\}_{j=1}^\infty \in \mathbb{R} \) and \( \lambda > 0 \).

25. Suppose \( \{X_n\}_{n=1}^\infty \) is a martingale difference sequence adopted to some filtration \( \{\mathcal{F}_n\}_{n=1}^\infty \). Show that
\[
\mathbb{P} \left( \left| \sum_{n=1}^{N} X_n \right| > \lambda \left( \sum_{n=1}^{N} \| X_n \|_\infty^2 \right)^{\frac{1}{2}} \right) \leq C e^{-c\lambda^2}
\]
for any \( N = 1, 2, \ldots, \lambda > 0 \). \( C, c > 0 \) are absolute constants.

26. Let \( f \in C^1(\mathbb{T}) \) be such that \( \|f\|_\infty \leq 1 \) and \( \|f'\|_\infty \leq K \) (with some \( K \geq 1 \)). Identifying \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) we let \( x \mapsto 2x \mod 1 \) be the doubling map on \( \mathbb{T} \). Using the previous exercise show that for any \( N = 1, 2, \ldots \)
\[
\text{mes} \left\{ x \in \mathbb{T} : \left| \frac{1}{N} \sum_{n=1}^{N} f(2^n x) - \int_{\mathbb{T}} f \right| > \lambda \right\} \leq C \exp \left( -c \frac{\lambda^2 N}{\log^2 (K/\lambda)} \right)
\]
for some absolute constants \( c, C \). Can you obtain \( \log(K/\lambda) \) instead of \( \log^2(K/\lambda) \)? Would this be optimal?
27. Let \( \{r_j\}_{j=1}^\infty \) be as in No. 24. Show that for any \( \{a_j\} \in \mathbb{C} \), and \( N \in \mathbb{Z}^+ \)

\[
P \left( \sup_{0 \leq \theta \leq 1} \left| \sum_{j=1}^N r_j a_j e^{2\pi i j \theta} \right| > C_0 \left( \sum_{j=1}^N |a_j|^2 \right)^{\frac{1}{2}} \sqrt{\log N} \right) \leq C_0 N^{-2}
\]

provided \( C_0 \) is a sufficiently large absolute constant.

28. Let \( T_N(x) = \sum_{n=0}^N [a_n \cos(2\pi n x) + b_n \sin(2\pi n x)] \) be an arbitrary trigonometric polynomial with real coefficients \( a_0, \ldots, a_N, b_0, \ldots, b_N \). Show that there is a polynomial \( P(z) = \sum_{\ell=0}^{2N} u_\ell z^\ell \) so that \( T_N(x) = e^{-2\pi i N x} P(e^{2\pi i x}) \) and such that \( P(z) = z^{2N} P(z^{-1}) \).

How are the zeros of \( P \) distributed in the complex plane?

29. Suppose \( T_N(x) = \sum_{n=0}^N [a_n \cos(2\pi n x) + b_n \sin(2\pi n x)] \) is such that \( T_N \geq 0 \) everywhere and \( a_n, b_n \in \mathbb{R} \) for all \( n = 0, 1, \ldots, N \). Show that there are \( c_0, \ldots, c_N \in \mathbb{C} \) such that

\[
T_N(x) = \left| \sum_{n=0}^N c_n e^{2\pi i n x} \right|^2 \text{ for all } x.
\]

30. Suppose that \( T(x) = a_0 + \sum_{h=1}^H a_h \cos(2\pi h x) \) satisfies \( T(x) \geq 0 \) for all \( x \) and \( T(0) = 1 \).

Show that for any complex numbers \( y_1, y_2, \ldots, y_N \),

\[
\left| \sum_{n=1}^N y_n \right|^2 \leq (N + H) \left( a_0 \sum_{n=1}^N |y_n|^2 + \sum_{h=1}^H |a_h| \left| \sum_{n=1}^{N-h} y_{n+h} \bar{y}_n \right| \right).
\]

31. Let \( \{x_n\}_{n=1}^\infty \) be an infinite sequence of real numbers. Show that the following three conditions are equivalent:

(a) For any \( f \in C(\mathbb{T}) \),

\[
\frac{1}{N} \sum_{n=1}^N f(x_n) \to \int f \, dx \quad \text{in } L^2(\mathbb{T}).
\]

(b) \( \frac{1}{N} \sum_{n=1}^N e(kx_n) \to 0 \) for all \( k \in \mathbb{Z}^+ \).

(c) \( \lim_{N \to \infty} \sup_{I \subset \mathbb{T}} |\frac{1}{N} \sum_{n=1}^N 1_{I} - \frac{1}{|I|}| = 0 \).

If these conditions hold we say that \( \{x_n\}_{n=1}^\infty \) is uniformly distributed modulo 1.
32. Using #30 with a suitable choice of $T$, prove the following: If $\{x_n\}_{n=1}^\infty$ is a sequence for which $\{x_{n+k} - x_n\}_{n=1}^\infty$ is u.d. modulo 1 for any $k \in \mathbb{Z}^+$, then $\{x_n\}_{n=1}^\infty$ is also u.d. mod 1. In particular, show that $\{n^d \omega\}_{n=1}^\infty$ is u.d. mod 1 for any irrational $\omega$ and $d \in \mathbb{Z}^+$.

33. (a) Let $p \geq 2$ be a positive integer. Show that for a.e. $x \in \mathbb{T}$ $\{p^n x\}_{n=1}^\infty$ is u.d. modulo 1.

(b) Can you characterize those $x$ which have this property?

34. Suppose $\sum_{n=1}^\infty a_n^2 n < \infty$ and $\sum_{n=1}^\infty a_n$ is Cesaro summable. Show that $\sum_{n=1}^\infty a_n$ converges. Use this to prove that for any $f \in C(\mathbb{T})$ for which $\sum_{n=1}^\infty |\hat{f}(n)|^2 \cdot |n| < \infty$ one has $S_n f \to f$ uniformly.

35. Show that there exists an absolute constant $C$ so that

$$C^{-1} \sum_{n \neq 0} |\hat{f}(n)|^2 |n| \leq \int_0^1 \int_0^1 \frac{|f(x) - f(y)|^2}{\sin^2(\pi(x - y))} \, dx \, dy \leq C \sum_{n \neq 0} |\hat{f}(n)|^2 |n|$$

for any $f \in H^{1/2}(\mathbb{T})$.

36. Use #34 and #35 to prove the following theorem of Pal-Bohr: For any $f \in C(\mathbb{T})$ there exists a homeomorphism $\phi : \mathbb{T} \to \mathbb{T}$ such that

$$S_n (f \circ \phi) \to f \circ \phi$$

uniformly. Hint: Wlog $f > 0$. Consider the domain defined in terms of polar coordinates by means of $r(\theta) = f(\theta/2\pi)$. Then apply the Riemann mapping theorem to the unit disc.

37. Show that

$$\|f * g\|_{L^2(\mathbb{R})}^2 \leq \|f \|_{L^2(\mathbb{R})}\|g \|_{L^2(\mathbb{R})} \quad \text{for all } f, g \in L^2(\mathbb{R}).$$

Can there be such an inequality with $L^1(\mathbb{R})$ instead of $L^2(\mathbb{R})$?

38. Prove Poincaré’s inequality:

$$\int_{D(0,R)} |f(x) - f_D(0,R)|^2 \, dx \leq C R^2 \int_{D(0,R)} |\nabla f(x)|^2 \, dx$$

for all $f \in S$. Here $C$ depends only on the dimension. $f_D(0,R)$ denotes the mean of $f$ over $D(0,R)$. 
39. Prove the following weak form of the Logvinenko-Sereda theorem by means of Poincaré’s inequality and Bernstein’s inequality:

Suppose $|F \cap D| \leq \gamma |D|$ for all disks of radius $R$. Show that for small $\gamma > 0$ there exists $\delta = \delta(\gamma)$ so that $\delta(\gamma) \to 0$ as $\gamma \to 0$, and such that

$$\|f\|_{L^2(F)} \leq \delta(\gamma)\|f\|_2 \quad \text{if } \text{supp}(\hat{f}) \subset D(0, R^{-1}).$$

40. Given $N$ arcs $\{I_\alpha\}_{\alpha=1}^N \subset \mathbb{T}$, set $f = \sum_{\alpha=1}^N \chi_{I_\alpha}$. Show that

$$\sum_{|\nu|>k} |\hat{f}(\nu)|^2 \lesssim \frac{N}{k}.$$

41. Given any function $\psi : \mathbb{Z}^+ \to \mathbb{R}^+$ so that $\psi(n) \to 0$ as $n \to \infty$, show that you can find a measurable set $E \subset \mathbb{T}$ for which

$$\limsup_{n \to \infty} \frac{|\hat{\chi}_E(n)|}{|\psi(n)|} = \infty.$$