INVERSION OF THE SPHERICAL RADON TRANSFORM BY A POISSON TYPE FORMULA

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Abstract. The article presents an analog of the Poisson summation formula for approximate reconstruction of an even smooth function on the unit sphere using a discrete set of values of its integrals along great sub-spheres.

1. Introduction and main results

The spherical Radon transform, also known as the Funk transform, associates to a function \( \varphi(\theta) \) its integral over great sub-spheres: 
\[
R \varphi(\theta^\perp \cap S^d) = \int_{\theta^\perp \cap S^d} d\mu,
\]
where \( \theta^\perp \cap S^d \) is the great (equatorial) subsphere of \( S^d \) whose plane has normal \( \theta \). This transform is of interest in pure and applied mathematics. A good deal of the analysis of the spherical Radon transform can be done using spherical harmonics and the representation theory of the orthogonal group. See, [3] and also, e.g., [1] for a discussion of projective spaces that applies equally well to spheres. Here we will deal with even functions on the sphere. The entire discussion could be phrased in terms of real projective spaces. But this usage is not common in applications and, in any case, does not add much to the present context. Analysis of the spherical Radon transform using spherical wavelets, the spherical Calderon reproducing formula and their applications to inversion of the Radon transform on symmetric spaces is considered in [7, 8, 9, 10].

To outline our approach to the inversion of the spherical Radon transform let us begin by recalling the classical Poisson formula for the Fourier transform. One of the forms of the classical Poisson summation formula for a function \( \varphi \) belonging to \( L^2(\mathbb{R}) \) with support in \([−\omega, \omega]\) is
\[
\varphi(t) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n\Omega) e^{i nt}, \quad \Omega = \pi/\omega.
\]
This shows that the Fourier coefficients of a function with compact support are regularly spaced samples of its Fourier transform. Taking the limit of the right hand side when \( \omega \) goes to infinity we obtain a formula that makes sense for any function belonging to \( L^2(\mathbb{R}) \). This formula can be treated as an inversion formula for the Fourier transform.

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A simple way to obtain formula (1.1) is by using the famous Shannon-Whittaker sampling theorem. This result states that if a function $f \in L^2(\mathbb{R})$ has band width $\omega$, i.e., its Fourier transform $\hat{f}$ has support in $[-\omega, \omega]$, then $f$ is completely determined by its values at points $n\Omega$, where $\Omega = \pi/\omega$ and, in the $L^2$-sense,

$$f(t) = \sum f(n\Omega) \frac{\sin(\pi(t-n\Omega))}{\pi(t-n\Omega)}.$$ 

It was shown in [4] that a band limited function belonging to $L^2(\mathbb{R}^d)$ can be reconstructed from an appropriate irregular set of points as a limit of polyharmonic splines. To give a precise formulation of this statement we will need the following notations.

$B_\sigma(\mathbb{R}^d)$:
The set of all band limited functions in $L^2(\mathbb{R}^d)$ with band width $\sigma$ is denoted by $B_\sigma(\mathbb{R}^d)$.

$X(\lambda)$:
The symbol $X(\lambda) = \{x_\nu\}$ is used for a set of sample points, or a knot set in $\mathbb{R}^d$. Mention of a knot set $X(\lambda)$ will carry tacitly the assumption such that there is a triangulation of $\mathbb{R}^d$ with the following properties:

1) Each element of the triangulation contains just one of the points $x_\nu$;
2) The diameter of every element of the triangulation is not greater than $\lambda$.

We consider the operator $\Delta + \varepsilon$ in the space $L^2(\mathbb{R}^d)$, where $\Delta$ is the Laplace operator in $L^2(\mathbb{R}^d)$. A polyharmonic Lagrangian spline $L^2_{\nu} \in L^2(\mathbb{R}^d)$ is a special linear combination of the fundamental solutions of the shifted Laplace operator $(\Delta + \varepsilon)^k$. (See more details in [4].) Note that Lagrangian splines have exponential decay at infinity. The following result can be found in [4], [2].

**Theorem 1.1 (Approximation by polyharmonic splines with bounds).**

There exists a constant $c = c(d, \varepsilon)$ that depends only on the dimension $d$ and the parameter $\varepsilon$ such that for any $\sigma > 0$, for every knot set $X(\lambda)$ with $\lambda < (c(\sigma + \varepsilon))^{-1}$, for every integer $r \geq \lfloor d/2 \rfloor + 1$, and for every band limited function $f \in B_\sigma(\mathbb{R}^d)$, we have the following interpolation by polyharmonic Lagrangian splines:

$$f = \lim_{l \to \infty} \sum_{\nu \in X(\lambda)} f(x_\nu)L^2_{\nu^r}, \quad l \in \mathbb{N},$$

(1.2)

whose approximations enjoy the error estimate below:

$$\|f - \sum f(x_\nu)L^2_{\nu^r}\| \leq 2(c\lambda(\sigma + \varepsilon))^{2l+1}||f||.$$

(1.3)
Allowing $\lambda$ to go to zero we obtain (1.3) for any function belonging to $L^2(\mathbb{R}^d)$ that is sufficiently smooth. This sampling theorem can be used to obtain the following irregular analog of the Poisson summation formula [2]

**Theorem 1.2.** If the band-limited function $\varphi \in L^2(\mathbb{R}^d)$ has support in the ball $B(\sigma, 0)$ and $\Lambda^k_\nu$ is the inverse Fourier transform of the spline function function $L^k_\nu$ then

\[
\varphi = \lim_{l \to \infty} \sum_{\xi \in \Xi(\lambda)} \hat{\varphi}(\xi) \Lambda^{2^r}_{2^r},
\]

assuming $l \in \mathbb{N}, r \geq \lfloor d/2 \rfloor + 1, \lambda < (c(\sigma + \varepsilon))^{-1}, c = c(d, \varepsilon)$ is the same as in the previous theorem and the $\Xi(\lambda)$ is an appropriate discrete set in the space of the dual variable $\xi$.

An error estimate for this approximation is

\[
\| \varphi - \sum_{\xi \in \Xi(\lambda)} \hat{\varphi}(\xi) \Lambda^{2^r}_{2^r} \| \leq 2(c(\sigma + \varepsilon))^{2^{d+1}+1} \| \varphi \|.
\]

The above result can also be considered as an inverse Fourier transform. The main objects of study in the present paper are: the unit sphere $S^d$, the corresponding space $L^2(S^d)$, the Laplace -Beltrami operator in the space $L^2(S^d)$, and the spherical Radon transform (also called the Funk transform). In this situation we obtain a very close analog of Theorem 1.2 with the spherical Radon transform in place of the Fourier transform. The method used to obtain such a Theorem is similar to that of $\mathbb{R}^d$. Namely, we go to the Radon Transform side and use an appropriate sampling theorem on the sphere [4]. For example, if $\varphi$ is an even harmonic polynomial on the unit sphere $S^d$ and $R\varphi(\xi_\nu)$ are the values of its Spherical Radon transform on an appropriate grid $\{\xi_\nu\}$ we obtain the formula

\[
R\varphi = \lim_{k \to \infty} \sum_{\nu} R\varphi(\xi_\nu) L^k_\nu,
\]

where $L^k_\nu$ is a Lagrange spline on the dual sphere i.e. $L^k_\nu$ is a specific linear combination of the fundamental solutions of the shifted Laplace -Beltrami operator on the sphere.

Taking the inverse Radon Transform of both sides of (1.5) we obtain our main formula:

\[
\varphi = \lim_{k \to \infty} \sum_{\nu} R\varphi(\xi_\nu) \Lambda^k_\nu,
\]

where $\Lambda^k_\nu$ is the inverse Radon transform of $L^k_\nu$.

Using a nested sequence of knots whose union is dense in the sphere $S^d$ we can extend (1.6) to any smooth function on $S^d$ by means of the formula
\[ \varphi = \lim_{\nu} \sum R\varphi(\xi_{\nu})\Lambda_{\nu}^{k}, \]  
where the limit is taken over a nested sequence of knots and $\lambda$ is fixed.

Formulas (1.6) and (1.7) are inversion formulas for the spherical Radon Transform. Their advantage is that they suggest a way to approximate a function using just a finite number of samples of its Radon transform. Note that according to the uncertainty principle we have the usual trade-off between smoothness and localization: the lower the index $k$ the better localization we obtain for the functions $\Lambda_{\nu}^{k}$. At the end of the paper we give a constructive way to determine the Fourier-Laplace coefficients of the functions $\Lambda_{\nu}^{k}$. We hope to conduct numerical experiments to test the reconstruction schemes presented here. The results of such experiments will be published in a future paper.

2. Splines on Spheres and a Poisson summation formula for the Spherical Radon Transform

In this section we recall the notion of splines on manifolds and their construction. We prove that a sequence of splines that interpolates a given smooth function converges to it in the appropriate Sobolev norms. Then we prove the inversion formula for the spherical Radon transform (1.6).

Let \( \{r_{\nu}\}, \nu = 1, \ldots, N \) be a simplicial cover of \( S^d \) such that every simplex \( r_{\nu} \) has a diameter not greater than $\lambda$. Let \( X(\lambda) = \{x_{\nu}\} \) be a set of points of \( S^d \) such that $x_{\nu}$ belongs to the interior of $r_{\nu}$. In this paper we will be interested in even functions on $S^d$ and as a result in even spline functions. Because of this it is natural to assume that the set of knots $X(\lambda)$ is symmetric or even, i.e., $X(\lambda) = -X(\lambda)$.

Given a knot set $X(\lambda)$ and a sequence of numbers \( \{t_{\gamma}\} \in l^2 \) we will be interested to find a function $t_k \in H^{2k}(S^d)$, for $k$ large enough, such that the following conditions hold.

a) **Interpolation:** \( t_k(x_{\gamma}) = t_{\gamma}, \quad x_{\gamma} \in X(\lambda); \)

b) **Minimality:** the function $t_k$ minimizes the functional \( u \rightarrow \| (1 + \Delta)^k u \| \).

Since $\|(1 + \Delta)^k u\|$ is a norm this minimization problem has a unique solution $t_k \in H^{2k}(S^d)$ which will be called a spline function (see [4]). We will often refer to this minimization problem below.

**Notation.** If $f$ is a smooth function on $S^d$ then the (unique) spline function belonging to $H^{2k}(S^d)$ that interpolates $f$ on $X(\lambda)$ is denoted by $t_k(f)$. Note that if $f$ is even and the set $X(\lambda)$ is symmetric, then the spline $t_k(f)$ is an even function.

It was shown in [4] that every smooth function $f$ can be interpolated and approximated by splines $t_k(f)$ in the $L^2$-norm. In order to realize our approach to the formula (1.6), we have to prove that the above convergence takes place not only in the $L^2$ norm but in fact in any Sobolev norm and even in the uniform norm. We begin with a lemma that will be useful in obtaining Sobolev norm estimates.
Lemma 2.1. Let $Q$ be any self-adjoint operator in a Hilbert space $E$. If $f$ in the domain of $Q$ satisfies

$$
\|f\| \leq A + a\|Qf\| \tag{2.1}
$$

for some $a > 0$ then for the same $f$ and all integers $m = 2^l_1, l_1 = 0, 1, 2, \ldots$

$$
\|f\| \leq mA + 8^{m-1}a^m\|Q^m f\|. \tag{2.2}
$$

Moreover, if $A = 0$ then for any nonnegative $r$ and every $m = 2^l_1 \geq r, l_1 = 0, 1, 2, \ldots$, there exists a positive constant $b(r, m)$ such that for all $n = 2^l_2, l_2 = l_1, r, l_2 = 0, 1, 2, \ldots$

$$
\|Q^r f\| \leq (b(r, m)a^{r-m})^n \|Q^{n(m-r)+r} f\| \tag{2.3}
$$

as long as $f$ belongs to the domain of $Q^{n(m-r)+r}$.

Proof. The assertions here are quite similar to those of Lemmas 3.1 and 3.2 of [4], so we merely sketch the proof. If $Q$ is a self-adjoint operator then $iQ$ generates a strongly continuous group of unitary operators $e^{itQ}$. Because of this we have the following Laplace transform representations for the resolvents of the operators $iQ$ and $-iQ$

$$
(\lambda I - iQ)^{-1} f = \int_0^\infty e^{-\lambda t} e^{itQ} f dt,
$$

$$
(\lambda I + iQ)^{-1} f = \int_0^\infty e^{-\lambda t} e^{-itQ} f dt.
$$

for any $\lambda > 0$. This gives us the following inequalities

$$
\|(\lambda I - iQ)^{-1}\| \leq \lambda^{-1}, \quad \|(\lambda I + iQ)^{-1}\| \leq \lambda^{-1}.
$$

In other words for any positive $\varepsilon$ we have

$$
\|(I - i\varepsilon Q)^{-1}\| \leq 1, \quad \|(I + i\varepsilon Q)^{-1}\| \leq 1.
$$

This gives

$$
\varepsilon\|Qf\| \leq \|(I - i\varepsilon Q)f\| + \|f\| \leq \|(I + \varepsilon^2 Q^2)f\| + \|f\| \leq \varepsilon^2\|Q^2f\| + 2\|f\|. \tag{2.4}
$$

So, for any $f$ from the domain of $Q^2$ we have the inequality

$$
\|Qf\| \leq \varepsilon\|Q^2f\| + 2/\varepsilon\|f\|, \varepsilon > 0.
$$

The claim is evident for $m = 1$ and combining this case with the last equation above verifies the case $m = 2$. Continuing in this way, or, equivalently, using the induction hypothesis for a given $m$ we conclude that

$$
\|f\| \leq mA + 2^{3m-3}a^m(\varepsilon\|Q^{2m}f\| + 2/\varepsilon\|f\|).
$$

Setting $\varepsilon = 2^{3m-1}(a)^m$, we obtain

$$
\|f\| \leq 2mA + 2^{6m-3}a^{2m}\|Q^{2m}f\|. \tag{2.5}
$$
Thus the induction continues and the first part of the lemma is proved. In particular the inequality (2.5) implies, for \( A = 0 \), that

\[
\| f \| \leq (8a)^m \| Q^m f \|, m = 2^l, l = 0, 1, \ldots
\]

Next, since

\[
\| Q^r f \| \leq c(m, r) \| Q^m f \|^{r/m} \| f \|^{r-m}, 0 \leq r \leq m
\]
we have

\[
\| Q^r f \| \leq c(m, r)(8a)^{r-m} \| Q^m f \|, m = 2^l, l = 0, 1, \ldots , 0 \leq r \leq m.
\]

For \( g = Q^r f \) this gives

\[
\| g \| \leq c(m, r)(8a)^{r-m} \| Q^m g \|
\]
and then by (2.5)

\[
\| Q^r f \| \leq (ba^{r-m})^n \| Q^m g \|, m = 2^l, n = 2L^2,
\]
where constant \( b \) is of the form

\[
b(m, r) = c(m, r)8^{r-m}.
\]

In other words with the same \( b \) as above we have for \( l_1, L^2 = 0, 1, \ldots \)

\[
\| Q^r f \| \leq (ba^{r-m})^n \| Q^m g \|, m = 2^l, n = 2L^2
\]

The following lemma is a consequence of a result from [4].

**Lemma 2.2.** There exist constants \( \lambda_0 \) and \( C \) such that for any smooth function \( f \in C_0^\infty(S^d) \) and any \( \lambda \) with \( 0 < \lambda < \lambda_0 \)

\[
\| f \| \leq C \left( \lambda^{d/2} \left( \sum_{\gamma} |f(x_\gamma)|^2 \right)^{1/2} + \lambda^d \| \Delta^{d/2} f \| \right).
\]

The Lemma 2.2 implies that if \( \lambda \) is small enough then for a suitable \( C \)

\[
\| f \| \leq C \left( \lambda^{d/2} \left( \sum_{\gamma} |f(x_\gamma)|^2 \right)^{1/2} + \lambda^d \| (1 + \Delta)^{d/2} f \| \right).
\]

Now we use Lemma 2.1 with \( Q = (1 + \Delta)^{d/2} \) which gives us
\[ \|f\| \leq C \left( m \lambda^{d/2} \left( \sum |f(x)|^2 \right)^{1/2} + 8^{m-1}(\lambda^d)^m \| (1 + \Delta)^{md/2} f \| \right), \]

where \( m = 2^l, l = 0, 1, 2, \ldots \). In particular if the restriction of \( f \) to the set \( X(\lambda) \) is zero then the second part of the Lemma 2.1 gives for \( m = 2r, 2k = rd, n = 2^l, l = 0, 1, \ldots \)

\[ \|(1 + \Delta)^k f\| \leq (C_1 \lambda^{2k})^n \| (1 + \Delta)^{k(n+1)} f \|. \tag{2.11} \]

Putting all this together we obtain the following estimate for convergence in the Sobolev norm

\[ \| f - t_{k(n+1)}(f) \|_{2k} \leq (C_2 \lambda^{2k})^n \| (1 + \Delta)^{k(n+1)} f \|. \tag{2.12} \]

Here the norm \( \| \cdot \|_{2k} \) is the Sobolev Norm \( \| \Delta^k (\cdot) \|_{L^2} \). The right hand side above goes to zero if \( n \) is fixed and \( \lambda \) goes to zero.

The formula (2.12) can be applied to interpolate and approximate any smooth function in \( L^2(S^d) \).

In particular, if \( f \) is a spherical harmonic polynomial of degree \( \omega \) then we also have another way to approximate \( f \) by keeping \( \lambda \) fixed. Indeed for a polynomial of degree \( \omega \) we have the estimate

\[ \| f - t_{k(n+1)}(f) \|_{2k} \leq (C_2 \lambda^{2k})^n (1 + \omega)^{k(n+1)} \| f \|. \tag{2.13} \]

and the right hand side goes to zero given that \( n \) goes to infinity and, for a fixed \( \lambda \), that the following condition holds true:

\[ C_2 \lambda (1 + \omega) < 1. \]

In what follows we will apply the estimates (2.12) and (2.13) to obtain our main result. Let us recall the basic properties of the spherical Radon transform. If a function \( \varphi \) belonging to \( L^2(S^d) \) has Fourier coefficients \( \varphi_j^m \) then its Radon Transform is given by the formula

\[ R\varphi = \pi^{-1/2} \Gamma((d + 1)/2) \sum_{j,i} c_j \varphi_j^m Y_j^i \] 

where \( Y_j^i \) is the standard notation for spherical harmonic polynomials and where

\[ c_j = (-1)^{j/2} \Gamma((j + 1)/2)) / \Gamma((j + d)/2) \]

if \( j \) is even and \( c_j = 0 \) otherwise.

Because the coefficients \( c_j \) have asymptotics \( (-1)^{j/2}(j/2)^{(1-d)/2} \) as \( j \) goes to infinity we have that \( R \) is a continuous operator from the Sobolev space of even functions \( H_{\text{even}}^a(S^d) \) onto the space \( H_{\text{even}}^{a+(d-1)/2}(S^d) \). Its inverse is a continuous operator from the space \( H_{\text{even}}^{a+(d-1)/2}(S^d) \) onto the space \( H_{\text{even}}^a(S^d) \).
Let \( \{ s_\nu \} \) be a set of equatorial subspheres on \( S^d \) of codimension one. We assume that the corresponding set of points \( \xi_\nu \) on the dual sphere form a set of type \( X(\lambda) \). According to the estimate (2.12) we have that for the spherical Radon transform \( R\varphi \) of the smooth function \( \varphi \),

\[
\| R\varphi - t_{k(n+1)}(R\varphi) \|_{2k} \leq (C_2\lambda^{2k})^n (1 + \Delta)^{k(n+1)}\| R\varphi \|
\]

where \( k \) is large enough. Taking the inverse Radon transform we obtain

\[
\| \varphi - R^{-1}t_{k(n+1)}(R\varphi) \|_{2k-(d-1)/2} \leq (C_3\lambda^{2k})^n (1 + \Delta)^{k(n+1)}\| R\varphi \|
\]

where the right side goes to zero for any smooth \( \varphi \) as long as \( \lambda \) goes to zero. If \( \varphi \) is a polynomial of degree \( \omega \) than we have the estimate

\[
\| \varphi - R^{-1}t_{k(n+1)}(R\varphi) \|_{2k-(d-1)/2} \leq (C_3\lambda^{2k})^n (1 + \omega)^{k(n+1)}\| R\varphi \|
\]

and the right hand side goes to zero also when \( \lambda \) is fixed and the following condition holds:

\[ C_3\lambda(1 + \omega) < 1. \]

The goal of the next section is to give an effective procedure to determine Fourier coefficients of the terms \( R^{-1}t_{k(n+1)}(R\varphi) \).

### 3. Fourier coefficients of \( R^{-1}t_{k(n+1)}(R\varphi) \)

A fundamental solution \( E^k_x \) of the operator \( (1 + \Delta)^k, k > d/2 \), is a solution of the equation

\[
(1 + \Delta)^k E^k_x = \delta(x),
\]

where \( \delta(x) \) is the Dirac measure at \( x \). Since \( (1 + \Delta)^k \) is an isomorphism between Sobolev spaces \( H^a \) and \( H^{a-k} \) and since \( \delta(x) \) belongs to \( H^{-d/2-\varepsilon} \) for any positive \( \varepsilon \), the fundamental solution \( E^k_x \) for \( k > d/2 \) belongs to a Sobolev space with a positive index.

Now let us consider the notion of even Lagrangian splines on the sphere. Even Lagrangian splines \( L^{2k}_\nu \) are the even functions that enjoy the same minimality condition as in section 2 above and also satisfy the following conditions:

1) if \(-x_\nu = x_\mu\) then \( L^{2k}_\nu(x_\nu) = L^{2k}_\mu(x_\mu) = 1\),
2) \( L^{2k}_\nu(x_\rho) = 0\), if \( \rho \) is different from \( \nu \) and \( \mu \).

The following theorem is consequence of a result from [4].

**Theorem 3.1.** For any smooth function \( f \) on the dual sphere the following representations hold true:

\[
t_k(f) = \sum_\nu f(\xi_\nu)L^{2k}_\nu
\]
(3.3) \[ t_k(f) = \sum \alpha_\nu E^k_\nu \]
where \( E^k_\nu = E^k_{x_\nu} \) and \( \{\alpha_\nu\} \) are suitable constants.

From this we obtain that

(3.4) \[ R^{-1} t_{k(n+1)}(R\varphi) = \sum_\nu (R\varphi)(\xi_\nu) \Lambda^{2k(n+1)}_\nu \]

where \( \Lambda^{2k(n+1)}_\nu = R^{-1} L^{2k(n+1)}_\nu \).

Since every function \( t_k(f) \) is a linear combination of Lagrangian splines it is enough to find the Fourier coefficients of \( L^{2k}_\nu \). Suppose that

(3.5) \[ L^{2k}_\nu = \sum_{j=0}^{\infty} \sum_i \beta^{k,i}_\nu Y^i_j. \]

Then we have the formula

(3.6) \[ (1 + \Delta)^{2k} L^{2k}_\nu = \sum_{j=0}^{\infty} \sum_i [1 + (j(j + 1))]^k \beta^{k,i}_\nu Y^i_j. \]

Using the following representation of the delta measure \( \delta_{x_\gamma} \)

(3.7) \[ \delta_{x_\gamma} = \sum_j \sum_i Y^i_j(x_\gamma) Y^i_j \]

and the equations (3.3) and (3.6) we obtain

(3.8) \[ \sum_{\gamma=1}^{N} \alpha_{\gamma,\nu} Y^i_j(x_\gamma) = [1 + (j(j + 1))]^k \beta^{k,i}_{\nu,j} \]

where

(1 + \Delta)^{2k} L^{2k}_\nu = \sum_{\gamma=1}^{N} \alpha_{\gamma,\nu} \delta(x_\gamma). \]

So for a fixed \( \nu \) we can express unknowns \( \beta^{k,i}_{\nu,j} \) in terms of \( N \) unknowns \( \alpha_{\gamma,\nu}, \gamma = 1, 2, \ldots, N. \) Using conditions 1) and 2) from the definition of the splines \( L^{2k}_\nu \) we obtain another \( N \) equations which help us to determine \( \alpha_{\gamma,\nu}, \gamma = 1, 2, \ldots, N. \) Note that such a system of \( N \) equations in \( N \) unknowns has a unique solution because the corresponding minimization problem (alluded to earlier) for the functions \( L^{2k}_\nu \) does have a unique solution [4].

All together this gives an algorithm for finding Fourier coefficients \( \beta^{i,j}_{\nu,n} \) of \( L^{2k}_\nu \) and then for the functions \( \Lambda^{2k}_\nu \) we have

(3.9) \[ \Lambda^{2k}_\nu = \sum_{j=\text{even}}^{\infty} \sum_i c^{-1}_j \beta^{k,i}_{\nu,j} Y^i_j. \]
where 
\[ c_j = (-1)^{j/2} \Gamma((j + 1)/2)/\Gamma((j + d)/2) \]
if \( j \) is even.

We can now give a precise formulation of our main result.

**Theorem 3.2.** If \( \xi_{\nu} \) is a discrete set of the type \( X(\lambda) \) on the dual sphere \( S^{d*} \) then, for any natural number \( k \), there are special functions \( \Lambda_{\nu}^{2k(n+1)}, n = 1, \ldots \), such that for any smooth even function \( \phi \) on the sphere \( S^d \) we have the following estimate for approximation of \( \phi \) by discrete samples of its spherical Radon transform:

\[
\| \phi - \sum_{\nu=1}^{N} R\phi(\xi_{\nu})\Lambda_{\nu}^{2k(n+1)} \|_{2k-(d-1)/2} \leq (C\lambda^{2k})^{n}\| (1 + \Delta)^{k(n+1)} R\phi \|.
\]

If \( \phi \) is an even harmonic spherical polynomial of (even) degree \( \omega \) then

\[
\| \phi - \sum_{\nu=1}^{N} R\phi(\xi_{\nu})\Lambda_{\nu}^{2k(n+1)} \|_{2k-(d-1)/2} \leq (C\lambda(1 + \omega))^{2kn}\| R\phi \|.
\]

The functions \( \Lambda_{\nu}^{2k(n+1)}, n = 1, \ldots \) are inverse spherical Radon transforms of Lagrangian splines on the sphere \( S^d \). Moreover, by solving \( N \) linear systems of size \( N \times N \) one can determine \( N^2 \) constants \( \alpha_{\gamma,\nu} \) so that the induced constants

\[
\beta_{\nu,j}^{k,i} = [1 + (j(j + 1))]^{-k} \sum_{\gamma=1}^{N} \alpha_{\gamma,\nu} Y_{j}^{i}(x_{\gamma})
\]

give the following presentation of the special functions \( \Lambda_{\nu}^{2k(n+1)} \):

\[
\Lambda_{\nu}^{2k(n+1)} = \sum_{j=\text{even}}^{\infty} \sum_{i} \beta_{\nu,j}^{k,i} R^{-1}Y_{j}^{i}.
\]

Note that the last linear combinations can be calculated in advance as long as the discrete set of points \( X(\lambda) \) is chosen. This could lead to a table lookup scheme for tomography on the sphere.

**References**


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