Traces and boundary value problems for elliptic wedge operators\textsuperscript{1}

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joint work with
Thomas Krainer

University of Arkansas Spring Lecture Series, 2012

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This talk reflects joint work, in progress, with T. Krainer of Penn State on boundary value problems for elliptic wedge operators.

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I will:
– first describe in a rather abstract and general way what one ought to regard as boundary values for differential operators,
– then narrow down the problem to elliptic wedge operators,
– then discuss boundary values (traces) as sections of a vector bundle.
Boundary values

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Nonzero boundary values (up to some order) are carried by elements in $\mathcal{D}_{\text{max}}(A)$ not in $\mathcal{D}_{\text{min}}(A)$. 

Let $M = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ and $A = \Delta$ (Laplacian).

$\mathcal{D}_{\text{min}}(\Delta) = H^2_0(\hat{M})$

$\mathcal{D}_{\text{max}}(\Delta) \neq H^2(\hat{M})$

In fact $\mathcal{D}_{\text{max}}(\Delta) \hookrightarrow L^2(M)$ is not compact!
\( \mathcal{D}_{\text{max}} = \{ u \in L^2(\mathcal{M}) : Au \in L^2(\mathcal{M}) \} \)

\[ \| u \|_A^2 = \| Au \|^2 + \| u \|^2 \]

\( \mathcal{D}_{\text{max}} \) is complete, \( \mathcal{D}_{\text{min}} \) is the closure of \( C^\infty_c(\mathcal{M}) \).
Elements in $\mathcal{D}_\text{min}$ vanish to some order on $\partial \mathcal{M}$. Nonvanishing boundary values (to some order) are produced by elements of $\mathcal{D}_\text{max}$.

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Elements in $\mathcal{D}_{\text{min}}$ vanish to some order on $\partial \mathcal{M}$. Nonvanishing boundary values (to some order) are produced by elements of $\mathcal{D}_{\text{max}}$. How do we get a hold on these?

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Elements in $D_{\min}$ vanish to some order on $\partial \mathcal{M}$. Nonvanishing boundary values (to some order) are produced by elements of $D_{\max}$. How do we get a hold on these?

$$\mathcal{E} = \text{ orthogonal of } D_{\min} \text{ in } D_{\max},$$
$$\pi_{\max} : D_{\max} \rightarrow D_{\max} \text{ orthogonal projection on } \mathcal{E}$$

If $A$ is elliptic, applying $\pi_{\max}$ is like taking traces (up to some order).

Pick $\chi \in C_c^{\infty}(\mathcal{M})$. If $u \in D_{\max}$, then $\chi u \in D_{\min}$.
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If $A$ is elliptic and $Au \in L^2$ then $u \in H^m_{\text{loc}}$.

Pick $\chi \in C_c^\infty(\mathcal{M})$. If $u \in D_{\text{max}}$, then $\chi u \in D_{\text{min}}$

so $\pi_{\text{max}}(\chi u) = 0$. 

For example, it may be that $D_{\text{min}} \hookrightarrow \to L^2$ is already compact. Then you would like to have $D \hookrightarrow \to L^2$ also compact and large enough that a parametrix exists and is a compact operator.
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\[ \omega = 1 \text{ "near } \infty." \]
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Arkansas Spring Lecture Series
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Choosing a boundary condition corresponds to choosing a subspace $D \subset E$.

The problem is: Solve $Au = f \in L^2$, $u \in D_{\text{min}} + D$. 

If $A$ is elliptic and $Au \in L^2$ then $u \in H_m^{\text{loc}}$. 

For example, it may be that $D_{\text{min}} \hookrightarrow \to L^2$ is already compact. Then you would like to have $D \to L^2$ also compact and large enough that a parametrix exists and is a compact operator.
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Part of the problem is to find criteria for detecting good spaces $D$. 

\[ \mathcal{D}_{\max} = \{ u \in L^2(\mathcal{M}) : Au \in L^2(\mathcal{M}) \} \]

$$\|u\|_A^2 = \|Au\|^2 + \|u\|^2$$

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Traces for elliptic wedge operators

Traces are a local issue, so we work on

\[ \mathcal{M} = [0, \infty) \times \mathcal{Y} \times \mathcal{Z} \]

\( \mathcal{Z} \) is a compact \( n \)-manifold without boundary, \( \mathcal{Y} \) an open set in \( \mathbb{R}^q \)

near \( \partial \mathcal{M} = \{0\} \times \mathcal{Y} \times \mathcal{Z} \).
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near \( \partial \mathcal{M} = \{0\} \times \mathcal{Y} \times \mathcal{Z} \). The measure is \( m_b = \frac{1}{x} \, m \), \( m = dx \, dy \, m_z \),

the \( L^2 \) space is \( L^2(\mathcal{M}, x^m m_b) = x^{-m/2} L^2_b(\mathcal{M}) \),

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\[
A = \frac{1}{x^m} \sum_{k+|\alpha|+|\beta|\leq m} a_{k,\alpha,\beta}(x, y, z)(xD_x)^k(xD_y)^{\alpha} D_z^{\beta}
\]

\( a_{k,\alpha,\beta} \in C^\infty(\mathcal{M}) \)

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\[ A = \frac{1}{x^m} \sum_{k+|\alpha|+|\beta| \leq m} a_{k,\alpha,\beta}(x, y, z) (xD_x)^k (xD_y)^\alpha D_z^\beta \]
with
\[ w_{\sigma}(A) = \sum_{k+|\alpha|+|\beta| = m} a_{k,\alpha,\beta}(x, y, z) \xi^k \eta^\alpha \zeta^\beta \]
invertible on \((\xi, \eta, \zeta) \neq 0\).
Traces for elliptic wedge operators

Traces are a local issue, so we work on
\[ M = [0, \infty) \times Y \times Z \]

near \( \partial M = \{0\} \times Y \times Z \). The measure is \( m_b = \frac{1}{x} m \), \( m = dx \, dy \, m_Z \), the \( L^2 \) space is \( L^2(M, x^m m_b) = x^{-m/2} L^2_b(M) \), and the operator is
\[ A = \frac{1}{x^m} \sum_{k+|\alpha|+|\beta| \leq m} a_{k,\alpha,\beta}(x, y, z)(xD_x)^k (xD_y)^\alpha D_z^\beta \]

with
\[ w_\sigma(A) = \sum_{k+|\alpha|+|\beta| = m} a_{k,\alpha,\beta}(x, y, z) \xi^k \eta^\alpha \zeta^\beta \]
invertible on \((\xi, \eta, \zeta) \neq 0\).

Let
\[ bA_y = \frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0, y, z)(xD_x)^k D_z^\beta \]

\[ b\hat{P}_y(\sigma) = \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0, y, z) \sigma^k D_z^\beta \]

\( \mathcal{Z} \) is a compact \( n \)-manifold without boundary, \( Y \) an open set in \( \mathbb{R}^q \)

\( a_{k,\alpha,\beta} \in C^\infty(M) \)

\( b\hat{P}_y(\sigma) \) is a family of elliptic operators on \( \mathcal{Z} \) depending smoothly on \( y, \sigma \), holomorphically in \( \sigma \in \mathbb{C} \).
The family of operators
\[ \hat{P}_y(\sigma) = \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z) \sigma^k D^\beta \]
on \mathbb{Z} is elliptic. For each \( y \),
\[ \text{spec}_b(A_y) = \{ \sigma : \hat{P}_y(\sigma) \text{ is not invertible} \} \]
is a discrete set such that \( \text{spec}_b(A_y) \cap \{ \sigma : |\Im \sigma| < r \} \) is finite for each \( r \).
The family of operators
\[
\hat{P}_y(\sigma) = \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)\sigma^k D_z^\beta
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This is a standard result from the elliptic theory of \( b \)-operators.
The family of operators
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We let
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We let

\[ \text{spec}_e(A) = \{ (y, \sigma) : \sigma \in \text{spec}_b(A_y) \} \]

This set is relevant in that if \( u \in \mathcal{D}_{\text{max}}(A) \), then

\[ \hat{u}(\sigma, y, z) = \int_{\Im \sigma = m/2} x^{-i\sigma} u(x, y, z) \frac{dx}{x} \, dy \, dz \]

is, for each \( y \), holomorphic in \( \Im \sigma > m/2 \),

This is a standard result from the elliptic theory of \( b \)-operators.
The family of operators

$$\hat{P}_y(\sigma) = \sum_{k + |\beta| \leq m} a_{k,0,\beta}(0,y,z)\sigma^k D_z^\beta$$
on $Z$ is elliptic. For each $y$,

$$\text{spec}_b(A_y) = \{ \sigma : \hat{P}_y(\sigma) \text{ is not invertible} \}$$
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$$\hat{u}(\sigma, y, z) = \int_{\Im \sigma = m/2} x^{-i\sigma} u(x, y, z) \frac{dx}{x} \ dy \ dz$$
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$$\Sigma = \{ \sigma \in \mathbb{C} : -m/2 < \Im \sigma < m/2 \}$$
The family of operators
\[ \hat{P}_y(\sigma) = \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)\sigma^k D_z^\beta \]
on $\mathcal{Z}$ is elliptic. For each $y$,
\[ \text{spec}_b(A_y) = \{\sigma : \hat{P}_y(\sigma) \text{ is not invertible}\} \]
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\[ \text{spec}_e(A) = \{(y,\sigma) : \sigma \in \text{spec}_b(A_y)\} \]
This set is relevant in that if $u \in \mathcal{D}_{\max}(A)$, then
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\[ \{\sigma - i\vartheta \in \Sigma : \sigma \in \text{spec}_b(A_y) \cap \Sigma, \, \vartheta \in \mathbb{N}_0\}, \]
The family of operators
\[ \hat{P}_y(\sigma) = \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)\sigma^k D_z^\beta \]
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We let
\[ \text{spec}_e(A) = \{ (y,\sigma) : \sigma \in \text{spec}_b(A_y) \} \]
This set is relevant in that if \(u \in D_{\max}(A)\), then
\[ \hat{u}(\sigma,y,z) = \int_{\Im \sigma = m/2} x^{-i\sigma} u(x,y,z) \frac{dx}{x} \, dy \, dz \]
is, for each \(y\), holomorphic in \(\Im \sigma > m/2\), meromorphic in \(\Sigma = \{ \sigma \in \mathbb{C} : -m/2 < \Im \sigma < m/2 \}\) with poles in
\[ \{ \sigma - i\vartheta \in \Sigma : \sigma \in \text{spec}_b(A_y) \cap \Sigma, \vartheta \in \mathbb{N}_0 \}, \]
This is not quite true: For \(\sigma > -m/2\) not in the local regularity of \(\hat{u}(\sigma,y,z)\) increases with
\[ \Im \sigma, \hat{u}(\sigma,y,z) \in H^\Im \sigma-m/2-\epsilon_{\text{loc}} \]
\[ bA_y = \frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)(xD_x)^k D_z^\beta \]
on [0, \infty) \times \{y\} \times \mathcal{Z}
\[ \mathcal{D}_{\text{min}}(bA_y) \]

\[ bA_y = \frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)(xD_x)^k D_z^\beta \]

on \([0, \infty) \times \{y\} \times \mathcal{Z}\)
\[ \mathcal{D}_{\text{max}}(bA_y) \mathcal{D}_{\text{min}}(bA_y) \]

\[
\frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)(xD_x)^k D_z^\beta
\]

on \([0, \infty) \times \{y\} \times \mathcal{Z}\)
For fixed $y$, the space $\mathcal{D}_{\text{max}}(bA_y)/\mathcal{D}_{\text{min}}(bA_y)$ is finite dimensional

$$
\frac{1}{x^m} \sum_{k+|\beta| \leq m} \frac{a_{k,0,\beta}(0,y,z)(xD_x)^k D_z^\beta}{k!} \\
\text{on } [0, \infty) \times \{y\} \times \mathcal{Z}
$$
For fixed $y$, the space $\mathcal{D}_{\text{max}}(bA_y)/\mathcal{D}_{\text{min}}(bA_y)$ is finite dimensional, isomorphic to the kernel, $\mathcal{T}_y$, of $bA_y$ on the space of functions of the form

$$\sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell=0}^{N_\sigma} a_{\sigma,\ell} x^{i\sigma} \log^\ell x$$

for some $a_{\sigma,\ell} \in C^\infty(\mathcal{Z})$ and some $N_\sigma$.

\[ bA_y = \frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z) (xD_x)^k D_z^\beta \]

on $[0, \infty) \times \{y\} \times \mathcal{Z}$
For fixed $y$, the space $\mathcal{D}_{\max}(bA_y)/\mathcal{D}_{\min}(bA_y)$ is finite dimensional, isomorphic to the kernel, $\mathcal{T}_y$, of $bA_y$ on the space of functions of the form
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bA_y = \frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)(xD_x)^k D_z^\beta
\]
on $[0, \infty) \times \{y\} \times \mathbb{Z}$

Example: (on $[0, \infty) \times \mathbb{R}$)
\[
\Delta = D_x^2 + D_y^2 = x^{-2}(x^2 D_x^2 + x^2 D_y^2)
\]
\[
b\Delta_y = x^{-2}(x^2 D_x^2) = D_x^2
\]
and so $\ker b\Delta_y = \text{span}_{\mathbb{C}}\{1, x\}$. 

Theorem (The trace bundle of $A_y$).

Suppose $\text{spec}_e(A_y) \cap \{(y,\sigma) : \Im \sigma = \pm m/2\} = \emptyset$ then $T = \bigoplus_{y \in Y} T_y$ with the canonical map $\pi : T \to Y$ is a smooth vector bundle over $Y$. The space $C^\infty(Y; T)$ consists of all sections $u : Y \to T$ which viewed as functions of $(x,y,z) \in (0, \infty) \times Y \times \mathbb{Z}$ are smooth.
For fixed $y$, the space $\mathcal{D}_{\text{max}}(bA_y)/\mathcal{D}_{\text{min}}(bA_y)$ is finite dimensional, isomorphic to the kernel, $\mathcal{T}_y$, of $bA_y$ on the space of functions of the form

$$a_{\sigma,\ell} x^{i\sigma} \log^\ell x$$

for some $a_{\sigma,\ell} \in C^\infty(\mathbb{Z})$ and some $N_{\sigma}$.

Example:

$$\Delta = D_x^2 + D_y^2 = x^{-2}(x^2 D_x^2 + x^2 D_y^2)$$

and so $\ker b\Delta_y = \text{span}_\mathbb{C}\{1, x\}$.

On the other hand

$$\mathcal{D}_{\text{min}}(b\Delta_y) = H_0^2(\mathbb{R}_+), \quad \mathcal{D}_{\text{max}}(b\Delta_y) = H^2(\mathbb{R}_+)$$

and $\mathcal{D}_{\text{max}}(b\Delta_y) / \mathcal{D}_{\text{min}}(b\Delta_y) \cong \text{span}\{1, x\}$.
For fixed \( y \), the space \( \mathcal{D}_{\text{max}}(bA_y)/\mathcal{D}_{\text{min}}(bA_y) \) is finite dimensional, isomorphic to the kernel, \( \mathcal{T}_y \), of \( bA_y \) on the space of functions of the form

\[
\sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell=0}^{N_\sigma} a_{\sigma,\ell} x^{i\sigma} \log^\ell x
\]

for some \( a_{\sigma,\ell} \in C^\infty(\mathbb{Z}) \) and some \( N_\sigma \).

**Theorem (The trace bundle of \( A \)).**

Suppose

\[
\text{spec}_e(A) \cap \{(y, \sigma) : \Im \sigma = \pm m/2\} = \emptyset
\]

Then \( \mathcal{T} = \bigsqcup_{y \in \mathcal{Y}} \mathcal{T}_y \) with the canonical map \( \pi : \mathcal{T} \to \mathcal{Y} \) is a smooth vector bundle over \( \mathcal{Y} \). The space \( C^\infty(\mathcal{Y}; \mathcal{T}) \) consists of all sections \( u : \mathcal{Y} \to \mathcal{T} \) which viewed as functions of \( (x, y, z) \in (0, \infty) \times \mathcal{Y} \times \mathcal{Z} \) are smooth.
For fixed $y$, the space $D_{\text{max}}(bA_y)/D_{\text{min}}(bA_y)$ is finite dimensional, isomorphic to the kernel, $\mathcal{T}_y$, of $bA_y$ on the space of functions of the form

$$
\sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell = 0}^{N_\sigma} a_{\sigma,\ell} x^{i\sigma} \log^\ell x
$$

for some $a_{\sigma,\ell} \in C^\infty(\mathcal{Z})$ and some $N_\sigma$.

**Theorem (The trace bundle of $A$).**

Suppose

$$\text{spec}_e(A) \cap \{(y, \sigma) : \Im \sigma = \pm m/2\} = \emptyset$$

Then $\mathcal{T} = \bigsqcup_{y \in \mathcal{Y}} \mathcal{T}_y$ with the canonical map $\pi : \mathcal{T} \to \mathcal{Y}$ is a smooth vector bundle over $\mathcal{Y}$. The space $C^\infty(\mathcal{Y}; \mathcal{T})$ consists of all sections $u : \mathcal{Y} \to \mathcal{T}$ which viewed as functions of $(x, y, z) \in (0, \infty) \times \mathcal{Y} \times \mathcal{Z}$ are smooth.

$$\mathcal{B}^\infty_{bA}(\mathcal{Y}) = \{u \in C^\infty(\mathcal{Y}; \mathcal{T}) : \pi(u) = 0\}$$

Since $bA_y(\phi(y)u) = \phi(y) bA_y(u)$, $\mathcal{B}^\infty_{bA}(\mathcal{Y})$ is a module over $C^\infty(\mathcal{Y})$.

\begin{align*}
\Delta &= D_x^2 + D_y^2 = x^{-2}(x^2 D_x^2 + x^2 D_y^2) \\
b\Delta_y &= x^{-2}(x^2 D_x^2) = D_x^2
\end{align*}

and so $\ker b\Delta_y = \text{span}_C \{1, x\}$.

On the other hand

$$D_{\text{min}}(b\Delta_y) = H^2_0(\mathbb{R}_+), \quad D_{\text{max}}(b\Delta_y) = H^2(\mathbb{R}_+)$$

and $D_{\text{max}}(b\Delta_y)/D_{\text{min}}(b\Delta_y) \cong \text{span}\{1, x\}$.
Suppose
\[ \text{spec}_e(A) \cap \{(y, \sigma) : \Re \sigma \neq \pm m/2\} \]
Then \( \mathcal{T} = \bigcup_{y \in Y} \mathcal{T}_y \) is a smooth vector bundle over \( Y \), and
\[ C^\infty(Y; \mathcal{T}) = \{ \text{all sections } u : Y \to \mathcal{T} \text{ which viewed as functions on } (0, \infty) \times Y \times Z \text{ are smooth} \}. \]
Suppose
\[ \text{spec}_e(A) \cap \{(y, \sigma) : \Im \sigma \neq \pm m/2 \} \]
Then \( \mathcal{T} = \bigcup_{y \in \mathcal{Y}} \mathcal{T}_y \) is a smooth
vector bundle over \( \mathcal{Y} \), and
\[ C^\infty(\mathcal{Y}; \mathcal{T}) = \{ \text{all sections} \ u : \mathcal{Y} \to \mathcal{T} \text{ which viewed as functions} \]
on \((0, \infty) \times \mathcal{Y} \times \mathcal{Z} \text{ are smooth} \}. \]
\[ \mathcal{T}_y = \ker A_y \text{ on} \]
\[ \left\{ \sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell=0}^{N_{\sigma}} a_{\sigma, \ell} x^{i\sigma} \log^\ell x \right\} \]
\(-m/2 < \Im \sigma < m/2\)
Suppose \( \text{spec}_e(A) \cap \{(y, \sigma) : \Im \sigma \neq \pm m/2\} \)

Then \( \mathcal{T} = \bigcup_{y \in \mathcal{Y}} \mathcal{T}_y \) is a smooth vector bundle over \( \mathcal{Y} \), and

\[
\mathcal{C}^\infty_b^{\infty}(\mathcal{Y}) = \{\text{all sections} \ u : \mathcal{Y} \to \mathcal{T} \text{ which viewed as functions on } (0, \infty) \times \mathcal{Y} \times \mathbb{Z} \text{ are smooth}\}.
\]

\( \mathcal{T}_y = \ker A_y \) on

\[
\{ \sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell=0}^{N_\sigma} a_{\sigma, \ell} x^{i\sigma} \log^\ell x \}
\]

such that for any \( \phi \in \mathcal{B}_b^{\infty}(U) \) there are unique elements \( f^k \in C^\infty(U) \)

with which

\[
\phi = \sum f^k \phi_k.
\]
Suppose
\[ \text{spec}_e(A) \cap \{(y, \sigma) : \Im \sigma \neq \pm m/2\} \]
Then \( \mathcal{T} = \bigcup_{y \in \mathcal{Y}} \mathcal{T}_y \) is a smooth vector bundle over \( \mathcal{Y} \), and
\[ C^\infty_B(\mathcal{Y}) = \{ \text{all sections} \ u : \mathcal{Y} \to \mathcal{T} \text{ which viewed as functions} \}
\[ \text{on } (0, \infty) \times \mathcal{Y} \times \mathcal{Z} \text{ are smooth} \}. \]
\[ \mathcal{T}_y = \ker A_y \text{ on } \]
\[ \left\{ \sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell=0}^{N_\sigma} a_{\sigma, \ell} x^{i\sigma} \log^\ell x \right\} \]
\[ -m/2 < \Im \sigma < m/2 \]

The proof consists of showing that every \( y_0 \in \mathcal{Y} \) has a neighborhood \( U \subset \mathcal{Y} \) for which there are elements
\[ \phi_k \in B^\infty_B(U) \quad k = 1, \ldots, d \]
such that for any \( \phi \in B^\infty_B(U) \) there are unique elements \( f^k \in C^\infty(U) \) with which
\[ \phi = \sum f^k \phi_k. \]
Declaring the \( \phi_k \) to be a frame over \( U \) gives the smooth vector bundle structure:
The proof consists of showing that every \( y_0 \in \mathcal{Y} \) has a neighborhood \( U \subset \mathcal{Y} \) for which there are elements

\[ \phi_k \in \mathcal{B}_b^\infty(A)(U) \quad k = 1, \ldots, d \]

such that for any \( \phi \in \mathcal{B}_b^\infty(A)(U) \) there are unique elements \( f^k \in C^\infty(U) \) with which

\[ \phi = \sum f^k \phi_k. \]

Declaring the \( \phi_k \) to be a frame over \( U \) gives the smooth vector bundle structure:

If \( \psi_\ell \in \mathcal{B}_b^\infty(U) \) is another such choice, then

\[ \phi_k = \sum g^\ell_k \psi_\ell \]
Suppose \( \text{spec}_e(A) \cap \{(y, \sigma) : \Re \sigma \neq \pm m/2 \} \)

Then \( \mathcal{T} = \bigcup_{y \in \mathcal{Y}} \mathcal{T}_y \) is a smooth vector bundle over \( \mathcal{Y} \), and

\[
\mathcal{C}^\infty_b(A) = \{ \text{all sections} \ u : \mathcal{Y} \to \mathcal{T} \text{ which viewed as functions on } (0, \infty) \times \mathcal{Y} \times \mathcal{Z} \text{ are smooth} \}.
\]

\[\mathcal{T}_y = \ker A_y \]
on

\[
\{ \sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell=0}^{N_{\sigma}} a_{\sigma, \ell} x^{i\sigma} \log^{\ell} x \}
\]

The proof consists of showing that every \( y_0 \in \mathcal{Y} \) has a neighborhood \( U \subset \mathcal{Y} \) for which there are elements \( \phi_k \in \mathcal{B}_b^\infty(U) \)

such that for any \( \phi \in \mathcal{B}_b^\infty(U) \) there are unique elements \( f^k \in C^\infty(U) \)

with which

\[
\phi = \sum f^k \phi_k.
\]

Declaring the \( \phi_k \) to be a frame over \( U \) gives the smooth vector bundle structure:

If \( \psi_\ell \in \mathcal{B}_b^\infty(U) \) is another such choice, then

\[
\phi_k = \sum g_k^\ell \psi_\ell
\]

\[
\psi_\ell = \sum f_\ell^m \phi_m
\]
The proof consists of showing that every $y_0 \in \mathcal{Y}$ has a neighborhood $U \subset \mathcal{Y}$ for which there are elements
\[ \phi_k \in B^\infty_{bA}(U) \quad k = 1, \ldots, d \]
such that for any $\phi \in B^\infty_{bA}(U)$ there are unique elements $f^k \in C^\infty(U)$ with which
\[ \phi = \sum f^k \phi_k. \]
Declaring the $\phi_k$ to be a frame over $U$ gives the smooth vector bundle structure:

If $\psi_\ell \in B^\infty_{bA}(U)$ is another such choice, then
\[ \phi_k = \sum g^\ell_k \psi_\ell = \sum g^\ell_k f^m \phi_m \]
\[ \psi_\ell = \sum f^m_\ell \phi_m \]

Suppose
\[ \text{spec}_e(A) \cap \{(y, \sigma) : \Im \sigma \neq \pm m/2\} \]
Then $\mathcal{I} = \bigcup_{y \in \mathcal{Y}} \mathcal{T}_y$ is a smooth vector bundle over $\mathcal{Y}$, and
\[ C^\infty B^\infty_{bA}(\mathcal{Y}) = \{\text{all sections} \quad u : \mathcal{Y} \to \mathcal{I} \quad \text{which viewed as functions} \quad \text{on } (0, \infty) \times \mathcal{Y} \times \mathcal{Z} \quad \text{are smooth}\}. \]
\[ \mathcal{T}_y = \ker A_y \quad \text{on} \]
\[ \{ \sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell=0}^{N_\sigma} a_{\sigma, \ell} x^{i_\sigma} \log^\ell x \} \]
\[ -m/2 < \Im \sigma < m/2 \]
The proof consists of showing that every \( y_0 \in \mathcal{Y} \) has a neighborhood \( U \subset \mathcal{Y} \) for which there are elements
\[
\phi_k \in \mathcal{B}_{bA}^\infty(U) \quad k = 1, \ldots, d
\]
such that for any \( \phi \in \mathcal{B}_{bA}^\infty(U) \) there are unique elements \( f^k \in C^\infty(U) \) with which
\[
\phi = \sum f^k \phi_k.
\]
Declaring the \( \phi_k \) to be a frame over \( U \) gives the smooth vector bundle structure:

If \( \psi_\ell \in \mathcal{B}_{bA}^\infty(U) \) is another such choice, then
\[
\phi_k = \sum g_k^\ell \psi_\ell = \sum g_k^\ell f^m \phi_m \quad :. \quad \sum g_k^\ell f^m = \delta_k^m.
\]
\[
\psi_\ell = \sum f_\ell^m \phi_m
\]
\[ \text{spec}_e(A) \cap \{(y, \sigma) : \Im \sigma \neq \pm m/2\}, \]
\[ \mathcal{I} = \bigcup_{y \in \mathcal{Y}} \mathcal{I}_y, \mathcal{I}_y = \ker A_y \text{ on } N_{\sigma} \]
\[ \{ \sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell=0}^{N_{\sigma}} a_{\sigma,\ell} x^{i \sigma} \log^\ell x \} \]
Suppose \( \phi_k \in B_{bA}^\infty(U), \ k = 1, \ldots, d, \) are such that \( \phi_k(y) \) is a basis of \( T_y \) for each \( y \). For each \( \phi \in B_{bA}^\infty(U) \) there are \( f^k \in C^\infty(U) \) such that

\[
\phi = \sum f^k \phi_k?
\]
Suppose $\phi_k \in \mathcal{B}_{bA}^\infty(U)$, $k = 1, \ldots, d$, are such that $\phi_k(y)$ is a basis of $\mathcal{T}_y$ for each $y$.

For each $\phi \in \mathcal{B}_{bA}^\infty(U)$ there are $f^k \in C^\infty(U)$ such that

$$\phi = \sum f^k \phi_k.$$  

Let $\psi^\ell \in \mathcal{B}_{bA^*}^\infty(U)$, $\ell = 1, \ldots, d$ have the same property, for $bA^*$

$$\mathcal{B}_{bA}^\infty(U) = \{\text{all sections } u : \mathcal{Y} \to \mathcal{T} \text{ which viewed as functions on } (0, \infty) \times \mathcal{Y} \times \mathcal{Z} \text{ are smooth}\}.$$  

$$\mathcal{B}_{bA^*}^\infty(U)$$ is the formal adjoint of $bA_y$.  

$$\text{spec}_e(A) \cap \{(y, \sigma) : \Im \sigma \neq \pm m/2\},$$  

$$\mathcal{T} = \bigcup_{y \in \mathcal{Y}} \mathcal{T}_y, \mathcal{T}_y = \ker A_y \text{ on } \{\sum_{\sigma \in \text{spec}_b(A_y)} \sum_{\ell=0}^{-m/2 < \Im \sigma < m/2} a_{\sigma,\ell} x^{i\sigma} \log^\ell x\}.$$  

$$\text{B}_{bA}^\infty(U) = \{\text{all sections } u : \mathcal{Y} \to \mathcal{T} \text{ which viewed as functions on } (0, \infty) \times \mathcal{Y} \times \mathcal{Z} \text{ are smooth}\}.\]
Suppose \( \phi_k \in \mathcal{B}_{bA}^\infty(U), \; k = 1, \ldots, d, \) are such that \( \phi_k(y) \) is a basis of \( \mathcal{T}_y \) for each \( y \).

For each \( \phi \in \mathcal{B}_{bA}^\infty(U) \) there are \( f^k \in C^\infty(U) \) such that

\[
\phi = \sum f^k \phi_k.
\]

Let \( \psi^\ell \in \mathcal{B}_{bA^*}^\infty(U), \; \ell = 1, \ldots, d \) have the same property, for \( bA^* \)

Then

\[
\alpha^\ell_k(y) = [\omega \phi_k(y), \omega \psi^\ell(y)]_{bA_y} = (bA_y(\omega \phi_k), \psi^\ell) - (\phi_k, bA^*_y(\omega \psi_k))
\]

depends smoothly on \( y \).
Suppose \( \phi_k \in \mathcal{B}_{bA}^\infty(U), \ k = 1, \ldots, d \), are such that \( \phi_k(y) \) is a basis of \( \mathcal{T}_y \) for each \( y \). For each \( \phi \in \mathcal{B}_{bA}^\infty(U) \) there are \( f^k \in C^\infty(U) \) such that

\[
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\]

Let \( \psi^\ell \in \mathcal{B}_{bA^*}^\infty(U), \ \ell = 1, \ldots, d \) have the same property, for \( bA^* \)

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\alpha_k^\ell(y) = [\omega \phi_k(y), \omega \psi^\ell(y)]_{bA_y} = (bA_y(\omega \phi_k), \psi^\ell) - (\phi_k, bA^*_y(\omega \psi_k))
\]

depends smoothly on \( y \). The pairing \( [\cdot, \cdot]_{bA_y} \) is nonsingular, so \( [\alpha_k^\ell(y)] \) is invertible for each \( y \).
Suppose \( \phi_k \in \mathcal{B}_b^{\infty}(U) \), \( k = 1, \ldots, d \), are such that \( \phi_k(y) \) is a basis of \( \mathcal{T}_y \) for each \( y \). For each \( \phi \in \mathcal{B}_b^{\infty}(U) \) there are \( f^k \in C^\infty(U) \) such that
\[
\phi = \sum f^k \phi_k?
\]
Let \( \psi^\ell \in \mathcal{B}_b^{\infty}(U) \), \( \ell = 1, \ldots, d \) have the same property, for \( bA^* \)

Then
\[
\alpha_k^\ell(y) = [\omega \phi_k(y), \omega \psi^\ell(y)]_{bA_y} = (bA_y(\omega \phi_k), \psi^\ell) - (\phi_k, bA^*_y(\omega \psi_k))
\]
depends smoothly on \( y \). The pairing \([\cdot, \cdot]\) \( bA_y \) is nonsingular, so \( [\alpha_k^\ell(y)] \) is invertible for each \( y \). Also
\[
h^\ell(y) = [\omega \phi(y), \omega \psi^\ell(y)]_{bA_y}
\]
is smooth.
Suppose $\phi_k \in \mathcal{B}_{bA}(U), k = 1, \ldots, d$, are such that $\phi_k(y)$ is a basis of $T_y$ for each $y$. For each $\phi \in \mathcal{B}_{bA}(U)$ there are $f^k \in C^\infty(U)$ such that

$$\phi = \sum f^k \phi_k?$$

Let $\psi^\ell \in \mathcal{B}_{bA^*}(U), \ell = 1, \ldots, d$ have the same property, for $bA^*$

Then

$$\alpha^\ell_k(y) = [\omega \phi_k(y), \omega \psi^\ell(y)]_{bA_y} = (bA_y(\omega \phi_k), \psi^\ell) - (\phi_k, bA^*_y(\omega \psi_k))$$

depends smoothly on $y$. The pairing $[\cdot, \cdot]_{bA_y}$ is nonsingular, so $[\alpha^\ell_k(y)]$ is invertible for each $y$. Also

$$h^\ell(y) = [\omega \phi(y), \omega \psi^\ell(y)]_{bA_y} = [\sum f^k(y) \omega \phi_k(y), \omega \psi^\ell(y)]_{bA_y}$$

is smooth.
Suppose $\phi_k \in B^\infty_{bA}(U), k = 1, \ldots, d$, are such that $\phi_k(y)$ is a basis of $T_y$ for each $y$. For each $\phi \in B^\infty_{bA}(U)$ there are $f^k \in C^\infty(U)$ such that

$$\phi = \sum f^k \phi_k?$$

Let $\psi^\ell \in B^\infty_{bA^*}(U), \ell = 1, \ldots, d$ have the same property, for $bA^*$.

Then

$$\alpha^\ell_k(y) = [\omega \phi_k(y), \omega \psi^\ell(y)]_{bA_y} = (bA_y(\omega \phi_k), \psi^\ell) - (\phi_k, bA^*_y(\omega \psi_k))$$

depends smoothly on $y$. The pairing $[\cdot, \cdot]_{bA_y}$ is nonsingular, so $[\alpha^\ell_k(y)]$ is invertible for each $y$. Also

$$h^\ell(y) = [\omega \phi(y), \omega \psi^\ell(y)]_{bA_y} = [\sum f^k(y)\omega \phi_k(y), \omega \psi^\ell(y)]_{bA_y} = \sum f^k(y)[\omega \phi_k(y), \omega \psi^\ell(y)]_{bA_y}$$

is smooth.
Suppose \( \phi_k \in \mathcal{B}_{bA}^\infty(U) \), \( k = 1, \ldots, d \), are such that \( \phi_k(y) \) is a basis of \( \mathcal{T}_y \) for each \( y \). For each \( \phi \in \mathcal{B}_{bA}^\infty(U) \) there are \( f^k \in C^\infty(U) \) such that
\[
\phi = \sum f^k \phi_k \, .
\]
Let \( \psi^\ell \in \mathcal{B}_{bA^*}^\infty(U) \), \( \ell = 1, \ldots, d \) have the same property, for \( bA^* \)

Then
\[
\alpha_k^\ell(y) = [\omega \phi_k(y), \omega \psi^\ell(y)]_{bA_y} = (bA_y(\omega \phi_k), \psi^\ell) - (\phi_k, bA^*_y(\omega \psi_k))
\]
depends smoothly on \( y \). The pairing \([\cdot, \cdot]_{bA_y}\) is nonsingular, so \([\alpha_k^\ell(y)]\) is invertible for each \( y \). Also
\[
h^\ell(y) = [\omega \phi(y), \omega \psi^\ell(y)]_{bA_y} = \left[ \sum f^k(y) \omega \phi_k(y), \omega \psi^\ell(y) \right]_{bA_y} = \sum f^k(y) \left[ \omega \phi_k(y), \omega \psi^\ell(y) \right]_{bA_y}
\]
is smooth. So \( f^k(y) \) is smooth.

\[
\mathcal{B}_{bA}^\infty(U) = \{ \text{all sections } u : Y \to \mathcal{T} \text{ which viewed as functions on } (0, \infty) \times Y \times Z \text{ are smooth} \}.
\]

\( bA^*_y \) is the formal adjoint of \( bA_y \).

\( \omega \in C_c^\infty(\mathbb{R}) \), \( \omega(x) = 1 \) near 0

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Suppose $\phi_k \in B_b^\infty(A)(U)$, $k = 1, \ldots, d$, are such that $\phi_k(y)$ is a basis of $T_y$ for each $y$. For each $\phi \in B_b^\infty(A)(U)$ there are $f^k \in C^\infty(U)$ such that

$$\phi = \sum f^k \phi_k.$$ 

Let $\psi^\ell \in B_b^\infty(A^*)(U)$, $\ell = 1, \ldots, d$ have the same property, for $bA^*$.

Then

$$\alpha^\ell_k(y) = [\omega\phi_k(y), \omega\psi^\ell(y)]_{bA_y} = (bA_y(\omega\phi_k), \psi^\ell) - (\phi_k, bA^*_y(\omega\psi_k))$$

depends smoothly on $y$. The pairing $[\cdot, \cdot]_{bA_y}$ is nonsingular, so $[\alpha^\ell_k(y)]$ is invertible for each $y$. Also

$$h^\ell(y) = [\omega\phi(y), \omega\psi^\ell(y)]_{bA_y} = \sum f^k(y) [\omega\phi_k(y), \omega\psi^\ell(y)]_{bA_y} = \sum f^k(y) \alpha^\ell_k(y)$$

is smooth. So $f^k(y)$ is smooth.

$$= \sum f^k(y) \alpha^\ell_k(y)$$
How to get the $\phi_k \cdots$
How to get the $\phi_k \cdots$

For open $\Omega \subset \mathbb{C}$, let $\mathcal{H}ol(\Omega)$ and $\mathcal{Mero}(\Omega)$ be the spaces of $C^\infty(\mathbb{Z})$-valued holomorphic or meromorphic functions on $\Omega$. 

\[
bA_y = \frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)(xD_x)^k D_z^\beta
\]

\[
b\hat{P}_y(\sigma) = \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)\sigma^k D_z^\beta
\]
How to get the $\phi_k \cdots$

For open $\Omega \subset \mathbb{C}$, let $\mathfrak{Hol}(\Omega)$ and $\mathfrak{Mero}(\Omega)$ be the spaces of $C^\infty(\mathbb{Z})$-valued holomorphic or meromorphic functions on $\Omega$. Fix $y_0$, let

$$\sigma_s, s = 1, \ldots, S_{y_0}$$

be the points in $\text{spec}_b(bA_{y_0}) \cap \Sigma$. Pick pairwise disjoint discs $D_s \subset \Sigma$ centered at the $\sigma_s$. 

$$bA_y = \frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)(xD_x)^k D_z^\beta$$

$$b\hat{P}_y(\sigma) = \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)\sigma^k D_z^\beta$$

$$\Sigma = \{ \sigma \in \mathbb{C} : -m/2 < \Im \sigma < m/2 \}$$
How to get the $\phi_k \cdots$

For open $\Omega \subset \mathbb{C}$, let $\mathcal{H}ol(\Omega)$ and $\mathcal{M}er\mathcal{O}(\Omega)$ be the spaces of $C^\infty(\mathbb{Z})$-valued holomorphic or meromorphic functions on $\Omega$. Fix $y_0$, let

$$\sigma_s, \ s = 1, \ldots, S_{y_0}$$

be the points in $\text{spec}_b(bA_{y_0}) \cap \Sigma$. Pick pairwise disjoint discs $D_s \subset \Sigma$ centered at the $\sigma_s$. View

$$(\dagger) \quad b\hat{P}_y(\sigma) : \mathcal{M}er\mathcal{O}(D_s)/\mathcal{H}ol(D_s) \to \mathcal{M}er\mathcal{O}(D_s)/\mathcal{H}ol(D_s).$$

$$bA_y = \frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)(xD_x)^{k} D_z^\beta$$

$$b\hat{P}_y(\sigma) = \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0,y,z)\sigma^k D_z^\beta$$

$$\Sigma = \{ \sigma \in \mathbb{C} : -m/2 < \Im \sigma < m/2 \}$$
How to get the $\phi_k \cdots$

For open $\Omega \subset \mathbb{C}$, let $\mathcal{H}\mathcal{O}\mathcal{L}(\Omega)$ and $\mathcal{M}\mathcal{E}\mathcal{R}\mathcal{O}(\Omega)$ be the spaces of $C^\infty(\mathbb{Z})$-valued holomorphic or meromorphic functions on $\Omega$. Fix $y_0$, let

$$\sigma_s, s = 1, \ldots, S_{y_0}$$

be the points in $\text{spec}_b(\,^bA_{y_0}) \cap \Sigma$. Pick pairwise disjoint discs $D_s \subset \Sigma$ centered at the $\sigma_s$. View

$$(\dagger) \quad \hat{b}\mathcal{P}_y(\sigma) : \mathcal{M}\mathcal{E}\mathcal{R}\mathcal{O}(D_s)/\mathcal{H}\mathcal{O}\mathcal{L}(D_s) \to \mathcal{M}\mathcal{E}\mathcal{R}\mathcal{O}(D_s)/\mathcal{H}\mathcal{O}\mathcal{L}(D_s).$$

The kernel of $(\dagger)$ when $y = y_0$ consists of classes generated by elements
How to get the φk · · ·

For open Ω ⊂ C, let Ξol(Ω) and Mero(Ω) be the spaces of $\mathcal{C}^\infty(\mathcal{Z})$-valued holomorphic or meromorphic functions on Ω. Fix $y_0$, let

$$\sigma_s, s = 1, \ldots, S_{y_0}$$

be the points in spec$_b$($^bA_{y_0}$) ∩ Σ. Pick pairwise disjoint discs $D_s \subset \Sigma$ centered at the $\sigma_s$. View

$$(\dagger) \quad ^b\hat{P}_y(\sigma) : \text{Mero}(D_s)/\text{Hol}(D_s) \rightarrow \text{Mero}(D_s)/\text{Hol}(D_s).$$

The kernel of (\dagger) when $y = y_0$ consists of classes generated by elements

$$\hat{\phi}_{s,j,\ell}(y_0) = (\sigma - \sigma_s)^\ell \hat{\phi}_{s,j,0}(y_0), \quad \hat{\phi}_{s,j,0}(y_0) = \sum_{\nu=0}^{L_s,j-1} a_{s,j,\nu}(y_0) \frac{(\sigma - \sigma_s)^\nu}{(\sigma - \sigma_s)^\nu}, \quad a_{s,j,\nu} \in \mathcal{C}^\infty(\mathcal{Z})$$
How to get the $\phi_k \cdots$

For open $\Omega \subset \mathbb{C}$, let $\mathcal{H}\mathcal{O}l(\Omega)$ and $\mathcal{M}ero(\Omega)$ be the spaces of $C^\infty(\mathcal{Z})$-valued holomorphic or meromorphic functions on $\Omega$. Fix $y_0$, let

$$\sigma_s, s = 1, \ldots, S_{y_0}$$

be the points in $\text{spec}_b(bA_{y_0}) \cap \Sigma$. Pick pairwise disjoint discs $D_s \subset \Sigma$ centered at the $\sigma_s$. View

$$(\dagger) \quad b\hat{P}_y(\sigma) : \mathcal{M}ero(D_s)/\mathcal{H}\mathcal{O}l(D_s) \to \mathcal{M}ero(D_s)/\mathcal{H}\mathcal{O}l(D_s).$$

The kernel of (\dagger) when $y = y_0$ consists of classes generated by elements

$$\hat{\phi}_{s,j,\ell}(y_0) = (\sigma - \sigma_s)\ell \hat{\phi}_{s,j,0}(y_0), \quad \hat{\phi}_{s,j,0}(y_0) = \sum_{\nu=0}^{L_s,j-1} \frac{a_{s,j,\nu}(y_0)}{(\sigma - \sigma_s)^\nu}, \quad a_{s,j,\nu} \in C^\infty(\mathcal{Z})$$

In a small neighborhood $U$ of $y_0$ there are elements $\hat{\phi}_{s,j,\ell}(y) \in \mathcal{M}ero(D_s)$ whose classes span the kernel of (\dagger) and are smooth in $U \times D_s$ off of $(U \times D_s) \cap \text{spec}_e(A)$. 

$$bA_y = \frac{1}{x^m} \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0, y, z)(xD_x)^k D_z^\beta$$

$$b\hat{P}_y(\sigma) = \sum_{k+|\beta| \leq m} a_{k,0,\beta}(0, y, z)\sigma^k D_z^\beta$$

$$\Sigma = \{ \sigma \in \mathbb{C} : -m/2 < \Im \sigma < m/2 \}$$
In a small neighborhood $U$ of $y_0$ there are elements $\hat{\phi}_{s,j,\ell}(y) \in \text{Mero}(D_s)$ whose classes span the kernel of

$$b\hat{P}_y(\sigma) : \text{Mero}(D_s)/\text{Hol}(D_s) \rightarrow \text{Mero}(D_s)/\text{Hol}(D_s).$$

and are smooth in $U \times D_s$ off of $(U \times D_s) \cap \text{spec}_e(A)$. 

Computes the singular part of $\phi$. Now let $\hat{\phi}_{s,j,\ell}(y) = \frac{1}{2\pi i} \int_{\partial D_s} \hat{\phi}_{s,j,\ell}(y,\sigma) d\sigma$ These are the $\phi_k$. 

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In a small neighborhood $U$ of $y_0$ there are elements $\hat{\phi}_{s,j,\ell}(y) \in \text{Mero}(D_s)$ whose classes span the kernel of

$$b^\mathcal{P}_y(\sigma) : \text{Mero}(D_s)/\text{Hol}(D_s) \to \text{Mero}(D_s)/\text{Hol}(D_s).$$

and are smooth in $U \times D_s$ off of $(U \times D_s) \cap \text{spec}_e(A)$. Let

$$\hat{\phi}_{s,j,\ell}^s(y) = s_{D_s}(\hat{\phi}_{s,j,\ell}(y))$$

where

$$s_{D_s}(\hat{\phi}) = \frac{i}{2\pi} \int_{|\zeta - \sigma| = \epsilon_s} \frac{\hat{\phi}(\zeta)}{\zeta - \sigma} d\zeta, \quad \sigma \notin D_s$$
In a small neighborhood $U$ of $y_0$ there are elements $\hat{\phi}_{s,j,\ell}(y) \in \text{Mero}(D_s)$ whose classes span the kernel of

$$b\hat{P}_y(\sigma) : \text{Mero}(D_s)/\text{Hol}(D_s) \to \text{Mero}(D_s)/\text{Hol}(D_s).$$

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$$\hat{\phi}_{s,j,\ell}^s(y) = s_{D_s}(\hat{\phi}_{s,j,\ell}(y))$$

where

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Computes the singular part of $\phi$.
In a small neighborhood $U$ of $y_0$ there are elements $\hat{\phi}_{s,j,\ell}(y) \in \text{Mero}(D_s)$ whose classes span the kernel of
\[
b\hat{P}_y(\sigma) : \text{Mero}(D_s)/\text{Hol}(D_s) \to \text{Mero}(D_s)/\text{Hol}(D_s).\]

and are smooth in $U \times D_s$ off of $(U \times D_s) \cap \text{spec}_e(A)$. Let
\[
\hat{\phi}^5_{s,j,\ell}(y) = \mathcal{s}_{D_s}(\hat{\phi}_{s,j,\ell}(y))
\]
where
\[
\mathcal{s}_{D_s}(\hat{\phi}) = \frac{i}{2\pi} \oint_{|\zeta - \sigma| = \epsilon_s} \frac{\hat{\phi}(\zeta)}{\zeta - \sigma} \, d\zeta, \quad \sigma \notin D_s
\]

Now let
\[
\phi^5_{s,j,\ell} = \frac{1}{2\pi} \int_{\partial D_s} \chi^i\sigma \hat{\phi}^5_{s,j,\ell}(y, \sigma) \, d\sigma
\]

These are the $\phi_k$. 
The next ingredient involves the normal family $A_{\wedge,\eta}$. This has its own minimal and maximal extensions:

$$D_{\wedge,\text{min}}(\eta), \quad D_{\wedge,\text{max}}(\eta)$$
The next ingredient involves the normal family $A_{\wedge,\eta}$. This has its own minimal and maximal extensions:

$$D_{\wedge,\text{min}}(\eta), \quad D_{\wedge,\text{max}}(\eta)$$

$$x^m A = \sum a_{k,\alpha,\beta}(x, y, z)(xD_x)^k(xD_y)^\alpha D_z^\beta$$

$$x^m A_{\wedge,\eta} = \sum a_{k,\alpha,\beta}(0, y, z)(xD_x)^k(x\eta)^\alpha D_z^\beta$$

Our standing assumption

$$\text{spec}_e(A) \cap (\mathcal{Y} \times \{\Re \sigma = \pm m/2\}) = \emptyset$$

implies $D_{\wedge,\text{min}}(\eta)$ is independent of $\eta$
The next ingredient involves the normal family $A_{\wedge, \eta}$. This has its own minimal and maximal extensions:

$$D_{\wedge, \text{min}}(\eta), \ D_{\wedge, \text{max}}(\eta)$$

Condition on conormal family:

$$x^mA = \sum a_{k, \alpha, \beta}(x, y, z)(xD_x)^k(xD_y)^\alpha D_z^\beta$$

$$x^mA_{\wedge, \eta} = \sum a_{k, \alpha, \beta}(0, y, z)(xD_x)^k(x\eta)^\alpha D_z^\beta$$

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The next ingredient involves the normal family $A_{\wedge,\eta}$. This has its own minimal and maximal extensions:

$$D_{\wedge,\min}(\eta), \quad D_{\wedge,\max}(\eta)$$

Condition on conormal family:

For $\eta \neq 0$: $A_{\wedge,\eta}$ is injective on $D_{\wedge,\min}(\eta)$ and surjective on $D_{\wedge,\max}(\eta)$
The next ingredient involves the normal family $A_{\land, \eta}$. This has its own minimal and maximal extensions:

$$D_{\land, \text{min}}(\eta), \quad D_{\land, \text{max}}(\eta)$$

Condition on conormal family:

For $\eta \neq 0$: $A_{\land, \eta}$ is injective on $D_{\land, \text{min}}(\eta)$ and surjective on $D_{\land, \text{max}}(\eta)$

Consequences: there are canonical operators

$$B_{\land, \text{min}}(\eta) : x^{-m/2}L_b^2 \to x^{-m/2}L_b^2, \quad B_{\land, \text{min}}(\eta)A_{\land}(\eta) = I \text{ on } D_{\land, \text{min}}(\eta)$$

$$B_{\land, \text{max}}(\eta) : x^{-m/2}L_b^2 \to x^{-m/2}L_b^2, \quad A_{\land}(\eta)B_{\land, \text{max}}(\eta) = I \text{ on } x^{-m/2}L_b^2.$$
The next ingredient involves the normal family $A_{\wedge,\eta}$. This has its own minimal and maximal extensions:

$$D_{\wedge,\text{min}}(\eta), \quad D_{\wedge,\text{max}}(\eta)$$

Condition on conormal family:

For $\eta \neq 0$: $A_{\wedge,\eta}$ is injective on $D_{\wedge,\text{min}}(\eta)$ and surjective on $D_{\wedge,\text{max}}(\eta)$

Consequences: there are canonical operators

$$B_{\wedge,\text{min}}(\eta): x^{-m/2}L^2_{b} \to x^{-m/2}L^2_{b}, \quad B_{\wedge,\text{min}}(\eta)A_{\wedge}(\eta) = I \text{ on } D_{\wedge,\text{min}}(\eta)$$

$$B_{\wedge,\text{max}}(\eta): x^{-m/2}L^2_{b} \to x^{-m/2}L^2_{b}, \quad A_{\wedge}(\eta)B_{\wedge,\text{max}}(\eta) = I \text{ on } x^{-m/2}L^2_{b}.$$
The next ingredient involves the normal family \( A_{\wedge, \eta} \). This has its own minimal and maximal extensions:

\[
\mathcal{D}_{\wedge, \min}(\eta), \quad \mathcal{D}_{\wedge, \max}(\eta)
\]

Condition on conormal family:

For \( \eta \neq 0 \): \( A_{\wedge, \eta} \) is injective on \( \mathcal{D}_{\wedge, \min}(\eta) \) and surjective on \( \mathcal{D}_{\wedge, \max}(\eta) \)

Consequences: there are canonical operators

\[
B_{\wedge, \min}(\eta) : x^{-m/2}L^2_b \to x^{-m/2}L^2_b, \quad B_{\wedge, \min}(\eta)A_{\wedge}(\eta) = I \text{ on } \mathcal{D}_{\wedge, \min}(\eta)
\]

\[
B_{\wedge, \max}(\eta) : x^{-m/2}L^2_b \to x^{-m/2}L^2_b, \quad A_{\wedge}(\eta)B_{\wedge, \max}(\eta) = I \text{ on } x^{-m/2}L^2_b.
\]

As maps into \( L^2 \) these are smooth in \( \eta \)

By an iterative process one extends the \( \phi_k \) (call them \( \phi_{k,0} \) now) as

\[
\sum_{\vartheta=0}^{N_k} \phi_{k,\vartheta}
\]

\[
x^m A = \sum a_{k,\alpha,\beta}(x, y, z)(xD_x)^k(xD_y)\alpha D_z^\beta
\]

\[
x^m A_{\wedge, \eta} = \sum a_{k,\alpha,\beta}(0, y, z)(xD_x)^k(x\eta)\alpha D_z^\beta
\]

Our standing assumption

\[
\text{spec}_e(A) \cap (\mathcal{Y} \times \{ \Im \sigma = \pm m/2 \}) = \emptyset
\]

implies \( \mathcal{D}_{\wedge, \min}(\eta) \) is independent of \( \eta \).
The next ingredient involves the normal family $A_{\wedge, \eta}$. This has its own minimal and maximal extensions:

$$D_{\wedge, \text{min}}(\eta), \quad D_{\wedge, \text{max}}(\eta)$$

Condition on conormal family:

For $\eta \neq 0$: $A_{\wedge, \eta}$ is injective on $D_{\wedge, \text{min}}(\eta)$ and surjective on $D_{\wedge, \text{max}}(\eta)$

Consequences: there are canonical operators

$$B_{\wedge, \text{min}}(\eta) : x^{-m/2} L^2_b \to x^{-m/2} L^2_b, \quad B_{\wedge, \text{min}}(\eta) A_{\wedge}(\eta) = I \text{ on } D_{\wedge, \text{min}}(\eta)$$

$$B_{\wedge, \text{max}}(\eta) : x^{-m/2} L^2_b \to x^{-m/2} L^2_b, \quad A_{\wedge}(\eta) B_{\wedge, \text{max}}(\eta) = I \text{ on } x^{-m/2} L^2_b.$$ 

As maps into $L^2$ these are smooth in $\eta$

By an iterative process one extends the $\phi_k$ (call them $\phi_{k,0}$ now) as

$$\sum_{\theta = 0}^{N_k} \phi_{k, \theta} \in D_{\wedge, \text{max}}(\eta)$$
The next ingredient involves the normal family $A_{\wedge, \eta}$. This has its own minimal and maximal extensions:

$$D_{\wedge, \text{min}}(\eta), \quad D_{\wedge, \text{max}}(\eta)$$

Condition on conormal family:

For $\eta \neq 0$: $A_{\wedge, \eta}$ is injective on $D_{\wedge, \text{min}}(\eta)$ and surjective on $D_{\wedge, \text{max}}(\eta)$

Consequences: there are canonical operators

$$B_{\wedge, \text{min}}(\eta) : x^{-m/2}L^2_b \to x^{-m/2}L^2_b, \quad B_{\wedge, \text{min}}(\eta)A_{\wedge}(\eta) = I \text{ on } D_{\wedge, \text{min}}(\eta)$$

$$B_{\wedge, \text{max}}(\eta) : x^{-m/2}L^2_b \to x^{-m/2}L^2_b, \quad A_{\wedge}(\eta)B_{\wedge, \text{max}}(\eta) = I \text{ on } x^{-m/2}L^2_b.$$ 

As maps into $L^2$ these are smooth in $\eta$

By an iterative process one extends the $\phi_k$ (call them $\phi_{k,0}$ now) as

$$\phi_k(\eta) = \sum_{\vartheta = 0}^{N_k} \phi_{k, \vartheta} \in D_{\wedge, \text{max}}(\eta)$$
The next ingredient involves the normal family $A_{\wedge, \eta}$. This has its own minimal and maximal extensions:

$$D_{\wedge, \text{min}}(\eta), \quad D_{\wedge, \text{max}}(\eta)$$

Condition on conormal family:

For $\eta \neq 0$: $A_{\wedge, \eta}$ is injective on $D_{\wedge, \text{min}}(\eta)$ and surjective on $D_{\wedge, \text{max}}(\eta)$

Consequences: there are canonical operators

$$B_{\wedge, \text{min}}(\eta) : x^{-m/2}L^2_b \rightarrow x^{-m/2}L^2_b, \quad B_{\wedge, \text{min}}(\eta)A_{\wedge}(\eta) = I \text{ on } D_{\wedge, \text{min}}(\eta)$$

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By an iterative process one extends the $\phi_k$ (call them $\phi_{k,0}$ now) as

$$\phi_k(\eta) = \sum_{\vartheta=0}^{N_k} \phi_{k,\vartheta} \in D_{\wedge, \text{max}}(\eta)$$

$$x^mA = \sum a_{k,\alpha,\beta}(x, y, z)(xD_x)^k(xD_y)^\alpha D_z^\beta$$

$$x^mA_{\wedge, \eta} = \sum a_{k,\alpha,\beta}(0, y, z)(xD_x)^k(x\eta)^\alpha D_z^\beta$$

Our standing assumption

$$\text{spec}_e(A) \cap (\mathcal{Y} \times \{\Im \sigma = \pm m/2\}) = \emptyset$$

implies $D_{\wedge, \text{min}}(\eta)$ is independent of $\eta$
The $\phi_k(\eta)$ are further corrected and then used to define a space

$$\mathcal{H}_A \subset D_{\text{max}}(A)$$

complementary to $D_{\text{min}}(A)$. The idea is that

$$H^m_A = \mathcal{H}_A \oplus D_{\text{min}}(A)$$

becomes an analogue of the regular $H^m(\mathcal{M})$ in the case where $A$ is a regular elliptic operator.
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A boundary value problem is now posed as

Find $u \in H_A$ such that

$$Au = f, \ f \in x^{-m/2}L^2_b$$

$$\beta \gamma u = 0.$$

where $\beta$ is some system of pseudodifferential operators on sections of $T_{bA}$. 
Cast

and so on . . .

The End
Cast

edges

and so on . . .

The End

Traces and boundary value problems

Arkansas Spring Lecture Series
Cast

- edges
- kinks
- "corners"
- conical points
- cracks
- singularities

and so on.

The End
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and so on...
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