Embedding theorems of manifolds with $\mathbb{R}$-action

Gerardo Mendoza

Temple University

Serra Negra, August 2011
I will discuss generalizations of the following two theorems concerning embeddings of compact manifolds in complex projective space determined by complex line bundles:

- Let $M$ be a compact manifold, let $E \to M$ be a complex line bundle. Then there is $\mathbb{P}^N$ and an embedding $\phi: M \to \mathbb{P}^N$ such that $E \approx \phi^* \Gamma$.

Recall: $\mathbb{P}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^N$. $\Gamma \to \mathbb{P}^N$ is the line bundle whose fiber at $p \in \mathbb{P}^N$ is the vector space $p$.

- If $M$ is a compact complex manifold and $E \to M$ is a positive line bundle, then there is a holomorphic embedding $\phi: M \to \mathbb{P}^N$. This is Kodaira's embedding theorem. Positive means that $E \to M$ has a holomorphic connection whose curvature $\sum_{\mu, \nu} \Omega_{\mu \nu} \, dz_{\mu} \wedge dz_{\nu}$ is such that $\sum_{\mu, \nu} \Omega_{\mu \nu} \, dz_{\mu} \otimes dz_{\nu}$ is positive definite. I'll remind you of this in a second.
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- Let $M$ be a compact manifold, let $E \to M$ be a complex line bundle. Then there is $N$ and an embedding

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- Let $\mathcal{M}$ be a compact manifold, let $E \rightarrow \mathcal{M}$ be a complex line bundle. Then there is $N$ and an embedding
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  \[ \text{without boundary} \]

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Recall: \( \mathbb{P} \mathbb{C}^N \) is the manifold whose points are the one-dimensional subspaces of \( \mathbb{C}^N \). \( \Gamma \to \mathbb{P} \mathbb{C}^N \) is the line bundle whose fiber at \( p \in \mathbb{P} \mathbb{C}^N \) is the vector space \( p \).

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Suppose $\pi : E \to M$ is a complex line bundle and $\phi : M \to \mathbb{P}\mathbb{C}^N$ an embedding such that $E$ is isomorphic to $\phi^*\Gamma$:

\[ \Phi : E \to \{ (x, v) \in M \times \mathbb{C}^N : v \in \phi(x) \} \]

\[ \Phi(\eta) = (\pi(\eta), F(\eta)) \in M \times \mathbb{C}^N \]

The map $F$ sends $E$ to $\mathbb{C}^N$ and $F|_{E_0}$ is linear injective for each $x$.

Suppose $E$ has a Hermitian metric such that $|\eta| = 1 \Rightarrow |F(\eta)| = 1$. That is:

\[ F : \{ \eta \in E : |\eta| = 1 \} \to S^2_{N-1} \subset \mathbb{C}^N \setminus 0 \]

\[ \text{Y} = SE \]

Since $F|_{E_0}$ is linear, $F(e^{it}\eta) = e^{it}F(\eta)$.

Let $T$ be the infinitesimal generator of the action $(t, \eta) \mapsto e^{it}\eta$.

This is a vector field on $SE$.

The infinitesimal generator of the "same" action on $S^2_{N-1}$ is $T' = i\sum_j (z_j \partial z_j - z_j \partial z_j)$.

$F : SE \to S^2_{N-1}$ is an embedding and $F^*T = i\sum_j (z_j \partial z_j - z_j \partial z_j)$.
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Embedding theorems 

Serra Negra, August 2011 4 / 13
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$$F : SE \to S^{2N-1} \text{ is an embedding and } F_*\mathcal{T} = i \sum_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}).$$
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ where

- $\mathcal{N}$ is a compact manifold;
- $\mathcal{T}$ is smooth real nowhere vanishing vector field
- there is a Riemannian metric $g$ with $\mathcal{L}_T g = 0$. 

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$\mathcal{N}$ is like $SE$ 
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- $\mathcal{T}$ is smooth real nowhere vanishing vector field
- there is a Riemannian metric $g$ with $\mathcal{L}_{\mathcal{T}} g = 0$.

Let $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$. There is an embedding
$$F : \mathcal{N} \rightarrow S^{2N-1}$$
for some $N$ such that
$$F_* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j})$$
for some positive $\tau_j$. 

$\mathcal{N}$ is like $SE$
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- $\mathcal{N}$ is a compact manifold;
- $\mathcal{T}$ is smooth real nowhere vanishing vector field;
- there is a Riemannian metric $g$ with $\mathcal{L}_T g = 0$.

Let $(\mathcal{N}, \mathcal{T}) \in F$. There is an embedding $F : \mathcal{N} \to S^{2N-1}$ for some $N$ such that

$$F_* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j})$$

for some positive $\tau_j$.

$\mathcal{N}$ is like $SE$; $\mathcal{T}$ gives an $\mathbb{S}^1$-action on $\mathcal{N}$.

Write $\mathcal{T}'$ for the vector field on the right.
Generalization

Let $\mathcal{F}$ be the family of pairs $(N, T)$ where
- $N$ is a compact manifold;
- $T$ is smooth real nowhere vanishing vector field;
- there is a Riemannian metric $g$ with $\mathcal{L}_T g = 0$.

Let $(N, T) \in \mathcal{F}$. There is an embedding

$$F : N \to S^{2N-1}$$

for some $N$ such that

$$F_* T = i \sum_j \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j})$$

for some positive $\tau_j$.

$N$ is like $SE$ gives an $\mathbb{R}^1$-action.

Write $T'$ for the vector field on the right. $T'$ is tangent to $S^{2N-1}$ and preserves the standard metric of $S^{2N-1}$. 
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- $\mathcal{N}$ is a compact manifold;
- $\mathcal{T}$ is smooth real nowhere vanishing vector field
- there is a Riemannian metric $g$ with $\mathcal{L}_T g = 0$.

Let $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$. There is an embedding

$$ F : \mathcal{N} \to S^{2N-1} $$

for some $N$ such that

$$ F_* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}) $$

for some positive $\tau_j$.

Write $\mathcal{T}'$ for the vector field on the right. $\mathcal{T}'$ is tangent to $S^{2N-1}$ and preserves the standard metric of $S^{2N-1}$. So the pair $(S^{2N-1}, \mathcal{T}')$ belongs to the class $\mathcal{F}$. 

$\mathcal{N}$ is like $SE$
$\mathcal{T}$ gives an $S^1$-action

$\mathbb{R}$
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle.
Suppose now that \( \mathcal{M} \) is a compact complex manifold; \( \pi : E \to \mathcal{M} \) is still a complex line bundle. Fix a Hermitian connection \( h \). Let
\[
\mathcal{V} = \{ v \in \mathbb{C}TSE : \pi_* v \in T^{0,1}\mathcal{M} \}.
\]
This is an elliptic structure on \( SE \).
Suppose now that \( \mathcal{M} \) is a compact complex manifold; \( \pi : E \to \mathcal{M} \) is still a complex line bundle. Fix a Hermitian connection \( h \). Let \( \overline{V} = \{ v \in \mathbb{C} TSE : \pi_* v \in T^{0,1} \mathcal{M} \} \). Let \( SE = \text{circle bundle} \) be a Hermitian connection. Its horizontal bundle \( \mathcal{H} \) is involutive and \( V + \overline{V} = \mathbb{C} TSE \). This is an elliptic structure on \( SE \).
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let

$$\mathcal{V} = \{ \nu \in \mathbb{C} TSE : \pi_* \nu \in T^{0,1} \mathcal{M} \}.$$ 

This is an elliptic structure on $SE$ such that

$$\mathcal{V} \cap \overline{\mathcal{V}} = \text{span } \mathcal{T}$$

$SE = \text{circle bundle}$

$\overline{\mathcal{V}}$ is involutive and $\mathcal{V} + \overline{\mathcal{V}} = \mathbb{C} TSE$
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let

$$\mathcal{V} = \{ v \in \mathbb{C} \mathcal{TSE} : \pi_* v \in \mathcal{T}^{0,1} \mathcal{M} \}. $$

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Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let

$$\mathcal{V} = \{ \nu \in \mathbb{C}TSE : \pi_* \nu \in T^{0,1} \mathcal{M} \}.$$ 

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Let

$$\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes \mathbb{C}T^* \mathcal{M})$$

be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$. 

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This is an elliptic structure on $SE$ such that $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span } \mathcal{T}$ where $\mathcal{T}$ is the infinitesimal generator of the $S^1$ action on $SE$.

Let

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$$\mathcal{H} = \{v \in TSE : \langle \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^* \omega, v \rangle = 0 \}$$

$\eta$ local frame over $U$, $|\eta| = 1$, $\bar{\zeta}\eta$ arbitrary element of $E|_U$, $\nabla \eta = \eta \otimes \omega$. 

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$$V = \{ v \in \mathbb{C}TSE : \pi_*v \in T^{0,1}\mathcal{M} \}. \quad SE = \text{circle bundle}$$

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Let $\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes \mathbb{C}T^*\mathcal{M})$ be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

Let $\mathcal{H} = \{ v \in TSE : \langle \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, v \rangle = 0 \}$

$\theta = 1$-form vanishing on $\mathcal{H}$ such that $\langle \theta, \mathcal{T} \rangle = 1$. $\eta$ local frame over $U$, $|\eta| = 1$, $\zeta \eta$ arbitrary element of $E|_U$, $\nabla \eta = \eta \otimes \omega$. 

$\overline{V}$ is involutive and $V + \overline{V} = \mathbb{C}TSE$
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let $V = \{ v \in \mathbb{C}TSE : \pi^* v \in T^{0,1} \mathcal{M} \}$. This is an elliptic structure on $SE$ such that $V \cap \overline{V} = \text{span } \mathcal{T}$.

Let $\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes \mathbb{C}T^* \mathcal{M})$ be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

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$$\mathcal{H} = \{ v \in TS\theta = (\frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^* \omega, v) = 0 \}$$

$\theta = 1$-form vanishing on $\mathcal{H}$ such that $\langle \theta, \mathcal{T} \rangle = 1$. Let $\beta = -i\theta|_{\overline{\mathcal{V}}}$. 

$\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes \mathcal{C} T^*\mathcal{M})$ be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$. 

Let $\mathcal{H}$ be the infinitesimal generator of the $S^1$ action on $SE$.

$\nabla \eta = \eta \otimes \omega$. 

$\eta$ local frame over $U$, $|\eta| = 1$, $\zeta \eta$ arbitrary element of $E|_U$. 

$D_{\beta} = \ker \beta$ is involutive iff $D_{\beta} = 0$: $X, Y \in K_{\beta}$

$\langle \mathcal{V}, X \rangle_\mathcal{H}$

$\beta$ is $D_{\beta}$-closed iff $\Omega_{\mathcal{V}, 2} = \partial \omega_{\mathcal{V}, 1} = 0$ (iff $\nabla$ is a holomorphic connection). 

$\mathcal{H}$

Levi $\theta(X, Y) = -i\theta(X, Y) = -((\pi^* d\omega)(X, Y)) = -((\pi^* \partial \omega_{0, 1})(X, Y))$. $X, Y \in K_{\beta}$

$\mathcal{H}$ is positive iff $E$ is negative.
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let

$$\mathcal{V} = \{ \nu \in \mathbb{C}TSE : \pi_* \nu \in T^{0,1}\mathcal{M} \}. \quad \text{SE = circle bundle}$$

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Let

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$$\mathcal{H} = \{ \nu \in T\mathcal{S}\theta = \langle \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, \nu \rangle = 0 \}$$

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Let $\mathcal{H} = \{ v \in TSE : \langle \frac{1}{2i} (\bar{\zeta} d\zeta - \zeta d\bar{\zeta}) - i \pi^* \omega, v \rangle = 0 \}$

$$\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.$$ 

Let $\beta = -i \theta |_{\overline{\mathcal{V}}}$. Then $\overline{\mathcal{K}}_\beta = \ker \beta$ is involutive iff $\overline{D} \beta = 0$.
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Let $\beta = -i\theta|_{\overline{\nabla}}$. Then $\overline{\mathcal{K}}_\beta = \text{ker } \beta$ is involutive iff $\overline{\mathcal{D}}\beta = 0$:

$$2(\overline{\mathcal{D}}\beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle$$

$X, Y \in \overline{\mathcal{K}}_\beta$
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi: E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let

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$$2(\overline{D}\beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -i(X\langle \theta, Y \rangle - Y\langle \theta, X \rangle - \langle \theta, [X, Y] \rangle)$$

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$$\mathcal{H} = \{ v \in T\mathcal{M} \theta = \langle \frac{1}{2i} (\overline{\zeta} d \zeta - \zeta d \overline{\zeta}) - i \pi^* \omega, v \rangle = 0 \}$$

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Let $\beta = -i \theta |_\mathcal{V}$. Then $\overline{\mathcal{K}}_\beta = \ker \beta$ is involutive iff $\bar{D}\beta = 0$:

$$2(\bar{D}\beta)(X, Y) = X \langle \beta, Y \rangle - Y \langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -i d \theta(X, Y)$$

$$= -i (X \langle \theta, Y \rangle - Y \langle \theta, X \rangle - \langle \theta, [X, Y] \rangle)$$

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$$\mathcal{H} = \{ v \in TSE \theta = \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, v \} = 0$$

$$\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.$$ 

Let $\beta = -i\theta|_\mathcal{V}$. Then $\overline{\mathcal{K}}_\beta = \ker \beta$ is involutive iff $\overline{D}\beta = 0$:

$$2(\overline{D}\beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -id\theta(X, Y)$$

$$= -(\pi^*d\omega)(X, Y)\langle \theta, X \rangle - \langle \theta, [X, Y] \rangle) \quad X, Y \in \overline{\mathcal{K}}_\beta$$
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let 

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$$\mathcal{H} = \{ v \in TSE : \theta = \langle \frac{1}{2i} (\overline{\zeta}d\zeta - \zeta d\overline{\zeta}) - i\pi^*\omega, v \rangle = 0 \}$$

$$\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.$$ 

Let $\beta = -i\theta|_V$. Then $K_\beta = \ker \beta$ is involutive iff $\overline{D} \beta = 0$:

$$2(\overline{D} \beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -i d\theta(X, Y)$$

$$= - (\pi^*d\omega)(X, Y) \langle \overline{\zeta} \rangle (\pi^*\overline{\partial_\omega}^0{}^1)(X, Y). \quad X, Y \in K_\beta$$
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let

$$\nabla = \{ v \in \mathbb{C}TSE : \pi_* v \in T^{0,1}\mathcal{M} \}. \quad \text{SE = circle bundle}$$

This is an elliptic structure on $SE$ such that $\nabla$ is involutive and $\nabla + \nabla = \mathbb{C}TSE$.

$$\nabla \cap \nabla = \text{span } \mathcal{T}$$

Let

$$\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes \mathbb{C}T^*\mathcal{M})$$

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Let

$$\mathcal{H} = \{ v \in TSE : \langle \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^* \omega, v \rangle = 0 \}$$

$$\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.$$ 

Let $\beta = -i\theta|_{\nabla}$. Then $\nabla_\beta = \ker \beta$ is involutive iff $\overline{\nabla} \beta = 0$:

$$2(\overline{\nabla} \beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -id\theta(X, Y)$$

$$= -(\pi^*d\omega)(X, Y) \langle \zeta \rangle (\pi^*\overline{\partial} \omega^{0,1})(X, Y).$$ 

$X, Y \in \nabla_\beta$

$\beta$ is $\overline{\nabla}$-closed iff $\Omega^{0,2} = \overline{\partial} \omega^{0,1} = 0$
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let

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Let

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Let

$$\mathcal{H} = \{ v \in TSE \theta = \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^* \omega, v \} = 0$$

$$\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.$$ 

Let $\beta = -i\theta|_\nabla$. Then $\overline{\mathcal{K}}_\beta = \ker \beta$ is involutive iff $\overline{\mathcal{D}} \beta = 0$:

$$2(\overline{\mathcal{D}} \beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -id \theta(X, Y)$$

$$= - (\pi^* d\omega)(X, Y) \langle \mathcal{F}, \pi^* \overline{\partial} \omega^{0,1} \rangle(X, Y). \quad X, Y \in \overline{\mathcal{K}}_\beta$$

$\beta$ is $\overline{\mathcal{D}}$-closed iff $\Omega^{0,2} = \overline{\partial} \omega^{0,1} = 0$ (iff $\nabla$ is a holomorphic connection).
Suppose now that \( \mathcal{M} \) is a compact complex manifold; \( \pi : E \to \mathcal{M} \) is still a complex line bundle. Fix a Hermitian connection \( h \). Let
\[
\mathcal{V} = \{ v \in \mathbb{C}TSE : \pi_* v \in T^{0,1}\mathcal{M} \}. \tag{SE = circle bundle}
\]
This is an elliptic structure on \( SE \) such that \( \mathcal{V} \) is involutive and \( \mathcal{V} + \overline{\mathcal{V}} = \mathbb{C}TSE \)
\[
\cdots \to C^\infty(SE; \wedge^q \mathcal{V}^*) \overset{\overline{\partial}}{\to} C^\infty(SE; \wedge^{q+1} \mathcal{V}^*) \to \cdots
\]
\[\mathcal{V} \cap \overline{\mathcal{V}} = \text{span } \mathcal{T}
\]
\( \mathcal{T} \) is the infinitesimal generator of the \( S^1 \) action on \( SE \)

Let\[
\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes \mathbb{C}T^*\mathcal{M})
\]
be a Hermitian connection. Its horizontal bundle \( \mathcal{H} \) is tangent to \( SE \).

Let\[
\mathcal{H} = \{ v \in TSE : \theta = \frac{1}{2i}(\zeta d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, v \} = \text{Levi}_\theta(X, Y) = -i d\theta(X, Y) \text{ over } U,
\]
\[\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.
\]

Let \( \beta = -i\theta|_{\overline{\mathcal{V}}} \). Then \( \overline{\mathcal{K}}_\beta = \ker \beta \) is involutive iff \( \overline{\partial}\beta = 0:\)
\[
2(\overline{\partial}\beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -id\theta(X, Y)
\]
\[= -(\pi^*d\omega)(X, Y) \langle \overline{\mathcal{V}}, (\pi^*\overline{\partial}\omega_{0,1})(X, Y) \rangle.
\]
\( X, Y \in \overline{\mathcal{K}}_\beta \)

\( \beta \) is \( \overline{\partial} \)-closed iff \( \Omega^{0,2} = \overline{\partial}\omega^{0,1} = 0 \) (iff \( \nabla \) is a holomorphic connection).
Suppose now that $\mathcal{M}$ is a compact complex manifold; $\pi : E \to \mathcal{M}$ is still a complex line bundle. Fix a Hermitian connection $h$. Let

$$\mathcal{V} = \{ v \in \mathbb{C}TSE : \pi_* v \in T^{0,1}\mathcal{M} \}. \quad \text{SE = circle bundle}$$

This is an elliptic structure on $SE$ such that $\mathcal{V} + \mathcal{V} = \mathbb{C}TSE$ such that

$$\mathcal{V} \cap \overline{\mathcal{V}} = \text{span } \mathcal{T}$$

Let

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Let

$$\mathcal{H} = \{ v \in TS \theta = \frac{1}{2i} (\zeta d\bar{\zeta} - \zeta d\bar{\zeta}) - i \pi^* \omega, v \} = \text{Levi}_\theta(X, Y) = \text{id}(X, Y)$$

$$\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.$$ 

Let $\beta = -i\theta|_{\mathcal{V}}$. Then $\overline{\mathcal{K}}_\beta = \ker \beta$ is involutive iff $\overline{\nabla} \beta = 0$:

$$2(\overline{\nabla} \beta)(X, Y) = X \langle \beta, Y \rangle - Y \langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -i \text{id}(X, Y)$$

$$= -(\pi^* d\omega)(X, Y) \langle \bar{\zeta}, \pi^* \bar{\omega}^{0,1} \rangle(X, Y). \quad X, Y \in \overline{\mathcal{K}}_\beta$$

$\beta$ is $\overline{\nabla}$-closed iff $\Omega^{0,2} = \overline{\partial} \omega^{0,1} = 0$ (iff $\nabla$ is a holomorphic connection).
Suppose $E \to \mathcal{M}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \text{ (m times);}$$
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$$E^m = E \otimes \cdots \otimes E \text{ (m times);}$$

$$\mathfrak{Hol}(\mathcal{M}; E^m) = \{ \eta \in C^\infty(\mathcal{M}; E^m) : \bar{\partial}\eta = 0 \}, \quad N = \dim \mathfrak{Hol}(\mathcal{M}; E^m);$$
Suppose $E \to \mathcal{M}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$

$$\mathcal{H} \mathcal{O} \mathcal{L}(\mathcal{M}; E^m) = \{ \eta \in C^\infty(\mathcal{M}; E^m) : \bar{\partial} \eta = 0 \}, \ N = \dim \mathcal{H} \mathcal{O} \mathcal{L}(\mathcal{M}; E^m);$$

$$\text{ev}_x : \mathcal{H} \mathcal{O} \mathcal{L}(\mathcal{M}; E^m) \to E^m_x, \ \text{ev}_x(\eta) = \eta(x);$$
Suppose $E \to \mathcal{M}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$

$$\mathcal{H}^\infty(\mathcal{M}; E^m) = \{ \eta \in C^\infty(\mathcal{M}; E^m) : \overline{\partial} \eta = 0 \}, \ N = \dim \mathcal{H}^\infty(\mathcal{M}; E^m);$$

$$\text{ev}_x : \mathcal{H}^\infty(\mathcal{M}; E^m) \to E^m_x, \ \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : \mathcal{M} \to \{ \text{subspaces of } \mathcal{H}^\infty(\mathcal{M}; E^m) \}, \ \Phi(x) = \ker \text{ev}_x$$
Suppose $E \to \mathcal{M}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \text{ (m times)};$$

$$\mathfrak{H}ol(\mathcal{M}; E^m) = \{ \eta \in C^\infty(\mathcal{M}; E^m) : \overline{\partial} \eta = 0 \}, \quad N = \dim \mathfrak{H}ol(\mathcal{M}; E^m);$$

$$\text{ev}_x : \mathfrak{H}ol(\mathcal{M}; E^m) \to E^m_x, \quad \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : \mathcal{M} \to \{ \text{subspaces of } \mathfrak{H}ol(\mathcal{M}; E^m) \}, \quad \Phi(x) = \ker \text{ev}_x$$

(Kodaira) If $m$ is large enough, then for all $x \in \mathcal{M}$ there is $\eta \in \mathfrak{H}ol(\mathcal{M}; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N - 1$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_{N-1}(\mathfrak{H}ol(\mathcal{M}; E^m))$ is an embedding.
Suppose $E \to \mathcal{M}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$

$$\mathfrak{hol}(\mathcal{M}; E^m) = \{\eta \in C^\infty(\mathcal{M}; E^m) : \bar{\partial}\eta = 0\}, \ N = \dim \mathfrak{hol}(\mathcal{M}; E^m);$$

$$\text{ev}_x : \mathfrak{hol}(\mathcal{M}; E^m) \to E_x^m, \ \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : \mathcal{M} \to \{\text{subspaces of } \mathfrak{hol}(\mathcal{M}; E^m)\}, \ \Phi(x) = \ker \text{ev}_x$$

(Kodaira) If $m$ is large enough, then for all $x \in \mathcal{M}$ there is $\eta \in \mathfrak{hol}(\mathcal{M}; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N - 1$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_{N-1}(\mathfrak{hol}(\mathcal{M}; E^m))$ is an embedding.
Suppose $E \to \mathcal{M}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$

$$\mathfrak{Hol}(\mathcal{M}; E^m) = \{ \eta \in C^\infty(\mathcal{M}; E^m) : \bar{\partial} \eta = 0 \}, \ N = \dim \mathfrak{Hol}(\mathcal{M}; E^m);$$

$$\text{ev}_x : \mathfrak{Hol}(\mathcal{M}; E^m) \to E^m_x, \ \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : \mathcal{M} \to \{ \text{subspaces of } \mathfrak{Hol}(\mathcal{M}; E^m) \}, \ \Phi(x) = \ker \text{ev}_x \ (\text{Kodaira})$$

If $m$ is large enough, then for all $x \in \mathcal{M}$ there is $\eta \in \mathfrak{Hol}(\mathcal{M}; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N - 1$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_{N-1}(\mathfrak{Hol}(\mathcal{M}; E^m))$ is an embedding.

If $p \in SE^*$ then $\varphi_m(p) = p \otimes \cdots \otimes p \in SE^{*m}$. 

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*because $E^m_x$ is one-dimensional*
Suppose $E \to \mathcal{M}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \text{ (m times)};$$

$$\mathcal{H} \sigma_l(M; E^m) = \{\eta \in C^\infty(M; E^m) : \bar{\partial} \eta = 0\}, \quad N = \dim \mathcal{H} \sigma_l(M; E^m);$$

$$\text{ev}_x : \mathcal{H} \sigma_l(M; E^m) \to E^m_x, \quad \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : \mathcal{M} \to \{\text{subspaces of } \mathcal{H} \sigma_l(M; E^m)\}, \quad \Phi(x) = \ker \text{ev}_x$$

(Kodaira) If $m$ is large enough, then for all $x \in \mathcal{M}$ there is

$\eta \in \mathcal{H} \sigma_l(M; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N - 1$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_{N-1}(\mathcal{H} \sigma_l(M; E^m))$ is an embedding.

If $p \in SE^*$ then $\wp_m(p) = p \otimes \cdots \otimes p \in SE^{*m}$. It makes sense to compose:

$$\eta \in \mathcal{H} \sigma_l(M; E^m), \quad f_\eta(p) = \langle \eta, \wp_m(p) \rangle$$
Suppose $E \rightarrow \mathcal{M}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$

$$\mathcal{H}\mathcal{o}\mathcal{l}(\mathcal{M}; E^m) = \{\eta \in C^{\infty}(\mathcal{M}; E^m) : \bar{\partial} \eta = 0\}, \ N = \dim \mathcal{H}\mathcal{o}\mathcal{l}(\mathcal{M}; E^m);$$

$$\text{ev}_x : \mathcal{H}\mathcal{o}\mathcal{l}(\mathcal{M}; E^m) \rightarrow E^m_x, \ \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : \mathcal{M} \rightarrow \{\text{subspaces of } \mathcal{H}\mathcal{o}\mathcal{l}(\mathcal{M}; E^m)\}, \ \Phi(x) = \ker \text{ev}_x$$

(Kodaira) If $m$ is large enough, then for all $x \in \mathcal{M}$ there is $\eta \in \mathcal{H}\mathcal{o}\mathcal{l}(\mathcal{M}; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N - 1$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_{N-1}(\mathcal{H}\mathcal{o}\mathcal{l}(\mathcal{M}; E^m))$ is an embedding.

If $p \in SE^*$ then $\varphi_m(p) = p \otimes \cdots \otimes p \in SE^{*m}$. It makes sense to compose:

$$\eta \in \mathcal{H}\mathcal{o}\mathcal{l}(\mathcal{M}; E^m), \ f_\eta(p) = \langle \eta, \varphi_m(p) \rangle$$

In Kodaira's theorem, if $\eta_1, \ldots, \eta_N$ is a basis of $\mathcal{H}\mathcal{o}\mathcal{l}(\mathcal{M}; E^m)$, then

$$SE^* \ni p \mapsto F(p) = (f_{\eta_1}(p), \ldots, f_{\eta_N}(p)) \in \mathcal{H}\mathcal{o}\mathcal{l}(\mathcal{M}; E^m) \setminus 0$$

is $m$-to-$1$ and such that $F_* \mathcal{T} = i \sum_j m(z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j}).$
Suppose $E \rightarrow \mathcal{M}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \ \text{times});$$

$$\mathfrak{hol}(\mathcal{M}; E^m) = \{ \eta \in C^\infty(\mathcal{M}; E^m) : \bar{\partial} \eta = 0 \}, \ N = \dim \mathfrak{hol}(\mathcal{M}; E^m);$$

$$\text{ev}_x : \mathfrak{hol}(\mathcal{M}; E^m) \rightarrow E^m_x, \ \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : \mathcal{M} \rightarrow \{ \text{subspaces of } \mathfrak{hol}(\mathcal{M}; E^m) \}, \ \Phi(x) = \ker \text{ev}_x$$

(Kodaira) If $m$ is large enough, then for all $x \in \mathcal{M}$ there is $\eta \in \mathfrak{hol}(\mathcal{M}; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N - 1$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_{N-1}(\mathfrak{hol}(\mathcal{M}; E^m))$ is an embedding.

If $p \in SE^*$ then $\wp_m(p) = \underbrace{p \otimes \cdots \otimes p}_{m \ \text{times}} \in SE^{*m}$. It makes sense to compose:

$$\eta \in \mathfrak{hol}(\mathcal{M}; E^m), \ f_\eta(p) = \langle \eta, \wp_m(p) \rangle \quad \wp(e^{it}p) = e^{imt}A \Rightarrow f_\eta(e^{it}p) = e^{it}f_\eta(p)$$

In Kodaira's theorem, if $\eta_1, \ldots, \eta_N$ is a basis of $\mathfrak{hol}(\mathcal{M}; E^m)$, then

$$SE^* \ni p \mapsto F(p) = (f_{\eta_1}(p), \ldots, f_{\eta_N}(p)) \in \mathfrak{hol}(\mathcal{M}; E^m) \setminus 0$$

is $m$-to-1 and such that $F_*T = i \sum_j m(z^j \partial_{\bar{z}^j} - \bar{z}^j \partial_{z^j})$. 

(Temple University)
Embedding theorems
Serra Negra, August 2011 7 / 13
Generalization

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$ such that:

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\overline{\mathcal{V}} \subset \mathbb{C} \mathcal{T} \mathcal{N}$ is an elliptic structure and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$;
- there is $\beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$, $\overline{\mathbb{D}} \beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$. 

Generalization

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$ such that:

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ is an elliptic structure and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_{\mathbb{C}} \mathcal{T}$;
- there is $\beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$, $\overline{D} \beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$. $\mathcal{K}_\beta = \ker \beta \subset \overline{\mathcal{V}}$ is a CR structure
Generalization

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \mathcal{V})$ such that:

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\mathcal{V} \subset \mathbb{C}T\mathcal{N}$ is an elliptic structure and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_{\mathbb{C}} T$;
- there is $\beta \in C^\infty(\mathcal{N}; \mathcal{V}^*)$, $\overline{D}\beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$. 

If $\beta, \beta' \in C^\infty(\mathcal{N}; \mathcal{V}^*)$ are two sections as described, say

$$\beta \sim \beta' \text{ iff there is } u \text{ real-valued such that } \beta' - \beta = \overline{D}u.$$

$\overline{K}_\beta = \ker\beta \subset \mathcal{V}$ is a CR structure.
Generalization

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \mathcal{V})$ such that:

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\mathcal{V} \subset \mathbb{C}T\mathcal{N}$ is an elliptic structure and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_{\mathbb{C}} \mathcal{T}$;
- there is $\beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$, $\overline{\mathbb{D}} \beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$. 

If $\beta, \beta' \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$ are two sections as described, say $\beta \sim \beta'$ iff there is $u$ real-valued such that $\beta' - \beta = \overline{\mathbb{D}} u$. 

$\overline{\mathcal{K}}_\beta = \ker \beta \subset \overline{\mathcal{V}}$ is a CR structure

$\overline{\mathbb{D}} u = du|_{\overline{\mathcal{V}}}$
Generalization

Let \( \mathcal{F}_{\text{ell}} \) be the set of triples \((\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})\) such that:

- \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\);
- \(\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}\) is an elliptic structure and \(\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_{\mathbb{C}} \mathcal{T}\);
- there is \(\beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)\), \(\overline{D}\beta = 0\), \(\langle \beta, \mathcal{T} \rangle = -i\).

If \(\beta, \beta' \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)\) are two sections as described, say \(\beta \sim \beta'\) iff there is \(u\) real-valued such that \(\beta' - \beta = \overline{Du}\).

Let \(\beta\) be the class of \(\beta\).

Let \((\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}\) with \(\dim \mathcal{N} \geq 5\). Fix \(\beta\). The following are equivalent:

- \(\exists \beta \in \beta\) such that \(\overline{K}_\beta\) is definite.
- \(\exists \beta \in \beta\) and an equivariant CR embedding \(F: \mathcal{N}, \overline{K}_\beta \rightarrow S^{2N-1}\) for some \(N\), with \(F_* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j})\) and all \(\tau_j\) of the same sign.
Outline of proofs
Outline of proofs

Embedding in $\mathbb{C}^N \backslash 0$ of $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$. 

Continuity, compactness and (1) give an immersion

Pick orthonormal bases
Outline of proofs

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute.
Outline of proofs

Pick $(N, T) \in \mathcal{F}$ and a $T$-invariant metric, let $\Delta$ be the Laplacian. Note that $\Delta$ and $T$ commute.

Embedding in $\mathbb{C}^N \setminus 0$ of $(N, T) \in \mathcal{F}$. $\mathcal{E}_\lambda = \ker(\Delta - \lambda)$ is invariant under $T$, $-iT$ is symmetric, so $\text{spec}(-iT|_{\mathcal{E}_\lambda}) \subset \mathbb{R}$

Continuity, compactness and (1) give an immersion $\phi$.

Use property (2) to ensure that the functions $\tau$ separate points of $N$. Pick orthonormal bases $\phi_i$. $\ell_i(2)$ the functions $\phi_i$. $\tau, \lambda, \ell_j \in E(t, p) \in N_0$. Replace $\phi_i$ by $\phi_i(t, p)$.

$\tau, \lambda, \ell_j \in E(t, p) \in N_0$.
Outline of proofs

Pick \((N, T) \in \mathcal{F}\) and a \(T\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(T\) commute. Let

\[ E_{\tau, \lambda} = \{ \phi \in C^\infty(N) : \Delta \phi = \lambda \phi, \quad -iT \phi = \tau \phi \}, \]

\[ \text{spec}( -iT, \Delta) = \{ (\tau, \lambda) : E_{\tau, \lambda} \neq 0 \}. \]
Outline of proofs

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[\mathcal{E}_\lambda = \ker(\Delta - \lambda)\]

be invariant under \(\mathcal{T}\), 

\[\mathcal{E}_\lambda\]

is invariant under \(\mathcal{T}\), 

\(-i\mathcal{T}\) is symmetric, so 

\[\text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\]

Pick orthonormal bases

\[\{\phi_{\tau,\lambda,j} \in \mathcal{E}_{\tau,\lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau,\lambda}\}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta)\]
Outline of proofs

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[\mathcal{E}_{\tau, \lambda} = \{ \phi \in C^\infty(\mathcal{N}) : \Delta \phi = \lambda \phi, \quad -i \mathcal{T} \phi = \tau \phi \},\]

\[\text{spec}(-i \mathcal{T}, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0 \} \text{.}\]

Pick orthonormal bases

\[\{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-i \mathcal{T}, \Delta)\]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C} \mathcal{T}_{p_0}^* \mathcal{N} = \text{span}\{ d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-i \mathcal{T}, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda} \} \);

(2) the functions \(\phi_{\tau, \lambda, j}, (\tau, \lambda) \in \text{spec}(-i \mathcal{T}, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda}\), separate points of \(\mathcal{N}\).
Outline of proofs

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[ \mathcal{E}_\lambda = \ker(\Delta - \lambda) \] is invariant under \(\mathcal{T}\),

\(-i\mathcal{T}\) is symmetric, so \(\text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\).

Pick orthonormal bases

\[ \{\phi_{\tau,\lambda,j} \in \mathcal{E}_{\tau,\lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau,\lambda}\}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta) \]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C}\mathcal{T}^*_{p_0} \mathcal{N} = \text{span}\{d\phi_{\tau,\lambda,j}(p_0) : (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), \ j = 1, \ldots, N_{\tau,\lambda}\};

(2) the functions \(\phi_{\tau,\lambda,j}\), \((\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), \ j = 1, \ldots, N_{\tau,\lambda}\), separate points of \(\mathcal{N}\).

Continuity, compactness and (1) give an immersion

\[ F = (\phi_{\tau_1,\lambda_1,j_1}, \ldots, \phi_{\tau_n,\lambda_n,j_n}) : SE \to \mathbb{C}^N \]
Outline of proofs

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[\mathcal{E}_\lambda = \ker(\Delta - \lambda)\]

is invariant under \(\mathcal{T}\), \(-i\mathcal{T}\) is symmetric, so \(\text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\).

\[\text{spec}(-i\mathcal{T}, \Delta) = \{(\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0\}\].

Pick orthonormal bases

\[\{\phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda}\}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta)\]

\((1)\) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C} T^*_{p_0} \mathcal{N} = \text{span}\{d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), \; j = 1, \ldots, N_{\tau, \lambda}\}\);

\((2)\) the functions \(\phi_{\tau, \lambda, j}\), \((\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), \; j = 1, \ldots, N_{\tau, \lambda}\), separate points of \(\mathcal{N}\).

Continuity, compactness and (1) give an immersion

\[F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : \mathcal{S}\mathcal{E} \rightarrow \mathbb{C}^N\]

If \(\tau_\ell < 0\), replace \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) by \(\overline{\phi_{\tau_\ell, \lambda_\ell, j_\ell}}\).
Outline of proofs

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[
\mathcal{E}_\lambda = \ker(\Delta - \lambda) \quad \text{is invariant under} \quad \mathcal{T},
\]

\[-i\mathcal{T}\text{ is symmetric, so } \text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}.
\]

Pick orthonormal bases

\[
\{\phi_{\tau,\lambda,j} \in \mathcal{E}_{\tau,\lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau,\lambda}\}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta)
\]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C} \mathcal{T}_{p_0}^* \mathcal{N} = \text{span}\{d\phi_{\tau,\lambda,j}(p_0) : (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), j = 1, \ldots, N_{\tau,\lambda}\};
\]

(2) the functions \(\phi_{\tau,\lambda,j}, (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), j = 1, \ldots, N_{\tau,\lambda},\)

separate points of \(\mathcal{N}\).

Continuity, compactness and (1) give an immersion

\[
F = (\phi_{\tau_1,\lambda_1,j_1}, \ldots, \phi_{\tau_N,\lambda_N,j_N}) : SE \to \mathbb{C}^N \setminus 0
\]

If \(\tau_ell < 0\), replace \(\phi_{\tau_ell,\lambda_ell,j_ell}\) by \(\overline{\phi_{\tau_ell,\lambda_ell,j_ell}}\).

Use property (2) to ensure that the functions \(\phi_{\tau_ell,\lambda_ell,j_ell}\) separate points.
Outline of proofs

Pick \((\mathcal{N}, T) \in \mathcal{F}\) and a \(T\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(T\) commute. Let \(\mathcal{E}_\lambda = \ker(\Delta - \lambda)\) is invariant under \(T\), \(-iT\) is symmetric, so \(\text{spec}(-iT|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\)

\[
\text{spec}(-iT, \Delta) = \{(\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0\}
\]

Pick orthonormal bases

\[
\{\phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda}\}, \quad (\tau, \lambda) \in \text{spec}(-iT, \Delta)
\]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C}T_{p_0}^* \mathcal{N} = \text{span}\{d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-iT, \Delta), j = 1, \ldots, N_{\tau, \lambda}\}\);

(2) the functions \(\phi_{\tau, \lambda, j}, (\tau, \lambda) \in \text{spec}(-iT, \Delta), j = 1, \ldots, N_{\tau, \lambda}\), separate points of \(\mathcal{N}\).

Continuity, compactness and (1) give an immersion

\[
F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \to \mathbb{C}^N \setminus 0
\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) separate points. 

\(a_t = \) one parameter group of diffeos generated by \(T\)
Outline of proofs

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[
\mathcal{E}_{\tau, \lambda} = \{ \phi \in C^\infty(\mathcal{N}) : \Delta \phi = \lambda \phi, \quad -i \mathcal{T} \phi = \tau \phi \},
\]

\[
\text{spec}(-i \mathcal{T}, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0 \}.
\]

Pick orthonormal bases

\[
\{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-i \mathcal{T}, \Delta)
\]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C} \mathcal{T}_{p_0}^* \mathcal{N} = \text{span}\{d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-i \mathcal{T}, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda}\};
\]

(2) the functions \(\phi_{\tau, \lambda, j}, \quad (\tau, \lambda) \in \text{spec}(-i \mathcal{T}, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda}, \)

separate points of \(\mathcal{N}\).

Continuity, compactness and (1) give an immersion

\[
F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \rightarrow \mathbb{C}^N \setminus 0
\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) separate points.
Outline of proofs

Pick \((N, T) \in \mathcal{F}\) and a \(T\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(T\) commute. Let

\[E_{\tau, \lambda} = \{ \phi \in C^\infty(N) : \Delta \phi = \lambda \phi, \quad -i T \phi = \tau \phi \},\]

\[
\text{spec}(-iT, \Delta) = \{(\tau, \lambda) : E_{\tau, \lambda} \neq 0 \}.
\]

Pick orthonormal bases

\[\{ \phi_{\tau, \lambda, j} \in E_{\tau, \lambda} : j = 1, \ldots, \text{dim } E_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-iT, \Delta)\]

(1) for all \(p_0 \in N\), \(\mathbb{C} T_{p_0}^* N = \text{span}\{ d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-iT, \Delta), j = 1, \ldots, N_{\tau, \lambda} \};\)

(2) the functions \(\phi_{\tau, \lambda, j}, (\tau, \lambda) \in \text{spec}(-iT, \Delta), j = 1, \ldots, N_{\tau, \lambda},\)

separate points of \(N\).

Continuity, compactness and (1) give an immersion

\[F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \rightarrow \mathbb{C}^N \setminus 0\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) separate points.

Embedding in \(\mathbb{C}^N \setminus 0\) of \((N, T) \in \mathcal{F}\).

\[\mathcal{E}_\lambda = \ker(\Delta - \lambda)\] is invariant under \(T,\)

\(-iT\) is symmetric, so \(\text{spec}(-iT|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\)

\(\alpha_t = \) one parameter group of diffeos generated by \(T\)

\(-iT \phi = \tau \phi \implies \phi(\alpha_t p) = e^{i \tau t} \phi(p)\)

If \(\tau_\ell < 0\), replace \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) by \(\overline{\phi_{\tau_\ell, \lambda_\ell, j_\ell}}\).
Outline of proofs

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[\mathcal{E}_\lambda = \ker(\Delta - \lambda)\] is invariant under \(\mathcal{T}\), \(-i\mathcal{T}\) is symmetric, so \(\text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\)

\[\mathcal{E}_{\tau,\lambda} = \{ \phi \in C^\infty(\mathcal{N}) : \Delta \phi = \lambda \phi, -i\mathcal{T} \phi = \tau \phi \}, \quad \text{spec}(-i\mathcal{T}, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau,\lambda} \neq 0 \} .\]

Pick orthonormal bases

\[\{ \phi_{\tau,\lambda,j} \in \mathcal{E}_{\tau,\lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau,\lambda}, (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta) \}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta) \]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C} T_{p_0}^* \mathcal{N} = \text{span}\{ d\phi_{\tau,\lambda,j}(p_0) : (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), j = 1, \ldots, N_{\tau,\lambda} \} ; \)

(2) the functions \(\phi_{\tau,\lambda,j}, (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), j = 1, \ldots, N_{\tau,\lambda} , \)

separate points of \(\mathcal{N}\).

Continuity, compactness and (1) give an immersion

\[F = (\phi_{\tau_1,\lambda_1,j_1}, \ldots, \phi_{\tau_N,\lambda_N,j_N}) : SE \rightarrow \mathbb{C}^N \setminus 0\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell,\lambda_\ell,j_\ell}\) separate points.
Outline of proofs

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[
\mathcal{E}_{\tau, \lambda} = \{ \phi \in C^\infty(\mathcal{N}) : \Delta \phi = \lambda \phi, -i \mathcal{T} \phi = \tau \phi \},
\]

\(
\text{spec}(-i \mathcal{T}, \Delta) = \{(\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq \emptyset \}.
\)

Pick orthonormal bases

\[
\{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-i \mathcal{T}, \Delta)
\]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C} T_{p_0}^* \mathcal{N} = \text{span}\{d \phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-i \mathcal{T}, \Delta), j = 1, \ldots, N_{\tau, \lambda} \} \)

(2) the functions \(\phi_{\tau, \lambda, j}, (\tau, \lambda) \in \text{spec}(-i \mathcal{T}, \Delta), j = 1, \ldots, N_{\tau, \lambda}, \)

separate points of \(\mathcal{N}\).

Continuity, compactness and (1) give an immersion

\[
F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : \text{SE} \rightarrow \mathbb{C}^N \setminus 0.
\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) separate points.

Since \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}(\alpha_t p) = e^{i \tau_\ell t} \phi_{\tau_\ell, \lambda_\ell, j_\ell}(p), F_* \mathcal{T} = i \sum \tau_\ell (z^\ell \partial_{z^\ell} - \bar{z}^\ell \partial_{\bar{z}^\ell})\). □
Outline of proofs

CR embedding in $S^{2N-1}$ of $(N, T, V) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

$\mathcal{F}_{\text{ell}} = \{ (N, T, V) : 
- (N, T) \in \mathcal{F}; 
- V \subset C_TN \text{ is an elliptic structure, } V \cap V = \text{span } C_TN; 
- \exists \beta \in C^\infty(N; V^*) \text{ s.t. } D\beta = 0, \langle \beta, T \rangle = -iK\beta \}

$K_\beta \subset V$ is a CR structure $\beta \sim \beta' \iff \exists u \text{ such that } \beta' = \beta + Du$.

Suppose $\beta \in \beta$, let $K_\beta = \ker \beta$.

From Cartan's formula

L$T_\beta = D(iT_\beta) + iT_D\beta = 0$

deduce $a_t$:

$K_\beta \to K_\beta$.

Let $H_\beta$ be the subbundle of $T_N$ s.t.

$C_{H_\beta} = K_\beta \oplus K_\beta$.

Let $J : H \to H$ be the complex structure such that $K_\beta = \{ v + iJv : v \in H_\beta \}$.

Pick a $T$-invariant metric $g$. Then $H_\beta \times H_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$ is a $T$-invariant hermitian metric on $H_\beta$.

Redefine $g$ to be this on $H_\beta$, and such that $T \perp H_\beta$, $|T| = 1$.

$H_\beta + \text{span } T = T_N$. Now $\langle \theta, v \rangle = g(T, v)$.

With this data (hermitian metric, Riemannian measure), let $\Box_b = \text{Laplacian of } \partial_b$ complex in any degree.

Note $L_T \Box_b = \Box_b L_T$. 

(Temple University)

Embedding theorems

Serra Negra, August 2011
Outline of proofs

CR embedding in $S^{2N-1}$ of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$. 

Suppose $\beta \in \mathcal{F}_{\text{ell}}$, let $
abla \beta \rightarrow \nabla \beta$.

From Cartan's formula $L_{\mathcal{T}} \beta = D_{\mathcal{T}} \beta + i \mathcal{T} D_{\mathcal{V}} \beta = 0$ deduce $\nabla \beta \rightarrow \nabla \beta$.

Let $H_{\beta}$ be the subbundle of $T\mathcal{N}$ s.t. $C_{H_{\beta}} = \nabla \beta \oplus \overline{\nabla \beta}$. Let $J : H \rightarrow H$ be the complex structure such that $\overline{\nabla \beta} = \{ v + iJv : v \in H_{\beta} \}$.

Pick a $\mathcal{T}$-invariant metric $g$. Then $H_{\beta} \times H_{\beta} \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$ is a $\mathcal{T}$-invariant hermitian metric on $H_{\beta}$.

Redefine $g$ to be this on $H_{\beta}$, and such that $\mathcal{T} \perp H_{\beta}$, $|\mathcal{T}| = 1$. $H_{\beta} + \text{span } \mathcal{T} = T\mathcal{N}$. Now $\langle \theta, v \rangle = g(\mathcal{T}, v)$.

With this data (hermitian metric, Riemannian measure), let $\square_{\beta}$ = Laplacian of $\partial_{\beta}$ complex in any degree.

Note $L_{\mathcal{T}} \square_{\beta} = \square_{\beta} L_{\mathcal{T}}$. 

(Temple University) 

Embedding theorems 

Serra Negra, August 2011 10 / 13
Outline of proofs

CR embedding in $S^{2N-1}$ of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

$\mathcal{F}_{\text{ell}} = \text{set of triples } (\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \text{ s.t.}:
- (\mathcal{N}, \mathcal{T}) \in \mathcal{F} ;
- \overline{\mathcal{V}} \subset \mathbb{C} \mathcal{T} \mathcal{N} \text{ is an elliptic structure, } \mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T} ;
- \mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T} ;
- \exists \beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*), \overline{\mathcal{D}} \beta = 0, \langle \beta, \mathcal{T} \rangle = -i.

$$\overline{\mathcal{K}}_\beta = \text{ker } \beta \subset \overline{\mathcal{V}} \text{ is a CR structure }$$

$\beta \sim \beta'$ iff there is $u$ real-valued such that
$$\beta' - \beta = \overline{\mathcal{D}} u.$$
Outline of proofs

Suppose $\beta \in \beta$, let $\overline{K}_\beta = \ker \beta$

From Cartan’s formula

$$\mathcal{L}_T \beta = \overline{D}(i_T \beta) + i_T \overline{D}_\beta = 0$$

deduce $\alpha_t : \overline{K}_\beta \to \overline{K}_\beta$.

CR embedding in $S^{2N-1}$ of $(N, T, \overline{V}) \in \mathcal{F}_{ell}$ with definite $\beta$.

$$\mathcal{F}_{ell} = \text{set of triples } (N, T, \overline{V}) \text{ s.t.:}$$
- $(N, T) \in \mathcal{F}$;
- $\overline{V} \subset \mathbb{C}T.N$ is an elliptic structure, $\mathcal{V} \cap \overline{V} = \text{span}_\mathbb{C} T$; and $\mathcal{V} \cap \overline{V} = \text{span}_\mathbb{C} T$;
- $\exists \beta \in C^\infty (N; \overline{V}^*)$, $\overline{D}_\beta = 0$, $\langle \beta, T \rangle = -i$.

$\overline{K}_\beta = \ker \beta \subset \overline{V}$ is a CR structure

$\beta \sim \beta'$ iff there is $u$ real-valued such that $\beta' - \beta = D_u$. 

Embedding theorems
Outline of proofs

Suppose $\beta \in \beta$, let $\overline{K}_\beta = \ker \beta$

From Cartan’s formula

$$\mathcal{L}_T \beta = \overline{D}(i_T \beta) + i_T \overline{D} \beta = 0$$

deduce $\alpha_t : \overline{K}_\beta \rightarrow \overline{K}_\beta$.

CR embedding in $S^{2N-1}$ of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

$\mathcal{F}_{\text{ell}} = \text{set of triples } (\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \text{ s.t.:}$

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\overline{\mathcal{V}} \subset \mathbb{C} \mathcal{T} \mathcal{N}$ is an elliptic structure, $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$; and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$;
- $\exists \beta \in C^\infty (\mathcal{N}; \overline{\mathcal{V}}^*)$, $\overline{D} \beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$.

$\overline{K}_\beta = \ker \beta \subset \overline{\mathcal{V}}$ is a CR structure

$\beta \sim \beta'$ iff there is $u$ real-valued such that $\beta' - \beta = \overline{D} u$. 

(Temple University)
Outline of proofs

Suppose $\beta \in \beta$, let $\mathcal{K}_\beta = \ker \beta$

From Cartan's formula

$$\mathcal{L}_T \beta = \overline{D}(i_T \beta) + i_T \overline{D} \beta = 0$$

deduce $\alpha_t : \mathcal{K}_\beta \rightarrow \mathcal{K}_\beta$.

Let $\mathcal{H}_\beta$ be the subbundle of $TN$ s.t.

$$\mathbb{C} \mathcal{H}_\beta = \mathcal{K}_\beta \oplus \mathcal{K}_\beta$$

Let $J : \mathcal{H} \rightarrow \mathcal{H}$ be the complex structure such that $\mathcal{K}_\beta = \{ v + iJv : v \in \mathcal{H}_\beta \}$

Pick a $T$-invariant metric $g$. Then

$$\mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$$

is a $T$-invariant hermitian metric on $\mathcal{H}_\beta$. 

CR embedding in $S^{2N-1}$ of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

$\mathcal{F}_{\text{ell}} = \text{set of triples } (\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \text{ s.t.:}$

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\overline{\mathcal{V}} \subset \mathbb{C} \mathcal{T} \mathcal{N}$ is an elliptic structure, $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$;
- $\forall \beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*), D\beta = 0, \langle \beta, \mathcal{T} \rangle = -i$.

$\mathcal{K}_\beta = \ker \beta \subset \overline{\mathcal{V}}$ is a CR structure

$\beta \sim \beta'$ iff there is $u$ real-valued such that $\beta' - \beta = Du$. 

(Temple University)  

Embedding theorems  

Serra Negra, August 2011
Outline of proofs

Suppose $\beta \in \beta$, let $\overline{K}_\beta = \ker \beta$

From Cartan’s formula
\[ \mathcal{L}_T \beta = \overline{D}(i_T \beta) + i_T \overline{D} \beta = 0 \]
deduce $a_t : \overline{K}_\beta \to \overline{K}_\beta$.

Let $\mathcal{H}_\beta$ be the subbundle of $TN$ s.t.
\[ \mathbb{C} \mathcal{H}_\beta = \mathcal{K}_\beta \oplus \overline{K}_\beta \quad \mathcal{H}_\beta + \text{span } T = TN \]

Let $J : \mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{K}_\beta = \{ v + iJv : v \in \mathcal{H}_\beta \}$

Pick a $T$-invariant metric $g$. Then
\[ \mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R} \]
is a $T$-invariant hermitian metric on $\mathcal{H}_\beta$. Redefine $g$ to be this on $\mathcal{H}_\beta$, and such that $T \perp \mathcal{H}_\beta$, $|T| = 1$
Outline of proofs

Suppose $\beta \in \beta$, let $\overline{\mathcal{K}}_\beta = \ker \beta$

From Cartan’s formula

$$\mathcal{L}_T \beta = \overline{D}(i_T \beta) + i_T \overline{D} \beta = 0$$

\[ \mathcal{L}_T \beta = \overline{D}(i_T \beta) + i_T \overline{D} \beta = 0 \]

deduce $\alpha_t : \overline{\mathcal{K}}_\beta \to \overline{\mathcal{K}}_\beta$.

Let $\mathcal{H}_\beta$ be the subbundle of $TN$ s.t.

$$\mathbb{C} \mathcal{H}_\beta = \mathcal{K}_\beta \oplus \overline{\mathcal{K}}_\beta \quad \mathcal{H}_\beta + \text{span} \ T = TN$$

Let $J:\mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{\mathcal{K}}_\beta = \{v + iJv : v \in \mathcal{H}_\beta\}$

Pick a $T$-invariant metric $g$. Then

$$\mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$$

is a $T$-invariant hermitian metric on $\mathcal{H}_\beta$. Redefine $g$ to be this on $\mathcal{H}_\beta$, and such that $T \perp \mathcal{H}_\beta$, $|T| = 1$. Now $\langle \theta, v \rangle = g(T, v)$.
Outline of proofs

Suppose $\beta \in \beta$, let $\overline{K}_\beta = \ker \beta$

From Cartan’s formula
\[ \mathcal{L}_{\mathcal{T}} \beta = \overline{D}(i_{\mathcal{T}} \beta) + i_{\mathcal{T}} \overline{\mathcal{D}} \beta = 0 \]
deduce $a_t : \overline{K}_\beta \to \overline{K}_\beta$.

Let $\mathcal{H}_\beta$ be the subbundle of $T\mathcal{N}$ s.t.
\[ \mathbb{C} \mathcal{H}_\beta = \mathcal{K}_\beta \oplus \overline{\mathcal{K}}_\beta \quad \mathcal{H}_\beta + \text{span } \mathcal{T} = T\mathcal{N} \]

Let $J : \mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{\mathcal{K}}_\beta = \{ v + iJv : v \in \mathcal{H}_\beta \}$

Pick a $\mathcal{T}$-invariant metric $g$. Then
\[ \mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R} \]
is a $\mathcal{T}$-invariant hermitian metric on $\mathcal{H}_\beta$. Redefine $g$ to be this on $\mathcal{H}_\beta$, and such that $\mathcal{T} \perp \mathcal{H}_\beta$, $|\mathcal{T}| = 1$. Now $\langle \theta, v \rangle = g(\mathcal{T}, v)$.

With this data (hermitian metric, Riemannian measure), let
\[ \square_b = \text{Laplacian of } \overline{\partial}_b \text{ complex in any degree.} \]
Outline of proofs

Suppose $\beta \in \beta$, let $\overline{K}_\beta = \ker \beta$

From Cartan’s formula

$$L_T \beta = i D(i_T \beta) + i_T D \beta = 0$$

deduce $a_t : \overline{K}_\beta \to \overline{K}_\beta$.

Let $\mathcal{H}_\beta$ be the subbundle of $TN$ s.t.

$$\mathbb{C} \mathcal{H}_\beta = K_\beta \oplus \overline{K}_\beta \quad \mathcal{H}_\beta + \text{span } T = TN$$

Let $J : \mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{K}_\beta = \{v + iJv : v \in \mathcal{H}_\beta\}$

Pick a $T$-invariant metric $g$. Then

$$\mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$$

is a $T$-invariant hermitian metric on $\mathcal{H}_\beta$. Redefine $g$ to be this on $\mathcal{H}_\beta$, and such that $T \perp \mathcal{H}_\beta$, $|T| = 1$. Now $\langle \theta, v \rangle = g(T, v)$.

With this data (hermitian metric, Riemannian measure), let

$$\Box_b = \text{Laplacian of } \overline{\partial}_b \text{ complex in any degree.}$$

Note $\mathcal{L}_T \Box_b = \Box_b \mathcal{L}_T$. 

CR embedding in $S^{2N-1}$ of $(\mathcal{N}, T, \overline{V}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

$\mathcal{F}_{\text{ell}} = \text{set of triples } (\mathcal{N}, T, \overline{V}) \text{ s.t.:}$

- $(\mathcal{N}, T) \in \mathcal{F}$;
- $\overline{V} \subset \mathbb{C}TN$ is an elliptic structure, $V \cap \overline{V} = \text{span}_\mathbb{C} T$; 
  and $V \cap \overline{V} = \text{span}_\mathbb{C} T$;
- $\exists \beta \in C^\infty(\mathcal{N}; \overline{V}^*)$, $D\beta = 0$, $\langle \beta, T \rangle = -i$.

$\overline{K}_\beta = \ker \beta \subset \overline{V}$ is a CR structure $\beta \sim \beta'$ iff there is $u$ real-valued such that $\beta' - \beta = Du$. 

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( Temple University )

Embedding theorems

Serra Negra, August 2011 10 / 13
Outline of proofs

Let $\mathcal{H}^q_{\partial_b} = \ker \Box_b \subset L^2(\mathcal{N}; \bigwedge^q \mathbb{C}^*)$.

Let $D = \{ \phi \in \mathcal{H}^q_{\partial_b} : L_T \phi \in L^2 \}$

$-iL_T : D \subset \mathcal{H}^q_{\partial_b} \to \mathcal{H}^q_{\partial_b}$

is a selfadjoint operator with compact parametrix.

CR embedding in $S^{2N-1}$ of $(\mathcal{N}, T, \overline{V}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$. 

Consequently $\text{spec}^q_0(-iL_T)$ is a discrete subset of $\mathbb{R}$.

The proof exploits $[\Box_b, L_T] = 0$ plus the fact that $\Box_b - L_T$ is elliptic (and that $\mathcal{N}$ is compact).

Theorem. Suppose that Levi $\theta$ is nondegenerate with $k$ positive and $n-k$ negative eigenvalues. Then

$\dim \mathcal{N} = 2n+1$

$\text{spec}^q_0(-iL_T)$ is finite if $q \neq k$, $n-k$;

$\text{spec}^k_0(-iL_T)$ contains only finitely many positive elements, and

$\text{spec}^{n-k}_0(-iL_T)$ contains only finitely many negative elements.

The case $k = n$ (or $k = 0$) is like Kodaira’s theorem on vanishing of cohomology.

If $k = n$ (or $0$), then $\text{spec}^0_0(iL_T)$ contains no negative (positive) elements.

This is because $\text{spec}^0_0(iL_T)$ is an additive subgroup of $\mathbb{R}$.
Outline of proofs

Let \( \mathcal{H}_{\partial_b}^q = \ker \Box_b \subset L^2(\mathcal{N}; \Lambda^q \mathcal{K}^*) \).

Let \( D = \{ \phi \in \mathcal{H}_{\partial_b}^q : \mathcal{L}_T \phi \in L^2 \} \)

\((*)\) \(-i\mathcal{L}_T : D \subset \mathcal{H}_{\partial_b}^q \to \mathcal{H}_{\partial_b}^q\)

is a selfadjoint operator with compact parametrix. Consequently

\( \text{spec}_0^q(-i\mathcal{L}_T) = \text{spectrum of } (*) \)

is a discrete subset of \( \mathbb{R} \).
Outline of proofs

Let \( \mathcal{H}^q_{\partial_b} = \ker \Box_b \subset L^2(\mathcal{N}; \bigwedge^q \mathcal{K}^*) \).

Let \( D = \{ \phi \in \mathcal{H}^q_{\partial_b} : \mathcal{L}_T \phi \in L^2 \} \)

\[(*) \quad -i \mathcal{L}_T : D \subset \mathcal{H}^q_{\partial_b} \to \mathcal{H}^q_{\partial_b} \]

is a selfadjoint operator with compact parametrix. Consequently

\[\text{spec}_0^q(-i \mathcal{L}_T) = \text{spectrum of (}*)\]

is a discrete subset of \( \mathbb{R} \).

CR embedding in \( S^{2N-1} \) of \((\mathcal{N}, \mathcal{T}, \mathcal{V}) \in \mathcal{F}_{\text{ell}}\) with definite \( \beta \).

The proof exploits \([\Box_b, \mathcal{L}_T] = 0\) plus the fact that \( \Box_b - \mathcal{L}_T^2 \) is elliptic (and that \( \mathcal{N} \) is compact).
Outline of proofs

Let \( \mathcal{H}^q = \ker \Box_b \subset L^2(\mathcal{N}; \wedge^q \overline{\mathcal{K}}^*) \).

Let \( D = \{ \phi \in \mathcal{H}^q : \mathcal{L}_T \phi \in L^2 \} \)

\[(*) \quad -i \mathcal{L}_T : D \subset \mathcal{H}^q \to \mathcal{H}^q \]

is a selfadjoint operator with compact parametrix. Consequently

\[ \text{spec}^q_0(-i \mathcal{L}_T) = \text{spectrum of (*)} \]

is a discrete subset of \( \mathbb{R} \).

**Theorem.** Suppose that \( \text{Levi}_\theta \) is nondegenerate with \( k \) positive and \( n - k \) negative eigenvalues. Then \( \dim \mathcal{N} = 2n + 1 \)

\[ \text{spec}^q_0(-i \mathcal{L}_T) \] is finite if \( q \neq k, n - k \);
\[ \text{spec}^k_0(-i \mathcal{L}_T) \] contains only finitely many positive elements, and
\[ \text{spec}^{n-k}_0(-i \mathcal{L}_T) \] contains only finitely many negative elements.

CR embedding in \( S^{2N-1} \) of \( (\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}} \) with definite \( \beta \).

The proof exploits \([\Box_b, \mathcal{L}_T] = 0\) plus the fact that \( \Box_b - \mathcal{L}_T^2 \) is elliptic (and that \( \mathcal{N} \) is compact).
Outline of proofs

Let $\mathcal{H}_q^{\partial_b} = \ker \Box_b \subset L^2(\mathcal{N}; \Lambda^q K^*)$.

Let $D = \{ \phi \in \mathcal{H}_q^{\partial_b} : L\mathcal{T}\phi \in L^2 \}$

\[ (*) \quad -iL_\mathcal{T} : D \subset \mathcal{H}_q^{\partial_b} \to \mathcal{H}_q^{\partial_b} \]

is a selfadjoint operator with compact parametrix. Consequently

\[ \text{spec}_0^q(-iL_\mathcal{T}) = \text{spectrum of (*)} \]

is a discrete subset of $\mathbb{R}$.

**Theorem.** Suppose that Levi $\theta$ is nondegenerate with $k$ positive and $n - k$ negative eigenvalues. Then $\dim \mathcal{N} = 2n + 1$

- \[ \text{spec}_0^q(-iL_\mathcal{T}) \text{ is finite if } q \neq k, \ n - k; \]
- \[ \text{spec}_0^k(-iL_\mathcal{T}) \text{ contains only finitely many positive elements, and} \]
- \[ \text{spec}_0^{n-k}(-iL_\mathcal{T}) \text{ contains only finitely many negative elements.} \]

The case $k = n$ (or $k = 0$) is like Kodaira's theorem on vanishing of cohomology.
Outline of proofs

CR embedding in $S^{2N-1}$ of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

Let $\mathcal{H}_b^q = \ker \Box_b \subset L^2(\mathcal{N}; \bigwedge^q \mathcal{V}^*)$.

Let $D = \{ \phi \in \mathcal{H}_b^q : \mathcal{L}_\mathcal{T} \phi \in L^2 \}$

\[
(*) \quad -i \mathcal{L}_\mathcal{T} : D \subset \mathcal{H}_b^q \to \mathcal{H}_b^q
\]

is a selfadjoint operator with compact parametrix. Consequently

\[
\text{spec}_0^q(-i \mathcal{L}_\mathcal{T}) = \text{spectrum of (*)}
\]

is a discrete subset of $\mathbb{R}$.

**Theorem.** Suppose that Levi$_\theta$ is nondegenerate with $k$ positive and $n - k$ negative eigenvalues. Then

\[
\dim \mathcal{N} = 2n + 1
\]

$\text{spec}_0^q(-i \mathcal{L}_\mathcal{T})$ is finite if $q \neq k$, $n - k$;

$\text{spec}_0^k(-i \mathcal{L}_\mathcal{T})$ contains only finitely many positive elements, and

$\text{spec}_0^{n-k}(-i \mathcal{L}_\mathcal{T})$ contains only finitely many negative elements.

The case $k = n$ (or $k = 0$) is like Kodaira’s theorem on vanishing of cohomology.

If $k = n$ (or 0), then $\text{spec}_0^0(i \mathcal{L}_\mathcal{T})$ contains no negative (positive) elements.
Outline of proofs

Let $\mathcal{H}^q = \ker \Box_{\partial_b} \subset L^2(\mathcal{N}; \bigwedge^q \mathcal{K}^*)$.

Let $D = \{ \phi \in \mathcal{H}^q_{\partial_b} : \mathcal{L}_T \phi \in L^2 \}$

\[ (*) \quad -i \mathcal{L}_T : D \subset \mathcal{H}^q_{\partial_b} \to \mathcal{H}^q_{\partial_b} \]

is a selfadjoint operator with compact parametrix. Consequently

\[ \text{spec}_0^q(-i \mathcal{L}_T) = \text{spectrum of} \ (*) \]

is a discrete subset of $\mathbb{R}$.

**Theorem.** Suppose that Levi$_\theta$ is nondegenerate with $k$ positive and $n - k$ negative eigenvalues. Then $\dim \mathcal{N} = 2n + 1$

\[ \text{spec}_0^q(-i \mathcal{L}_T) \text{ is finite if } q \neq k, \ n - k; \]
\[ \text{spec}_0^k(-i \mathcal{L}_T) \text{ contains only finitely many positive elements, and} \]
\[ \text{spec}_0^{n-k}(-i \mathcal{L}_T) \text{ contains only finitely many negative elements.} \]

This is because $\text{spec}_0^0(i \mathcal{L}_T)$ is an additive subgroup of $\mathbb{R}$.

The case $k = n$ (or $k = 0$) is like Kodaira’s theorem on vanishing of cohomology.

If $k = n$ (or 0), then $\text{spec}_0^0(i \mathcal{L}_T)$ contains no negative (positive) elements.
Outline of proofs

Let \( \{ \phi_\ell \}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}_0^0 \) consisting of eigenvectors of \(-iL_T\), \( \phi_\ell \in \mathcal{H}_q^q \). Then there are \( C, \mu > 0 \) such that

\[
\| \phi_\ell(p) \| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \; \ell \in \mathbb{N}_0.
\]

CR embedding in \( S^{2N-1} \) of \( (\mathcal{N}, T, \overline{V}) \in \mathcal{F}_{\text{ell}} \) with definite \( \beta \).
Outline of proofs

Let \( \{ \phi_\ell \}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}_0^0 \) consisting of eigenvectors of \( -i\mathcal{L}_T \), \( \phi_\ell \in \mathcal{H}_q^q \). Then there are \( C, \mu > 0 \) such that

\[
\| \phi_\ell(p) \| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \, \ell \in \mathbb{N}_0.
\]

This implies: The Fourier series of \( u \in C^\infty(\mathcal{N}) \cap \mathcal{H}_0^0 \) converges in \( C^\infty \).

CR embedding in \( S^{2N-1} \) of \((\mathcal{N}, T, \bar{V}) \in \mathcal{F}_{\text{ell}}\) with definite \( \beta \).
Outline of proofs

Let \( \{ \phi_\ell \}_{\ell=0}^\infty \) be an orthonormal basis of \( H^0_{\partial b} \) consisting of eigenvectors of \(-i\mathcal{L}_T\), \( \phi_\ell \in \mathcal{H}^q_{\partial b, \tau_\ell} \).

Then there are \( C, \mu > 0 \) such that

\[
\| \phi_\ell(p) \| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \; \ell \in \mathbb{N}_0.
\]

Consequence:

1. for all \( p_0 \in \mathcal{N} \), \( \text{span}\{ d\phi_\ell(p_0) : \ell = 0, 1, \ldots \} \) is the annihilator of \( \overline{K}_\beta \) in \( \mathbb{C} T^*_p \mathcal{N} \);
2. the functions \( \phi_\ell, \; \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

CR embedding in \( S^{2N-1} \) of \( (\mathcal{N}, T, \mathcal{V}) \in \mathcal{F}_{\text{ell}} \) with definite \( \beta \).

This implies: The Fourier series of \( u \in C^\infty(\mathcal{N}) \cap H^0_{\partial b} \) converges in \( C^\infty \).
Outline of proofs

Let \( \{\phi_\ell\}_{\ell=0}^{\infty} \) be an orthonormal basis of \( \mathcal{H}_0^{\partial_b} \) consisting of eigenvectors of \(-iL_T\), \( \phi_\ell \in \mathcal{H}_0^{q_{\partial_b,\tau_\ell}} \).

Then there are \( C, \mu > 0 \) such that

\[
\|\phi_\ell(p)\| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \ \ell \in \mathbb{N}_0.
\]

Consequence:

1. for all \( p_0 \in \mathcal{N} \), \( \text{span}\{d\phi_\ell(p_0) : \ell = 0, 1, \ldots\} \) is the annihilator of \( \overline{K}_\beta \) in \( \mathbb{C} T^*_p \mathcal{N} \);
2. the functions \( \phi_\ell, \ \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

This implies: The Fourier series of \( u \in C^\infty(\mathcal{N}) \cap \mathcal{H}_0^{\partial_b} \) converges in \( C^\infty \).

CR embedding in \( S^{2N-1} \) of \((\mathcal{N}, T, \overline{V}) \in \mathcal{F}_{\text{ell}}\) with definite \( \beta \).

The proofs of (1) and (2) use ideas of Boutet de Monvel.
Outline of proofs

Let \( \{ \phi_\ell \}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}^0_{\partial_b} \) consisting of eigenvectors of \( -iL_T \), \( \phi_\ell \in \mathcal{H}^q_{\partial_b, \tau_\ell} \).

Then there are \( C, \mu > 0 \) such that

\[
\| \phi_\ell(p) \| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \ \ell \in \mathbb{N}_0.
\]

This implies: The Fourier series of \( u \in C^\infty(\mathcal{N}) \cap \mathcal{H}^0_{\partial_b} \) converges in \( C^\infty \).

Consequence:

1. For all \( p_0 \in \mathcal{N} \), \( \text{span}\{ d\phi_\ell(p_0) : \ell = 0, 1, \ldots \} \) is the annihilator of \( \mathcal{K}_\beta \) in \( \mathbb{C} T^*_p \mathcal{N} \);

2. The functions \( \phi_\ell, \ \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

Theorem. Suppose \( \mathcal{K}_\beta \) is definite. There is an embedding \( F : \mathcal{N} \rightarrow \mathbb{C}^N \setminus 0 \) such that

\[
F_* T = i \sum_j \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j})
\]

and all \( \tau_j \) of the same sign.

\( F \) is constructed using eigenfunctions.
Outline of proofs

Let \( \{\phi_\ell\}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}_0^{\partial b} \) consisting of eigenvectors of \(-i\mathcal{L}_T\), \( \phi_\ell \in \mathcal{H}_0^{\partial b,\tau_\ell} \).

Then there are \( C, \mu > 0 \) such that

\[
\|\phi_\ell(p)\| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \; \ell \in \mathbb{N}_0.
\]

**Consequence:**

(1) for all \( p_0 \in \mathcal{N} \), \( \text{span}\{d\phi_\ell(p_0) : \ell = 0, 1, \ldots\} \) is the annihilator of \( \overline{K_\beta} \) in \( \mathbb{C} T_{p_0}^* \mathcal{N} \);

(2) the functions \( \phi_\ell, \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

**Theorem.** Suppose \( \overline{K_\beta} \) is definite. There is an embedding

\[
F : \mathcal{N} \rightarrow \mathbb{C}^N \setminus 0 \text{ such that}
\]

\[
F_* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j})
\]

and all \( \tau_j \) of the same sign.

\( F \) is constructed using eigenfunctions.
Outline of proofs

Let \( \{\phi_\ell\}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}^0_{\partial_b} \) consisting of eigenvectors of \(-iL_T\), \( \phi_\ell \in \mathcal{H}^q_{\partial_b,\tau_\ell} \).

Then there are \( C, \mu > 0 \) such that

\[
\|\phi_\ell(p)\| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \; \ell \in \mathbb{N}_0.
\]

This implies: The Fourier series of \( u \in C^\infty(\mathcal{N}) \cap \mathcal{H}^0_{\partial_b} \) converges in \( C^\infty \).

Consequence:

1. for all \( p_0 \in \mathcal{N} \), \( \text{span}\{d\phi_\ell(p_0) : \ell = 0, 1, \ldots\} \) is the annihilator of \( \overline{\mathcal{K}}_\beta \) in \( \mathbb{C} T_{p_0}^* \mathcal{N} \);
2. the functions \( \phi_\ell, \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

**Theorem.** Suppose \( \mathcal{K}_\beta \) is definite. There is an embedding

\[ F : \mathcal{N} \to \mathbb{C}^N \setminus 0 \text{ such that} \]

\[ F_* T = i \sum_j \tau_j (z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j}) \]

and all \( \tau_j \) of the same sign.

\( F \) is constructed using eigenfunctions.
End