The ideal boundary conditions of an elliptic complex of cone operators

Gerardo Mendoza

joint work with

Thomas Krainer

Geometric and Singular Analysis

Potsdam, February 2017

Outline

Task: Get a hold on the nature of the minimal and maximal domains of each of the operators of a first order elliptic complex of cone operators,

$$0 \to C_c^\infty(\mathring{\mathcal{X}}; E^0) \xrightarrow{A_0} C_c^\infty(\mathring{\mathcal{X}}; E^1) \to \cdots$$
$$\cdots \to C_c^\infty(\mathring{\mathcal{X}}; E^{m-1}) \xrightarrow{A_{m-1}} C_c^\infty(\mathring{\mathcal{X}}; E^m) \to 0,$$

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Fix a b-measure on $\mathcal{X}$, Hermitian forms on the vector bundles $E^q \to \mathcal{X}$ and a weight index $\gamma \in \mathbb{R}$, view the A_q as unbounded operators

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x def'n function for $\mathcal{Y} = \partial\mathcal{X}$,
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2. Analyze the structure of

$$\{(\mathcal{D}^0, \mathcal{D}^1 \dots, \mathcal{D}^{m-1}) : \mathcal{D}^q_{\min} \subset \mathcal{D}^q \subset \mathcal{D}^q_{\max},\ A_q(\mathcal{D}_q) \subset \mathcal{D}_{q+1}\ \forall q\}$$

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always compact

$\mathcal{X}$ is a manifold with boundary, $\mathfrak{m}_b$ a smooth positive b-density, the vector bundles $E^q \to \mathcal{X}$ are Hermitian, x is a positive defining function for $\mathcal{Y} = \partial\mathcal{X}$, and $\gamma \in \mathbb{R}$.

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Split $\mathcal{D}^q_{\max} = \mathcal{E}^q \oplus \mathcal{D}^q_{\min}$.

Task: Figure out $\mathcal{E}^q$ and $\mathcal{D}^q_{\min}$.

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one has[1]

$$\mathcal{D}_{\min}(A) = \mathcal{D}_{\max}(A) \cap \bigcap_{\varepsilon>0} x^{-\gamma+m-\varepsilon} H_b^m(\mathcal{X}; E)$$

$\mathcal{D}_{\min}(A) = x^{-\gamma+m} H_b^m(\mathcal{X}; E)$ iff $\mathrm{sp}_b(A) \cap \{\Im\sigma = \gamma - m\} = \emptyset$.

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With this knowledge of domains one can, e.g., determine the domain of the Friedrichs extension of a semibounded elliptic cone[1] or wedge[2] operator, set up and analyze boundary value problems[3] for elliptic wedge operators (simple edge),...

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Back to the complex

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Taylor → $A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$

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$\mathcal{H}^q_{\sigma_0}$ =cohomology of $(\dagger_q)$ in degree q,

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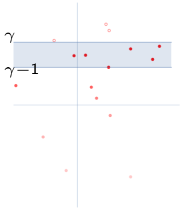

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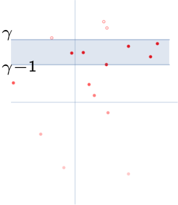

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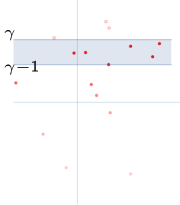

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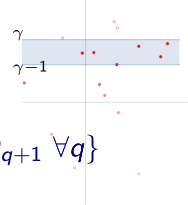

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cone-de Rham complex:

n even: $\mathcal{D}^q_{\min} = \mathcal{D}^q_{\max}$, all q;

n odd: $\mathcal{D}^q_{\min} = \mathcal{D}^q_{\max}$, $q \neq m$,
$\mathcal{D}^q_{\max}/\mathcal{D}^q_{\min} \approx H^m_{\mathrm{dR}}(\mathcal{Y})$, q=m

$[m = (n-1)/2]$

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$\mathcal{Y} = \partial\mathcal{X}$ Sobolev

Pick $\phi \in H^1(\mathcal{Y}; E^{q-1})$. Then $u = \omega(x) x^{\gamma + \frac{3}{2}} \phi \in \mathcal{D}_{\min}^{q-1}$, so $A_q u \in \mathcal{D}_{\min}^q$ but only L_b^2 regularity can be guaranteed.

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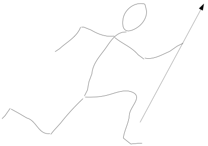

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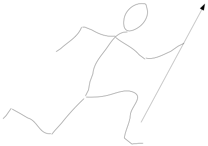

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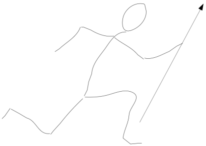

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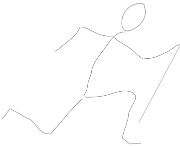

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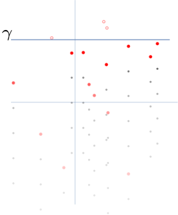

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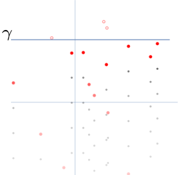

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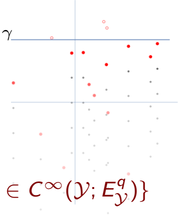

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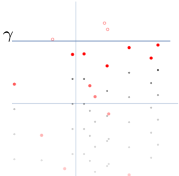

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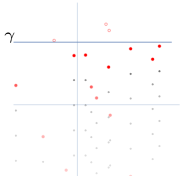

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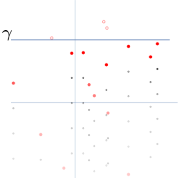
Taylor $\longrightarrow$ $A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$

$$A_{q+1}^{(0)} A_q^{(0)} = 0$$

$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A_{q-1}^{(0)}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A_q^{(0)}} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$



$$\mathcal{S}^q_{\sigma_0} = \{x^{i\sigma_0} \sum_{\text{finite}} a_\ell \log^\ell x, \ a_\ell \in C^\infty(\mathcal{Y}; E^q_{\mathcal{Y}})\}$$

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$u \in \mathcal{E}^q$. Ellipticity of $\Box_q$ gives expansion $u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j - ik} \in \mathcal{S}^q_{\sigma_j - ik}$

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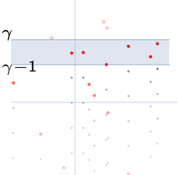

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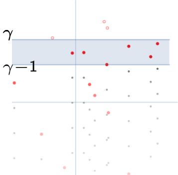

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$$u = \omega \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \gamma - 1 \leq \Im\sigma_j < \gamma}} u_{\sigma_j} + v, \quad v \in x^{-\gamma+1+\varepsilon} H_b^\infty \subset \mathcal{D}^q_{\min}$$

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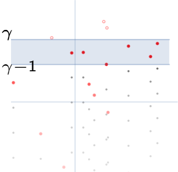

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$\omega \in C_c^\infty(\mathbb{R})$, $\gamma - 1 \leq \Im\sigma_j < \gamma$ ← $\underline{\Sigma}$

$\omega = 1$ near 0

$\underline{\Sigma} = \{\sigma \in \mathbb{C} : \gamma - 1 \leq \Im\sigma < \gamma\}$

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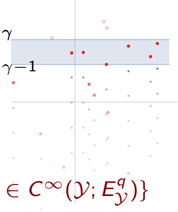

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also $u \in x^{-\gamma+\varepsilon} H_b^\infty$ $\Rightarrow$ $u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j - ik} \in \mathcal{S}^q_{\sigma_j - ik}$

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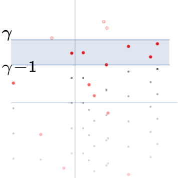

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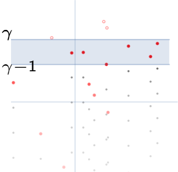

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$x\tilde{A}_q^{(1)}(u-v) \in x^{-\gamma+\varepsilon} H_b^\infty$ because $x\tilde{A}_q^{(1)} \in \mathrm{Diff}^1_b$

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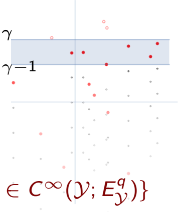

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$A_q(u-v)$ in $x^{-\gamma} L_b^2$

$x\tilde{A}_q^{(1)}(u-v) \in x^{-\gamma+\varepsilon} H_b^\infty$ because $x\tilde{A}_q^{(1)} \in \mathrm{Diff}^1_b$

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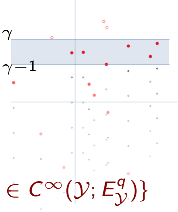

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$A_q(u-v)$ in $x^{-\gamma} L^2_b$; $A_q^{(0)}(u-v)$ $\therefore$ in $x^{-\gamma} L^2_b$; $x\tilde{A}_q^{(1)}(u-v) \in x^{-\gamma+\varepsilon} H_b^\infty$ because $x\tilde{A}_q^{(1)} \in \mathrm{Diff}^1_b$

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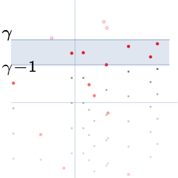

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$\therefore$ in $x^{-\gamma} L_b^2$

Consequence:

$$A_q^{(0)} u_{\sigma_j} = 0$$

because $\omega \mathcal{S}^{q+1}_{\sigma_j+i} \cap x^{-\gamma} L_b^2 = 0$ if $\Im\sigma_j \ge \gamma - 1$

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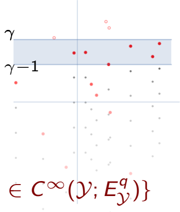

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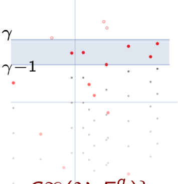
$x^{i\sigma_j+1} \qquad x^{i\sigma_j} \qquad x^{i\sigma_j - 1}$



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$u - v \in x^{-\gamma+\varepsilon} H_b^\infty$ and $A_q(u-v) \in x^{-\gamma} L_b^2$

$\implies u - v \in \mathcal{D}^q_{\max}$.

$\underline{\Sigma} = \{\sigma \in \mathbb{C} : \gamma - 1 \leq \Im\sigma < \gamma\}$

$$A_q(u-v) = A_q^{(0)}(u-v) + x\tilde{A}_q^{(1)}(u-v)$$

$\therefore$ in $x^{-\gamma} L_b^2$

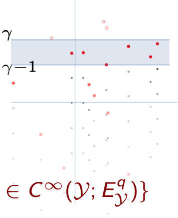
Consequence:

$$A_q^{(0)} u_{\sigma_j} = 0$$

because $\omega\mathcal{S}^{q+1}_{\sigma_j+i} \cap x^{-\gamma} L_b^2 = 0$ if $\Im\sigma_j \geq \gamma - 1$

$\gamma - 1 \leq \Im\sigma_j < \gamma \implies u_{\sigma_j}$ is a cocycle of

$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j-i} \xrightarrow{A_{q-1}^{(0)}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A_q^{(0)}} \mathcal{S}^{q+1}_{\sigma_j+i} \to \cdots$$



Conversely:

u_{σ_j} cocycle $\implies \omega u_{\sigma_j} \in \mathcal{D}^q_{\max}$.

$\mathcal{S}^q_{\sigma_0} = \{x^{i\sigma_0} \sum_{\text{finite}} a_\ell \log^\ell x, \ a_\ell \in C^\infty(\mathcal{Y}; E^q_{\mathcal{Y}})\}$

$\mathcal{E}^q$

$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j - ik} \in \mathcal{S}^q_{\sigma_j - ik}$

also $u \in x^{-\gamma+\varepsilon} H_b^\infty$

$u \in \mathcal{E}^q$. Ellipticity of $\Box_q$ gives expansion

$$u = \omega \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \gamma - 1 \leq \Im\sigma_j < \gamma}} u_{\sigma_j} + v, \quad v \in x^{-\gamma+1+\varepsilon} H_b^\infty \subset \mathcal{D}^q_{\min}$$

$\leftarrow \underline{\Sigma}$

$A_q \in \frac{1}{x}\mathrm{Diff}^1_b(\mathcal{X}; E^q, E^{q+1})$

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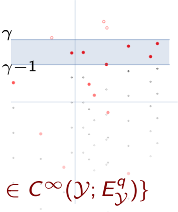

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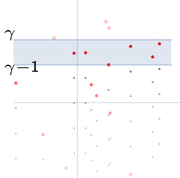

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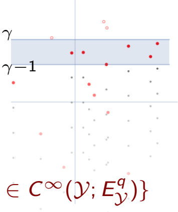

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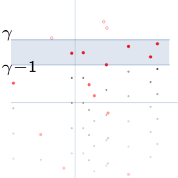

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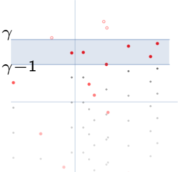

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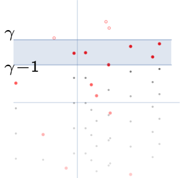

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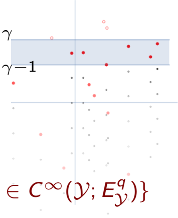

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Is that all?

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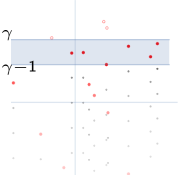

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Is that all? Show:

$$\sigma_0 \in \Sigma,\ u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}, \text{ and } \omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A^{(0)}_{q-1}\mathcal{S}^{q-1}_{\sigma_0 - i}$$

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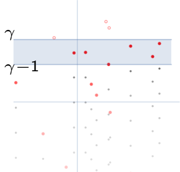

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$$\mathcal{D}^q_{\max} \cap \left(\bigcap_{\varepsilon > 0} x^{-\gamma+1-\varepsilon} H^1_b(\mathcal{X}; E^q)\right) \subset \mathcal{D}^q_{\min}$$

Next: If $v \in \mathcal{S}^{q-1}_{\sigma_0 - i}$, then $\omega v \in \mathcal{D}^{q-1}_{\min}$,

so $A_{q-1}(\omega v) \in \mathcal{D}^q_{\min}$, also $\omega A^{(0)}_q v \in \mathcal{D}^q_{\min}$

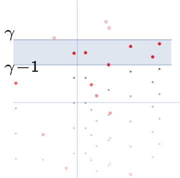
$A_q(\mathcal{D}^q_{\min}) \subset \mathcal{D}^{q+1}_{\min}$

$A_q(\mathcal{D}^q_{\max}) \subset \mathcal{D}^{q+1}_{\max}$

◎ Dismiss terms $u_{\sigma_j} \in A^{(0)}_{q-1}\mathcal{S}_{\sigma_0 - i}$.

Is that all? Show:

$$\sigma_0 \in \Sigma,\ u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}, \text{ and } \omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A^{(0)}_{q-1}\mathcal{S}^{q-1}_{\sigma_0 - i}$$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A^{(0)}_{q-1}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A^{(0)}_q} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$

$$\bigoplus_{\sigma_0 \in \Sigma} \mathcal{H}^q_{\sigma_0}(A^{(0)}) \approx \mathcal{D}^q_{\max}/\mathcal{D}^q_{\min}$$

$$\mathcal{S}^q_{\sigma_0} = \{x^{i\sigma_0} \sum_{\text{finite}} a_\ell \log^\ell x,\ a_\ell \in C^\infty(\mathcal{Y}; E^q_{\mathcal{Y}})\}$$

$\mathcal{E}^q$ Is that it?

$$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j-ik} \in \mathcal{S}^q_{\sigma_j-ik}$$

$u_0 \in \mathcal{S}^q_{\sigma_0}$, $\sigma_0 \in \underline{\Sigma}$ & $A_q^{(0)} u_0 = 0$

$$\Longrightarrow \ \omega u_0 \in \mathcal{D}^q_{\max}$$

$A_q \in \frac{1}{x}\mathrm{Diff}^1_b(\mathcal{X}; E^q, E^{q+1})$

$A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$

$A_{q+1}^{(0)} A_q^{(0)} = 0$

But if $\Im\sigma_0 = \gamma - 1$, then $\omega u_0 \in \mathcal{D}^q_{\min}$

◎ Dismiss terms $u_{\sigma_j} \in S^q_{\sigma_j}$ with $\Im\sigma_j = \gamma - 1$:

$$\omega \sum_{\sigma_j \in \underline{\Sigma}} u_{\sigma_j} \in \mathcal{D}^q_{\max} \Longleftrightarrow A_q^{(0)} u_{\sigma_j} = 0$$

$$u = \omega \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \gamma-1 < \Im\sigma_j < \gamma}} u_{\sigma_j} + v, \quad v \in \mathcal{D}^q_{\min}. \quad \longleftarrow \Sigma$$

$$\underline{\Sigma} = \{\sigma \in \mathbb{C} : \gamma - 1 \leq \Im\sigma < \gamma\}$$

$$\mathcal{D}^q_{\max} \cap \left(\bigcap_{\varepsilon>0} x^{-\gamma+1-\varepsilon} H^1_b(\mathcal{X}; E^q)\right) \subset \mathcal{D}^q_{\min}$$

Next: If $v \in \mathcal{S}^{q-1}_{\sigma_0 - i}$, then $\omega v \in \mathcal{D}^{q-1}_{\min}$,

so $A_{q-1}(\omega v) \in \mathcal{D}^q_{\min}$, also $\omega A_q^{(0)} v \in \mathcal{D}^q_{\min}$

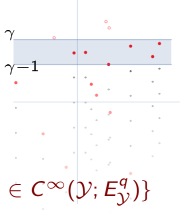
$A_q(\mathcal{D}^q_{\min}) \subset \mathcal{D}^{q+1}_{\min}$

$A_q(\mathcal{D}^q_{\max}) \subset \mathcal{D}^{q+1}_{\max}$

◎ Dismiss terms $u_{\sigma_j} \in A_{q-1}^{(0)} \mathcal{S}_{\sigma_0 - i}$.

Is that all? Show:

$$\sigma_0 \in \Sigma,\ u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0},\ \text{and}\ \omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \Longrightarrow u_{\sigma_0} \in A_{q-1}^{(0)} \mathcal{S}^{q-1}_{\sigma_0 - i}$$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A_{q-1}^{(0)}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A_q^{(0)}} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$

$$\bigoplus_{\sigma_0 \in \Sigma} \mathcal{H}^q_{\sigma_0}(A^{(0)}) \approx \mathcal{D}^q_{\max} / \mathcal{D}^q_{\min} \quad (\approx \mathcal{E}^q)$$

$$\mathcal{S}^q_{\sigma_0} = \{x^{i\sigma_0} \sum_{\text{finite}} a_\ell \log^\ell x,\ a_\ell \in C^\infty(\mathcal{Y}; E^q_{\mathcal{Y}})\}$$

$\mathcal{E}^q$

$$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j - ik} \in \mathcal{S}^q_{\sigma_j - ik}$$

$$A_q \in \tfrac{1}{x}\mathrm{Diff}^1_b(\mathcal{X}; E^q, E^{q+1})$$

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$$A_{q+1}^{(0)} A_q^{(0)} = 0$$

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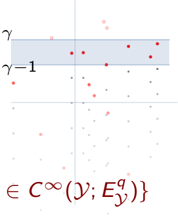
$$\Sigma = \{\sigma \in \mathbb{C} : \gamma - 1 < \Im\sigma < \gamma\}$$

$$\sigma_0 \in \Sigma,\ u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0},\ \text{and } \omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A_{q-1}^{(0)} \mathcal{S}^{q-1}_{\sigma_0 - i}$$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A_{q-1}^{(0)}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A_q^{(0)}} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$

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$\mathcal{E}^q$

$$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q)\\ \Im\sigma_j<\gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j-ik} \in \mathcal{S}^q_{\sigma_j-ik}$$

Suppose $\sigma_0 \in \Sigma$, $u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}$, $A^{(0)}_q u_{\sigma_0} = 0$.

$(\dagger)$ $[A_q(\omega u_{\sigma_0}), w]_q = 0$ for all $w \in \mathcal{D}_{\max}(A_q^\star)$

implies u_{σ_0} is $A^{(0)}$-exact

$$A_q \in \tfrac{1}{x}\mathrm{Diff}^1_b(\mathcal{X}; E^q, E^{q+1})$$

$$A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$$

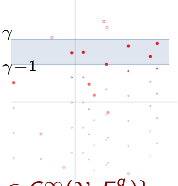
$$A^{(0)}_{q+1}A^{(0)}_q = 0$$

$$\omega \sum_{\sigma_j\in\Sigma} u_{\sigma_j} \in \mathcal{D}^q_{\max} \iff A^{(0)}_q u_{\sigma_j} = 0$$

$$\Sigma = \{\sigma \in \mathbb{C} : \gamma - 1 < \Im\sigma < \gamma\}$$

$$[u,w]_q = (A_q u, w) - (u, A_q^\star w), \quad u \in \mathcal{D}_{\max}(A_q),\ w \in \mathcal{D}_{\max}(A_q^\star)$$

$$\sigma_0 \in \Sigma,\ u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}, \text{ and } \omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A^{(0)}_{q-1}\mathcal{S}^{q-1}_{\sigma_0-i}$$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j-i} \xrightarrow{A^{(0)}_{q-1}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A^{(0)}_q} \mathcal{S}^{q+1}_{\sigma_j+i} \to \cdots$$

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Suppose $\sigma_0 \in \Sigma$, $u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}$, $A^{(0)}_q u_{\sigma_0} = 0$.

$A_q \in \frac{1}{x}\mathrm{Diff}^1_b(\mathcal{X}; E^q, E^{q+1})$

$A_q = A^{(0)}_q + x\tilde{A}^{(1)}_q$

$A^{(0)}_{q+1} A^{(0)}_q = 0$

(†) $[A_q(\omega u_{\sigma_0}), w]_q = 0$ for all $w \in \mathcal{D}_{\max}(A^\star_q)$

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$\mathfrak{D}^{q+1}_{\min}$, $\mathfrak{D}^{q+1}_{\max}$, $\mathfrak{E}^{q+1}$

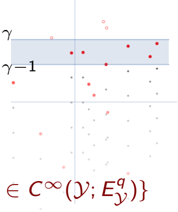
$$\omega \sum_{\sigma_j \in \Sigma} u_{\sigma_j} \in \mathcal{D}^q_{\max} \iff A^{(0)}_q u_{\sigma_j} = 0$$

$\omega u_{\sigma_0} \in \mathcal{D}_{\min} \implies$ (†), so $u_{\sigma_0} = A^{(0)}_{q-1} v$

$\Sigma = \{\sigma \in \mathbb{C} : \gamma - 1 < \Im\sigma < \gamma\}$

$[u, w]_q = (A_q u, w) - (u, A^\star_q w)$,
$u \in \mathcal{D}_{\max}(A_q)$, $w \in \mathcal{D}_{\max}(A^\star_q)$

$\sigma_0 \in \Sigma$, $u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}$, and $\omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A^{(0)}_{q-1}\mathcal{S}^{q-1}_{\sigma_0 - i}$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A^{(0)}_{q-1}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A^{(0)}_q} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$

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$\mathcal{E}^q$

$$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j - ik} \in \mathcal{S}^q_{\sigma_j - ik}$$

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$$\omega \sum_{\sigma_j \in \Sigma} u_{\sigma_j} \in \mathcal{D}^q_{\max} \iff A^{(0)}_q u_{\sigma_j} = 0$$

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$$[u, w]_q = (A_q u, w) - (u, A^\star_q w),$$
$$u \in \mathcal{D}_{\max}(A_q),\ w \in \mathcal{D}_{\max}(A^\star_q)$$

The standard pairing $[\cdot,\cdot]_q$ on $\mathcal{D}^q_{\max} \times \mathfrak{D}^{q+1}_{\max}$ descends to a nondegenerate pairing on

$$\bigoplus_{\sigma_0 \in \Sigma} \mathcal{H}^q_{\sigma_0}(A^{(0)}) \times \bigoplus_{\sigma'_0 \in \Sigma} \mathcal{H}^{q+1}_{\sigma'_0}(A^{\star(0)}).$$

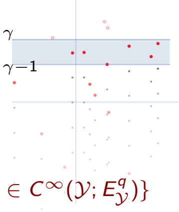
$$\sum_{\sigma_0 \in \Sigma} \mathbf{u}_{\sigma_0} \in \bigoplus_{\sigma_0 \in \Sigma} \mathcal{H}^q(A^{(0)}),$$

For the resulting pairing one has

$$\sum_{\sigma'_0 \in \Sigma} \mathbf{w}_{\sigma_0} \in \bigoplus_{\sigma'_0 \in \Sigma} \mathcal{H}^{q+1}(A^{\star(0)})$$

$$[\sum_{\sigma_0 \in \Sigma} \mathbf{u}_{\sigma_0}, \sum_{\sigma'_0 \in \Sigma} \mathbf{w}_{\sigma'_0}]_q = \sum_{\sigma_0 \in \Sigma} [\mathbf{u}_{\sigma_0}, \mathbf{w}_{\sigma_0^\star}],\ \sigma_0^\star = \overline{\sigma - i(2\gamma - 1)}$$

$$\sigma_0 \in \Sigma,\ u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}, \text{ and } \omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A^{(0)}_{q-1} \mathcal{S}^{q-1}_{\sigma_0 - i}$$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A^{(0)}_{q-1}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A^{(0)}_q} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$

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$(\approx \mathcal{E}^q)$

$$\mathcal{S}^q_{\sigma_0} = \{x^{i\sigma_0} \sum_{\text{finite}} a_\ell \log^\ell x,\ a_\ell \in C^\infty(\mathcal{Y}; E^q_{\mathcal{Y}})\}$$

$\mathcal{D}^q_{\mathrm{min}}$?

$\mathcal{D}^q_{\min}$?

Pick $\phi \in H^1(\mathcal{Y}; E^{q-1})$. Then $u = \omega(x)x^{\gamma+\frac{3}{2}}\phi \in \mathcal{D}^{q-1}_{\min}$, so $A_q u \in \mathcal{D}^q_{\min}$ but only L^2_b regularity can be guaranteed.

$\mathcal{D}^q_{\min}$

$u \in \mathcal{D}^q_{\min}$: there is $v \in \mathcal{D}^{q-1}_{\min}$ such that

$$u - A_{q-1} v \in \bigcap_{\varepsilon>0} x^{-\gamma+1-\varepsilon} H^1_b.$$

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$A \in x^{-1}\,\mathrm{Diff}^1_b$ elliptic, on $x^{-\gamma} L^2_b$:

a. $\mathcal{D}_{\min}(A) = \mathcal{D}_{\max}(A) \cap \Big(\bigcap_{\varepsilon>0} x^{-\gamma+1-\varepsilon} H^1_b \Big)$

b. $\mathcal{D}_{\min}(A) = x^{-\gamma+1} H^m_b \Longleftrightarrow \mathrm{sp}_b(A) \cap \{\Im\sigma = -1/2\} = \emptyset$.

$$\cdots \to \mathcal{D}^{q-1}_{\min} \to \mathcal{D}^{q}_{\min} \to \mathcal{D}^{q+1}_{\min} \to \cdots$$
$$x^{-\gamma} L^2_b = \operatorname{rg} A_{q-1,\min} \oplus \ker A^\star_{q-1,\max}.$$

$\mathcal{D}^q_{\min}$

$u \in \mathcal{D}^q_{\min}$: there is $v \in \mathcal{D}^{q-1}_{\min}$ such that
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$\mathcal{D}^q_{\min}$

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$$\therefore \mathcal{D}^q_{\min} = \left[\ker A^\star_{q-1,\max} \cap \mathcal{D}^q_{\min}\right] \oplus \operatorname{rg} A_{q-1,\min}.$$

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$(A_q + A^*_{q-1}) u' \in x^{-\gamma} L^2_b$

$u' \in \ker A^\star_{q-1,\max} \cap \mathcal{D}^q_{\min}$.

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$(A_q + A^*_{q-1})u' \in x^{-\gamma} L^2_b$

$$u' \in \ker A^\star_{q-1,\max} \cap \mathcal{D}^q_{\min}. \quad \Box_q u' = A^\star_q A_q u' \in x^{-\gamma-1} H^{-1}_b$$

Pick $\phi \in H^1(\mathcal{Y}; E^{q-1})$. Then $u = \omega(x) x^{\gamma+\frac{3}{2}} \phi \in \mathcal{D}^{q-1}_{\min}$, so $A_q u \in \mathcal{D}^q_{\min}$ but only L^2_b regularity can be guaranteed.

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$$\mathcal{D}^q_{\min} \ni u = u' + A_{q-1}v \in \left[\ker A^\star_{q-1,\max}\right] \oplus \operatorname{rg} A_{q-1,\min} \implies u' \in \mathcal{D}^q_{\min}$$
$$\therefore \mathcal{D}^q_{\min} = \left[\ker A^\star_{q-1,\max} \cap \mathcal{D}^q_{\min}\right] \oplus \operatorname{rg} A_{q-1,\min}. \qquad (A_q + A^*_{q-1})u' \in x^{-\gamma}L^2_b$$

$u' \in \ker A^\star_{q-1,\max} \cap \mathcal{D}^q_{\min}$. $\quad \square_q u' = A^\star_q A_q u' \in x^{-\gamma-1}H^{-1}_b$ so
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Pick $\phi \in H^1(\mathcal{Y}; E^{q-1})$. Then $u = \omega(x)x^{\gamma+\frac{3}{2}}\phi \in \mathcal{D}^{q-1}_{\min}$, so $A_q u \in \mathcal{D}^q_{\min}$ but only L^2_b regularity can be guaranteed.

$A \in x^{-1}\operatorname{Diff}^1_b$ elliptic, on $x^{-\gamma}L^2_b$:

a. $\mathcal{D}_{\min}(A) = \mathcal{D}_{\max}(A) \cap \left(\bigcap_{\varepsilon>0} x^{-\gamma+1-\varepsilon}H^1_b\right)$

b. $\mathcal{D}_{\min}(A) = x^{-\gamma+1}H^m_b \iff \operatorname{sp}_b(A) \cap \{\Im\sigma = -1/2\} = \emptyset$.

$\mathcal{D}^q_{\min}$

$$\cdots \to \mathcal{D}^{q-1}_{\min} \to \mathcal{D}^q_{\min} \to \mathcal{D}^{q+1}_{\min} \to \cdots$$
$$x^{-\gamma} L^2_b = \operatorname{rg} A_{q-1,\min} \oplus \ker A^\star_{q-1,\max}.$$

$u \in \mathcal{D}^q_{\min}$: there is $v \in \mathcal{D}^{q-1}_{\min}$ such that

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$\omega u_s \in \mathcal{D}^q_{\max}$. Then $u_c = u' - \omega u_s \in \mathcal{D}^q_{\min}$.

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Thus $\omega u_s \in \mathcal{D}^q_{\min}$, so u_s is $A^{(0)}$-exact: $u_s = A^{(0)}_q v_s$,

$$\omega u_s = A_{q-1} \omega v_s + w,$$

Get

$$u' = (u_c + w) + A_{q-1} \omega v_s.$$

$$\begin{aligned} A_{q-1} \omega v_s &= \omega A_{q-1} v_s - i a_{q-1}(d\omega)(v_s) \\ &= \omega u_s + \omega x A^{(1)}_{q-1} v_s - i a_{q-1}(d\omega)(v_s) \\ &= \omega u_s - w, \quad w \in x^{-\gamma+1} H^\infty_b \end{aligned}$$

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$$\mathfrak{G} = \bigcup_{\mathbf{d} \in \mathbb{N}_0^m} \mathfrak{G}_{\mathbf{d}} \leftrightarrow \text{Hilbert complexes associated with } A.$$

For each $\mathbf{d} \in \mathbb{N}_0^m$, the set $\mathfrak{G}_{\mathbf{d}}$ is an algebraic subvariety of $\prod_{q=0}^{m-1} \mathrm{Gr}_{d_q}(\mathcal{E}^q)$.

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Let V = finite-dimensional vector space, $\mathfrak{a} : V \to V$ linear,
$d_0, d_1 \in \mathbb{N}$. Then $\mathscr{V} = \{(X, Y) \in \mathrm{Gr}_{d_0}(V) \times \mathrm{Gr}_{d_1}(V) : \mathfrak{a}X \subset Y\}$
is an algebraic subvariety of $\mathrm{Gr}_{d_0}(V) \times \mathrm{Gr}_{d_1}(V)$.

End