Line bundles, $\mathbb{R}$-actions, complex and CR *b*-manifolds

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Geometric and Singular Analysis

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– In the presence of a positivity condition, embedding and vanishing theorems analogous to Kodaira's embedding theorem.

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The interior of a complex (or CR) b-manifold is a complex (of CR) manifold.

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The associated complex

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if X, Y are sections of ${}^bT\mathcal{M}$, then $[X, Y]|_{\mathcal{N}} = [X|_{\mathcal{N}}, Y|_{\mathcal{N}}]$

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Let $\iota_{\mathcal{N}}^*\beta_{\mathfrak{r}} = \mathfrak{r}^{-1}\overline{\mathbb{D}}\mathfrak{r}|_{\mathcal{N}}$. ${}^b\overline{\mathbb{D}}\beta_{\mathfrak{r}} = 0$ because $\overline{\mathbb{D}}(\mathfrak{r}^{-1}\overline{\mathbb{D}}\mathfrak{r}) = 0$.

$\iota_{\mathcal{N}}[X, Y] = [\iota_{\mathcal{N}}X, \iota_{\mathcal{N}}Y]$

if X, Y are sections of ${}^bT\mathcal{M}$, then $[X, Y]|_{\mathcal{N}} = [X|_{\mathcal{N}}, Y|_{\mathcal{N}}]$

CR b-structures

CR b-structures: Levi form

Suppose $\xi \in {}^bT^*_p\mathcal{M}$ and $\langle \xi, v \rangle = 0$ for all $v \in \overline{\mathcal{V}}_p$.

CR b-structures: Levi form

$\xi \in \mathrm{Char}(\mathcal{V})$

Suppose $\xi \in {}^bT^*_p\mathcal{M}$ and $\langle \xi, v\rangle = 0$ for all $v \in \overline{\mathcal{V}}_p$. If $\xi \neq 0$ then

${}^{\mathrm{b}}\mathcal{L}_\xi(X, Y) = \langle \xi, [X, \overline{Y}]_p\rangle$, X, Y sections of $\mathcal{V}$ near p.

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$$[aX, \overline{bY}] = a\overline{b}[X, \overline{Y}] + aX(\overline{b})\overline{Y} - \overline{bY}(a)X \Rightarrow$$

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$\mathcal{V}$ is strictly pseudoconvex at $\xi \in \mathrm{Char}\,\mathcal{V}$ if ${}^{\mathrm{b}}\mathcal{L}_\xi$ is positive definite.

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If $\mathcal{V}$ is a CR b-structure of CR codimension 1, then $\mathrm{Char}(\mathcal{V})$ is an orientable $\mathbb{R}$-line subbundle of ${}^bT^*\mathcal{M}$.

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CR b-structures: boundary

For a while $\mathcal{V}$ will be a strictly pseudoconvex CR b-structure on $\mathcal{M}$.

$\mathcal{V}_b = \iota\mathcal{V}|_{\mathcal{N}}$ has a characteristic variety: $\{\xi \in T^*\mathcal{N} : \iota^*_{\mathcal{N}}\xi \in \operatorname{Char}\mathcal{V}\}$.

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$$p \in \mathcal{C} \iff \exists \tau \in T_p\mathcal{N} \text{ s.t. } \mathfrak{r}\partial_{\mathfrak{r}} + i\tau \in \mathcal{V} \text{ at } p$$

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$\dim \mathcal{M} = 2n+1$, $\operatorname{rk} \mathcal{V} = n$.

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$\mathcal{V}_b + \overline{\mathcal{V}}_b$ drops dimension by 1 on $\mathcal{C}$

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$\mathcal{V}_b$ is a complex structure on $\mathcal{N}\backslash\mathcal{C}$

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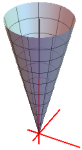

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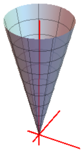

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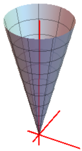

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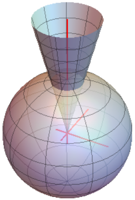

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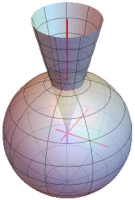

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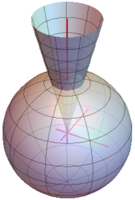

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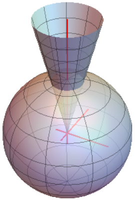

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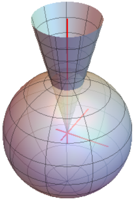
Then each component of $\mathcal{N}\backslash\mathcal{C}$ is Stein, therefore their boundaries are connected.



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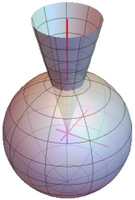
N. Hanges, H. Jacobowitz, The Euclidean elliptic complex, Indiana Univ. Math. J. **46** (1997), 753–770.



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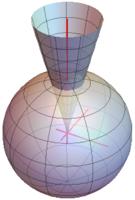

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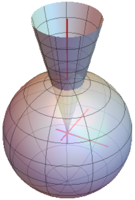
G.M., Strictly Pseudoconvex b-CR Manifolds, Comm. in PDE, **29** 9–10 (2005), 1437–1503



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Example

Let $\mathcal{N}' = S^{2n+1} \subset \mathbb{C}^{2(n+1)}$, $\mathcal{K}$ the span of

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The orbits of $\mathfrak{a}_t$ need not be closed. The closures of orbits are tori of possibly varying dimensions depending on the τ_j.

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$C^\infty(U, \mathfrak{g})$ is the space of smooth functions $f : \pi^{-1}(U) \to \mathfrak{g}$ such that f is constant on orbits of $\mathcal{T}$. $\{C^\infty(U, \mathfrak{g})\}_{\{U \text{ open}\}}$ is an abelian presheaf over $\mathcal{B}$ under addition

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$\mathcal{I}^\infty(U)$ is the space of smooth equivariant maps $h : \pi^{-1}(U) \to \pi^{-1}(U)$ such that $h(p) \in \overline{\mathcal{O}}_p$ for each $p \in \pi^{-1}(U)$.

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– Hence, by the long exact sequence in cohomology,

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Suppose $(\mathcal{N}, \mathcal{T}) \in \mathscr{F}$. There are $N \in \mathbb{N}$, $\tau_1, \ldots, \tau_N$ positive and an embedding

$$\Phi : \mathcal{N} \to S^{2N-1}$$

such that

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Suppose $\mathcal{N}$ arises as the boundary of a complex b-manifold and there is $\mathfrak{r}$ such that the CR structure $\ker \beta_{\mathfrak{r}} \subset \overline{\mathcal{V}}$ is strictly pseudoconvex and

$$\iota^* \beta_{\mathfrak{r}} = \mathfrak{r}^{-1} {}^b\overline{\partial} \mathfrak{r}$$
$$\langle \beta_{\mathfrak{r}}, \mathcal{T} \rangle = -i + a_{\mathfrak{r}}$$

$$\lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} a_{\mathfrak{r}}(\mathfrak{a}_s(p))\, ds = 0.$$

Then there is $\mathfrak{r}'$ and an embedding Φ as above, now a CR embedding of the CR manifold $\mathcal{N}$ with CR structure $\ker \beta_{\mathfrak{r}'}$.

metric g and a chosen Hermt

The trick in both cases is to eigenfunctions of $-i\mathcal{T}$ which are also eigenfunctions of suitable Laplacians. In the first use the metric g, in the second use CR functions (kernel of $\Box_b$)

G.M., Two embedding theorems, in Developments in Mathematics **28** (2013), pp. 399–429.

Vanishing theorems

There is a $\mathcal{T}$-invariant metric on $\overline{\mathcal{V}}_{\mathfrak{r}}$, with it define $\square_b$ on $C^\infty(\mathcal{N}\bigwedge^q\mathcal{K}^*_{\mathfrak{r}})$

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Theorem. Suppose that $\mathcal{L}_\theta$ is nondegenerate with k positive and $n-k$ negative eigenvalues. Then

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If $k=n$ (or 0), then $\operatorname{spec}^0_0(i\mathscr{L}_\mathcal{T})$ contains no negative (positive) elements.

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This is because $\operatorname{spec}^0_0(-i\mathscr{L}_\mathcal{T})$ is an additive sub-semigroup of $\mathbb{R}$.

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G.M., Hypoellipticity and vanishing theorems, Bull. Inst. Math. Acad. Sin. **8** (2013), 231–258.

Thank you