First order elliptic complexes of cone operators

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joint work with

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Geometric and Spectral Methods in Partial Differential Equations

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Outline

Task: Get a hold on the nature of the minimal and maximal domains of each of the operators of a first order elliptic complex of cone operators,

$$0 \to C_c^\infty(\mathring{\mathcal{X}}; E^0) \xrightarrow{A_0} C_c^\infty(\mathring{\mathcal{X}}; E^1) \to \cdots$$
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$$A_q : C_c^\infty(\mathring{\mathcal{X}}; E^q) \subset x^{-\gamma} L^2(\mathcal{X}; E^q) \to x^{-\gamma} L^2(\mathcal{X}; E^{q+1}).$$

x def'n function for $\mathcal{Y} = \partial\mathcal{X}$,
$x > 0$ in $\mathring{\mathcal{X}}$
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Maximal domain:

inner product $\langle u, v\rangle_q = \langle A_q u, A_q v\rangle + \langle u, v\rangle$

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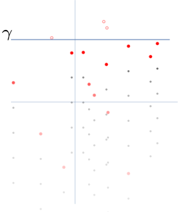

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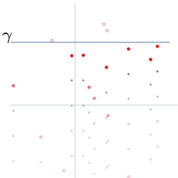

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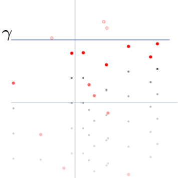

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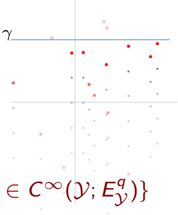

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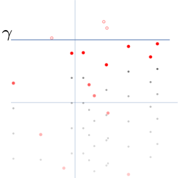

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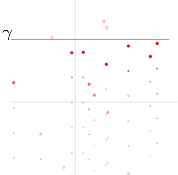

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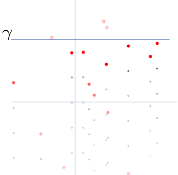

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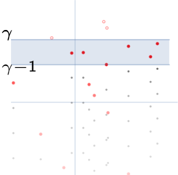

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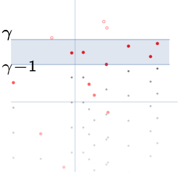

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$\underset{\approx \mathcal{D}^{q\,\perp}_{\min}}{\mathcal{E}^q} = \ker(A_q^\star A_q + \mathrm{I}) \cap \mathcal{D}^q_{\max} \subset \ker(\Box_q + \mathrm{I}) \cap \mathcal{D}^q_{\max}$

$$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j - ik} \in \mathcal{S}^q_{\sigma_j - ik}$$

Ellipticity of $\Box_q$:

$$u \in \mathcal{E}^q \implies u \in x^{-\gamma+\varepsilon} H_b^\infty,$$

$A_q \in \frac{1}{x}\mathrm{Diff}^1_b(\mathcal{X}; E^q, E^{q+1})$

Taylor $\longrightarrow$ $A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$

$A_{q+1}^{(0)} A_q^{(0)} = 0$

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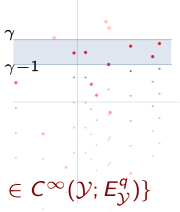
$$\cdots \to \underset{x^{i\sigma_j+1}}{\mathcal{S}^{q-1}_{\sigma_j - i}} \xrightarrow{A_{q-1}^{(0)}} \underset{x^{i\sigma_j}}{\mathcal{S}^q_{\sigma_j}} \xrightarrow{A_q^{(0)}} \underset{x^{i\sigma_j-1}}{\mathcal{S}^{q+1}_{\sigma_j + i}} \to \cdots$$



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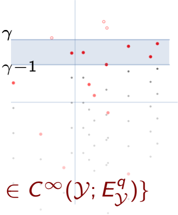

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$\omega \in C^\infty_c(\mathbb{R})$, $\omega = 1$ near 0

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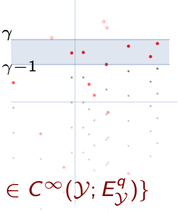
$x^{i\sigma_j + 1}$ $x^{i\sigma_j}$ $x^{i\sigma_j - 1}$



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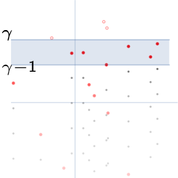

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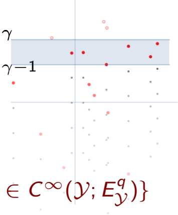

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$$A_q(u-v) = A_q^{(0)}(u-v) + x\tilde{A}_q^{(1)}(u-v)$$

$\in x^{-\gamma+\varepsilon} L^2_b$ because $x\tilde{A}_q^{(1)} \in \mathrm{Diff}^1_b$

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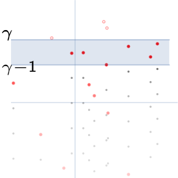
$x^{i\sigma_j + 1}$ $\quad x^{i\sigma_j}$ $\quad x^{i\sigma_j - 1}$



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$$A_q(u-v) = A_q^{(0)}(u-v) + x\tilde{A}_q^{(1)}(u-v)$$

$A_q(u-v)$: in $x^{-\gamma} L^2_b$

$x\tilde{A}_q^{(1)}(u-v)$: $\in x^{-\gamma+\varepsilon} L^2_b$ because $x\tilde{A}_q^{(1)} \in \mathrm{Diff}^1_b$

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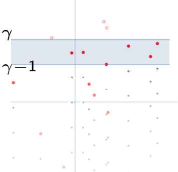
$$x^{i\sigma_j+1} \qquad x^{i\sigma_j} \qquad x^{i\sigma_j-1}$$



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$$A_q(u-v) = A_q^{(0)}(u-v) + x\tilde{A}_q^{(1)}(u-v)$$

$A_q(u-v)$ in $x^{-\gamma}L^2_b$; $A_q^{(0)}(u-v)$ $\therefore$ in $x^{-\gamma}L^2_b$; $x\tilde{A}_q^{(1)}(u-v)$ $\in x^{-\gamma+\varepsilon}L^2_b$ because $x\tilde{A}_q^{(1)}\in\mathrm{Diff}^1_b$

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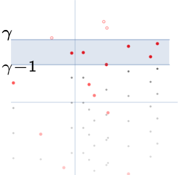

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$$A_q(u-v) = A_q^{(0)}(u-v) + x\tilde{A}_q^{(1)}(u-v)$$

$\therefore$ in $x^{-\gamma} L^2_b$

Consequence:

$$A_q^{(0)} u_{\sigma_j} = 0$$

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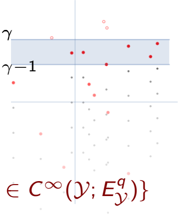
$x^{i\sigma_j + 1}$ $\quad x^{i\sigma_j}$ $\quad x^{i\sigma_j - 1}$



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Taylor $\to$ $A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$

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$\therefore$ in $x^{-\gamma} L^2_b$

Consequence:

$$A_q^{(0)} u_{\sigma_j} = 0$$

because $\omega \mathcal{S}^{q+1}_{\sigma_j + i} \cap x^{-\gamma} L^2_b = 0$ if $\Im\sigma \geq \gamma - 1$

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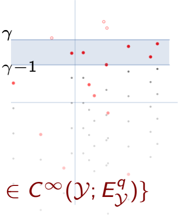
$$x^{i\sigma_j + 1} \qquad x^{i\sigma_j} \qquad x^{i\sigma_j - 1}$$



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$$u - v \in x^{-\gamma+\varepsilon} H^\infty_b \text{ and } A_q(u-v) \in x^{-\gamma} L^2_b \implies u - v \in \mathcal{D}^q_{\max}.$$

$$A_q(u-v) = A_q^{(0)}(u-v) + x\tilde{A}_q^{(1)}(u-v)$$

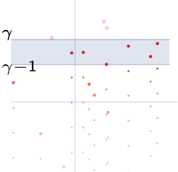
$\therefore$ in $x^{-\gamma} L^2_b$

Consequence:

$$A_q^{(0)} u_{\sigma_j} = 0 \qquad \text{because } \omega\mathcal{S}^{q+1}_{\sigma_j+i} \cap x^{-\gamma} L^2_b = 0 \text{ if } \Im\sigma \geq \gamma - 1$$

$\gamma - 1 \leq \Im\sigma_j < \gamma \implies u_{\sigma_j}$ is a cocycle of

$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A_{q-1}^{(0)}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A_q^{(0)}} \mathcal{S}^{q+1}_{\sigma_j+i} \to \cdots$$



Conversely:

$$u_{\sigma_j} \text{ cocycle} \implies \omega u_{\sigma_j} \in \mathcal{D}^q_{\max}.$$

$$\mathcal{S}^q_{\sigma_0} = \{x^{i\sigma_0} \sum_{\text{finite}} a_\ell \log^\ell x, \ a_\ell \in C^\infty(\mathcal{Y}; E^q_{\mathcal{Y}})\}$$

$\mathcal{E}^q = \ker(A_q^\star A_q + \mathrm{I}) \cap \mathcal{D}^q_{\max} \subset \ker(\Box_q + \mathrm{I}) \cap \mathcal{D}^q_{\max}$ $\quad \approx \mathcal{D}^{q\,\perp}_{\min}$

$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j - ik} \in \mathcal{S}^q_{\sigma_j - ik}$

Ellipticity of $\Box_q$:

$A_q \in \frac{1}{x}\mathrm{Diff}^1_b(\mathcal{X}; E^q, E^{q+1})$

$u \in \mathcal{E}^q \implies u \in x^{-\gamma+\varepsilon} H^\infty_b,$

Taylor $\to$ $A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$

$A_{q+1}^{(0)} A_q^{(0)} = 0$

$$u = \omega \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \gamma - 1 \leq \Im\sigma_j < \gamma}} u_{\sigma_j} + v, \quad v \in x^{-\gamma+1+\varepsilon} H^\infty_b \subset \mathcal{D}^q_{\min}$$

$\leftarrow \underline{\Sigma}$

$\omega \sum_{\sigma_j \in \underline{\Sigma}} u_{\sigma_j} \in \mathcal{D}^q_{\max} \iff A_q^{(0)} u_{\sigma_j} = 0$

$$u - v \in x^{-\gamma+\varepsilon} H^\infty_b \text{ and } A_q(u - v) \in x^{-\gamma} L^2_b \implies u - v \in \mathcal{D}^q_{\max}.$$

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Consequence:

$$A_q^{(0)} u_{\sigma_j} = 0$$

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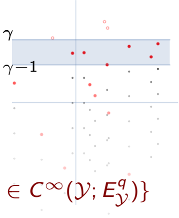

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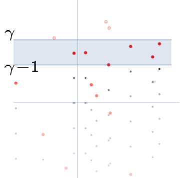

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$$u_0 \in \mathcal{S}^q_{\sigma_0},\ \sigma_0 \in \underline{\Sigma}\ \&\ A_q^{(0)} u_0 = 0$$

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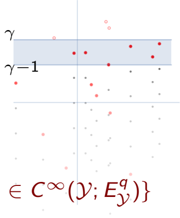

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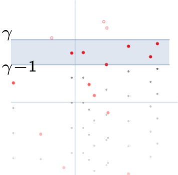

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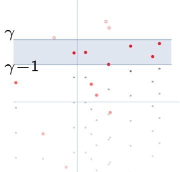

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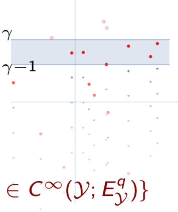

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So $A_{q-1}(\omega v) \in \mathcal{D}^q_{\min}$, also $\omega A^{(0)}_q v \in \mathcal{D}^q_{\min}$

$$A_q(\mathcal{D}^q_{\min}) \subset \mathcal{D}^q_{\min}$$

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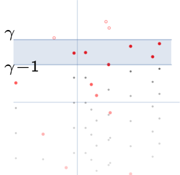

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$u_0 \in \mathcal{S}^q_{\sigma_0}$, $\sigma_0 \in \underline{\Sigma}$ & $A^{(0)}_q u_0 = 0$

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Is that all?

$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A^{(0)}_{q-1}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A^{(0)}_q} \mathcal{S}^{q+1}_{\sigma_j+i} \to \cdots$$

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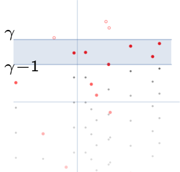

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$u_0 \in \mathcal{S}^q_{\sigma_0}$, $\sigma_0 \in \underline{\Sigma}$ & $A_q^{(0)} u_0 = 0$

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But if $\Im\sigma_0 = \gamma - 1$, then $\omega u_0 \in \mathcal{D}^q_{\min}$

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So $A_{q-1}(\omega v) \in \mathcal{D}^q_{\min}$, also $\omega A_q^{(0)} v \in \mathcal{D}^q_{\min}$

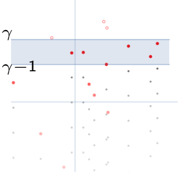
$A_q(\mathcal{D}^q_{\min}) \subset \mathcal{D}^q_{\min}$

$A_q(\mathcal{D}^q_{\max}) \subset \mathcal{D}^q_{\max}$

◎ Dismiss terms $u_{\sigma_j} \in A_{q-1}^{(0)} \mathcal{S}_{\sigma_0 - i}$.

Is that all? <u>Show</u>:

$\sigma_0 \in \Sigma$, $u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}$, and $\omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A_{q-1}^{(0)} \mathcal{S}^{q-1}_{\sigma_0 - i}$



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$\underline{\Sigma} = \{\sigma \in \mathbb{C} : \gamma - 1 \leq \Im\sigma < \gamma\}$

$\mathcal{D}^q_{\max} \cap \left(\bigcap_{\varepsilon > 0} x^{-\gamma+1-\varepsilon} H^1_b(\mathcal{X}; E^q)\right) \subset \mathcal{D}^q_{\min}$

Next: If $v \in \mathcal{S}^{q-1}_{\sigma_0 - i}$, then $\omega v \in \mathcal{D}^{q-1}_{\min}$.

So $A_{q-1}(\omega v) \in \mathcal{D}^q_{\min}$, also $\omega A^{(0)}_q v \in \mathcal{D}^q_{\min}$

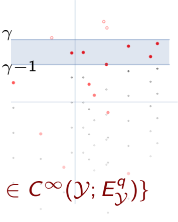
$A_q(\mathcal{D}^q_{\min}) \subset \mathcal{D}^q_{\min}$

$A_q(\mathcal{D}^q_{\max}) \subset \mathcal{D}^q_{\max}$

◎ Dismiss terms $u_{\sigma_j} \in A^{(0)}_{q-1} \mathcal{S}_{\sigma_0 - i}$.

Is that all? Show:

$\sigma_0 \in \Sigma$, $u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}$, and $\omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A^{(0)}_{q-1} \mathcal{S}^{q-1}_{\sigma_0 - i}$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A^{(0)}_{q-1}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A^{(0)}_q} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$

$$\bigoplus_{\sigma_0 \in \Sigma} \mathcal{H}^q_{\sigma_0}(A^{(0)}) \approx \mathcal{D}^q_{\max} / \mathcal{D}^q_{\min}$$

$\mathcal{S}^q_{\sigma_0} = \{x^{i\sigma_0} \sum_{\text{finite}} a_\ell \log^\ell x, \ a_\ell \in C^\infty(\mathcal{Y}; E^q_{\mathcal{Y}})\}$

$\mathcal{E}^q$

$$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q)\\ \Im\sigma_j<\gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j-ik}\in\mathcal{S}^q_{\sigma_j-ik}$$

$$A_q \in \tfrac{1}{x}\mathrm{Diff}^1_b(\mathcal{X};E^q,E^{q+1})$$

$$A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$$

$$A_{q+1}^{(0)}A_q^{(0)} = 0$$

$u_0 \in \mathcal{S}^q_{\sigma_0}$, $\sigma_0 \in \underline{\Sigma}$ & $A_q^{(0)}u_0 = 0$

$$\implies \omega u_0 \in \mathcal{D}^q_{\max}$$

But if $\Im\sigma_0 = \gamma - 1$, then $\omega u_0 \in \mathcal{D}^q_{\min}$

◎ Dismiss terms $u_{\sigma_j} \in S^q_{\sigma_j}$ with $\Im\sigma_j = \gamma - 1$:

$$\omega\sum_{\sigma_j\in\underline{\Sigma}} u_{\sigma_j}\in\mathcal{D}^q_{\max} \iff A_q^{(0)}u_{\sigma_j} = 0$$

$$u = \omega \sum_{\substack{\sigma_j\in\mathrm{sp}_b(\Box_q)\\ \gamma-1<\Im\sigma_j<\gamma}} u_{\sigma_j} + v, \quad v \in \mathcal{D}^q_{\min}.$$

$\leftarrow \Sigma$

$$\underline{\Sigma} = \{\sigma\in\mathbb{C} : \gamma - 1 \leq \Im\sigma < \gamma\}$$

$$\mathcal{D}^q_{\max}\cap\Big(\bigcap_{\varepsilon>0} x^{-\gamma+1-\varepsilon}H^1_b(\mathcal{X};E^q)\Big)\subset\mathcal{D}^q_{\min}$$

Next: If $v \in \mathcal{S}^{q-1}_{\sigma_0 - i}$, then $\omega v \in \mathcal{D}^{q-1}_{\min}$.

So $A_{q-1}(\omega v) \in \mathcal{D}^q_{\min}$, also $\omega A_q^{(0)} v \in \mathcal{D}^q_{\min}$

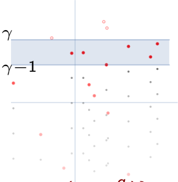
$$A_q(\mathcal{D}^q_{\min}) \subset \mathcal{D}^q_{\min}$$

$$A_q(\mathcal{D}^q_{\max}) \subset \mathcal{D}^q_{\max}$$

◎ Dismiss terms $u_{\sigma_j} \in A^{(0)}_{q-1}\mathcal{S}_{\sigma_0 - i}$.

Is that all? Show:

$$\sigma_0 \in \Sigma,\ u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}, \text{ and } \omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A^{(0)}_{q-1}\mathcal{S}^{q-1}_{\sigma_0 - i}$$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A^{(0)}_{q-1}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A^{(0)}_q} \mathcal{S}^{q+1}_{\sigma_j+i} \to \cdots$$

$$\bigoplus_{\sigma_0\in\Sigma} \mathcal{H}^q_{\sigma_0}(A^{(0)}) \approx \mathcal{D}^q_{\max}/\mathcal{D}^q_{\min}$$

$(\approx \mathcal{E}^q)$

$$\mathcal{S}^q_{\sigma_0} = \{x^{i\sigma_0}\sum_{\text{finite}} a_\ell \log^\ell x,\ a_\ell \in C^\infty(\mathcal{Y};E^q_{\mathcal{Y}})\}$$

$\mathcal{E}^q$

$$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j - ik} \in \mathcal{S}^q_{\sigma_j - ik}$$

$$A_q \in \tfrac{1}{x}\mathrm{Diff}^1_b(\mathcal{X}; E^q, E^{q+1})$$

$$A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$$

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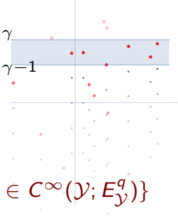
$$\Sigma = \{\sigma \in \mathbb{C} : \gamma - 1 < \Im\sigma < \gamma\}$$

$$\sigma_0 \in \Sigma,\ u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0},\ \text{and } \omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A_{q-1}^{(0)} \mathcal{S}^{q-1}_{\sigma_0 - i}$$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A_{q-1}^{(0)}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A_q^{(0)}} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$

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$A_q = A^{(0)}_q + x\tilde{A}^{(1)}_q$

$A^{(0)}_{q+1}A^{(0)}_q = 0$

Suppose $\sigma_0 \in \Sigma$, $u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}$, $A^{(0)}_q u_{\sigma_0} = 0$.

(†) $[A_q(\omega u_{\sigma_0}), w]_q = 0$ for all $w \in \mathcal{D}_{\max}(A^\star_q)$

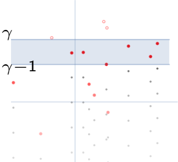
implies u_{σ_0} is $A^{(0)}$-exact

$$\omega \sum_{\sigma_j \in \Sigma} u_{\sigma_j} \in \mathcal{D}^q_{\max} \iff A^{(0)}_q u_{\sigma_j} = 0$$

$$\Sigma = \{\sigma \in \mathbb{C} : \gamma - 1 < \Im\sigma < \gamma\}$$

$$[u, w]_q = (A_q u, w) - (u, A^\star_q w),$$
$$u \in \mathcal{D}_{\max}(A_q),\ w \in \mathcal{D}_{\max}(A^\star_q)$$

$$\sigma_0 \in \Sigma,\ u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}, \text{ and } \omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A^{(0)}_{q-1}\mathcal{S}^{q-1}_{\sigma_0 - i}$$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A^{(0)}_{q-1}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A^{(0)}_q} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$

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$$u \sim \sum_{\substack{\sigma_j \in \mathrm{sp}_b(\Box_q) \\ \Im\sigma_j < \gamma}} \sum_{k=0}^{\infty} u_{\sigma_j - ik}, \quad u_{\sigma_j - ik} \in \mathcal{S}^q_{\sigma_j - ik}$$

Suppose $\sigma_0 \in \Sigma$, $u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}$, $A_q^{(0)} u_{\sigma_0} = 0$.

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$A_q = A_q^{(0)} + x\tilde{A}_q^{(1)}$

$A_{q+1}^{(0)} A_q^{(0)} = 0$

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implies u_{σ_0} is $A^{(0)}$-exact $\quad \mathfrak{D}^{q+1}_{\min}, \mathfrak{D}^{q+1}_{\max}, \mathfrak{E}^{q+1}$

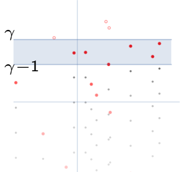
$$\omega \sum_{\sigma_j \in \Sigma} u_{\sigma_j} \in \mathcal{D}^q_{\max} \iff A_q^{(0)} u_{\sigma_j} = 0$$

$\omega u_{\sigma_0} \in \mathcal{D}_{\min} \implies$ (†), so $u_{\sigma_0} = A_{q-1}^{(0)} v$

$$\Sigma = \{\sigma \in \mathbb{C} : \gamma - 1 < \Im\sigma < \gamma\}$$

$$[u, w]_q = (A_q u, w) - (u, A_q^\star w), \quad u \in \mathcal{D}_{\max}(A_q),\ w \in \mathcal{D}_{\max}(A_q^\star)$$

$\sigma_0 \in \Sigma$, $u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}$, and $\omega u_{\sigma_0} \in \mathcal{D}^q_{\min} \implies u_{\sigma_0} \in A_{q-1}^{(0)} \mathcal{S}^{q-1}_{\sigma_0 - i}$



$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A_{q-1}^{(0)}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A_q^{(0)}} \mathcal{S}^{q+1}_{\sigma_j + i} \to \cdots$$

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Suppose $\sigma_0 \in \Sigma$, $u_{\sigma_0} \in \mathcal{S}^q_{\sigma_0}$, $A^{(0)}_q u_{\sigma_0} = 0$.

$(\dagger)$ $[A_q(\omega u_{\sigma_0}), w]_q = 0$ for all $w \in \mathcal{D}_{\max}(A^\star_q)$

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$\mathfrak{D}^{q+1}_{\min}$, $\mathfrak{D}^{q+1}_{\max}$, $\mathfrak{E}^{q+1}$

$$A_q \in \tfrac{1}{x}\mathrm{Diff}^1_b(\mathcal{X};E^q,E^{q+1})$$
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$$\omega u_{\sigma_0} \in \mathcal{D}_{\min} \implies (\dagger), \text{ so } u_{\sigma_0} = A^{(0)}_{q-1}v$$

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$$\Sigma = \{\sigma\in\mathbb{C} : \gamma - 1 < \Im\sigma < \gamma\}$$

The standard pairing $[\cdot,\cdot]_q$ on $\mathcal{D}^q_{\max} \times \mathfrak{D}^{q+1}_{\max}$ descends to a nondegenerate pairing on

$$\bigoplus_{\sigma_0\in\Sigma} \mathcal{H}^q_{\sigma_0}(A^{(0)}) \times \bigoplus_{\sigma_0'\in\Sigma} \mathcal{H}^{q+1}_{\sigma_0'}(A^{\star(0)}).$$

For the resulting pairing one has

$$[\sum_{\sigma_0\in\Sigma} \mathbf{u}_{\sigma_0}, \sum_{\sigma_0'\in\Sigma} \mathbf{w}_{\sigma_0'}]_q = \sum_{\sigma_0\in\Sigma} [\mathbf{u}_{\sigma_0}, \mathbf{w}_{\sigma_0^\star}],\ \sigma_0^\star = \overline{\sigma - i(2\gamma-1)}$$

$$[u,w]_q = (A_q u, w) - (u, A^\star_q w),$$
$$u \in \mathcal{D}_{\max}(A_q),\ w\in\mathcal{D}_{\max}(A^\star_q)$$

$$\sum_{\sigma_0\in\Sigma}\mathbf{u}_{\sigma_0} \in \bigoplus_{\sigma_0\in\Sigma}\mathcal{H}^q(A^{(0)}),$$

$$\sum_{\sigma_0'\in\Sigma}\mathbf{w}_{\sigma_0} \in \bigoplus_{\sigma_0'\in\Sigma}\mathcal{H}^{q+1}(A^{\star(0)})$$

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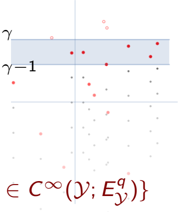

$$\cdots \to \mathcal{S}^{q-1}_{\sigma_j - i} \xrightarrow{A^{(0)}_{q-1}} \mathcal{S}^q_{\sigma_j} \xrightarrow{A^{(0)}_q} \mathcal{S}^{q+1}_{\sigma_j+i} \to \cdots$$

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$$\mathcal{S}^q_{\sigma_0} = \{x^{i\sigma_0}\sum_{\text{finite}} a_\ell \log^\ell x,\ a_\ell \in C^\infty(\mathcal{Y}; E^q_{\mathcal{Y}})\}$$

Hilbert complexes

$\pi^q_{\max}$ = orthoprojection $\mathcal{D}^q_{\max} \to \mathcal{E}^q$

$\mathfrak{a}^q = \pi^{q+1}_{\max} A_q$

$\boldsymbol{\mathcal{D}} = (\mathcal{D}^0, \ldots, \mathcal{D}^{m-1})$ with $\mathcal{D}^q_{\min} \subset \mathcal{D}^q \subset \mathcal{D}^q_{\max}$, $q = 0, \ldots, m-1$,

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decompose $\mathcal{D}^q = D^q + \mathcal{D}^q_{\min}$, $D^q \subset \mathcal{E}^q$,

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Set

$$\mathfrak{G}_{\mathbf{d}} = \{\boldsymbol{D} \in \prod_{q=0}^{m-1} \mathrm{Gr}_{d_q}(\mathcal{E}^q) : \mathfrak{a}_q D^q \subset D^{q+1},\ q = 0, \ldots, m-1\},$$

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Then

$$\mathfrak{G} = \bigcup_{\mathbf{d} \in \mathbb{N}_0^m} \mathfrak{G}_{\mathbf{d}} \leftrightarrow \text{Hilbert complexes associated with } A.$$

For each $\mathbf{d} \in \mathbb{N}_0^m$, the set $\mathfrak{G}_{\mathbf{d}}$ is an algebraic subvariety of $\prod_{q=0}^{m-1} \mathrm{Gr}_{d_q}(\mathcal{E}^q)$.

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For each $\mathbf{d} \in \mathbb{N}_0^m$, the set $\mathfrak{G}_{\mathbf{d}}$ is an algebraic subvariety of $\prod_{q=0}^{m-1} \mathrm{Gr}_{d_q}(\mathcal{E}^q)$.

Let V = finite-dimensional vector space, $\mathfrak{a} : V \to V$ linear,
$d_0, d_1 \in \mathbb{N}$. Then $\mathscr{V} = \{(X, Y) \in \mathrm{Gr}_{d_0}(V) \times \mathrm{Gr}_{d_1}(V) : \mathfrak{a}X \subset Y\}$
is an algebraic subvariety of $\mathrm{Gr}_{d_0}(V) \times \mathrm{Gr}_{d_1}(V)$.

$\mathcal{D}_{\min}$

$u \in \mathcal{D}^q_{\min}$: there is $v \in \mathcal{D}^{q-1}_{\min}$ such that

$$u - A_{q-1} v \in \bigcap_{\varepsilon > 0} x^{-\gamma+1-\varepsilon} H^1_b.$$

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$A \in x^{-1}\,\mathrm{Diff}^1_b$ elliptic, on $x^{-\gamma} L^2_b$:

a. $\mathcal{D}_{\min}(A) = \mathcal{D}_{\max}(A) \cap \big(\bigcap_{\varepsilon>0} x^{-\gamma+1-\varepsilon} H^1_b \big)$

b. $\mathcal{D}_{\min}(A) = x^{-\gamma+1} H^m_b \iff \mathrm{sp}_b(A) \cap \{\Im\sigma = -\nu/2\} = \emptyset.$

$\mathcal{D}_{\min}$

$$\cdots \to \mathcal{D}^{q-1}_{\min} \to \mathcal{D}^{q}_{\min} \to \mathcal{D}^{q+1}_{\min} \to \cdots$$
$$x^{-\gamma}L^2_b = \operatorname{rg} A_{q-1,\min} \oplus \ker A^\star_{q-1,\max}.$$

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$$A_q \omega u_s = A_q u' - A_q u_c \in \textstyle\bigcap_{\varepsilon>0} x^{-\gamma-\varepsilon} L^2_b \implies A^{(0)}_q(u_s) = 0 \text{ so}$$

$\omega u_s \in \mathcal{D}^q_{\max}$. Then $u_c = u' - \omega u_s \in \mathcal{D}^q_{\min}$.

$$\mathcal{D}^q_{\max} \cap \bigcap_{\varepsilon>0} x^{-\gamma+1-\varepsilon} H^1_b \subset \mathcal{D}^q_{\min}$$

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Thus $\omega u_s \in \mathcal{D}^q_{\min}$, so u_s is $A^{(0)}$-exact: $u_s = A^{(0)}_q v_s$,

$$\omega u_s = A_{q-1} \omega v_s + w,$$

$$\begin{aligned} A_{q-1}\omega v_s &= \omega A_{q-1} v_s - i a_{q-1}(d\omega)(v_s) \\ &= \omega u_s + \omega x A^{(1)}_{q-1} v_s - i a_{q-1}(d\omega)(v_s) \\ &= \omega u_s - w, \qquad w \in x^{-\gamma+1} H^\infty_b \end{aligned}$$

Get

$$u' = (u_c + w) + A_{q-1} \omega v_s.$$

End