Domains of closed extensions of ODEs
Part IV

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Overview

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- Then I’ll describe the Friedrichs extension. This is a classical procedure to construct a selfadjoint extension of a symmetric semibounded operator.

Semibounded means symmetric on $\mathcal{C}_\infty(M, \mathcal{C}_n)$, plus there is $C \in \mathbb{R}$ such that $(A\phi, \phi) \geq C\|\phi\|^2$ for all $\phi \in \mathcal{C}_\infty(M)$. We will assume the inequality as stated.
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Semibounded means symmetric on $C_c^\infty(\hat{M}, \mathbb{C}^n)$, plus there is $C \in \mathbb{R}$ such that

$$(A\phi, \phi) \geq C\|\phi\|^2 \quad \text{for all } \phi \in C_c^\infty(\hat{M})$$

or the opposite inequality. We will assume the inequality as stated.
A crash course on completions

Let $\mathcal{N}$ be a normed space over $\mathbb{C}$, assume it is not already complete. Consider the family $\mathcal{C}$ whose elements are the Cauchy sequences in $\mathcal{N}$.

Define $\sim$ on $\mathcal{C}$ by declaring

\[
\{x_\mu\}_{\mu=1}^\infty \sim \{x'_\mu\}_{\mu=1}^\infty \quad \text{iff} \quad \|x_\mu - x'_\mu\| \rightarrow 0 \quad \text{as} \quad \mu \rightarrow \infty.
\]

Let $\mathcal{N}$ be the set of equivalence classes. Suppose

\[
\{x_\mu\}_{\mu=1}^\infty \sim \{x'_\mu\}_{\mu=1}^\infty \quad \text{and} \quad \{y_\mu\}_{\mu=1}^\infty \sim \{y'_\mu\}_{\mu=1}^\infty.
\]

Then:

\[
\{x_\mu + y_\mu\}_{\mu=1}^\infty \sim \{x'_\mu + y'_\mu\}_{\mu=1}^\infty \quad \text{and} \quad \{\lambda x_\mu\}_{\mu=1}^\infty \sim \{\lambda x'_\mu\}_{\mu=1}^\infty
\]

so there are well defined operations of sum and multiplication by scalar on $\mathcal{N}$.

$\mathcal{N}$ has a norm:

The limits $\lim_{\mu \rightarrow \infty} \|x_\mu\|$, $\lim_{\mu \rightarrow \infty} \|x'_\mu\|$ exist and are equal. This defines a norm on $\mathcal{N}$.
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Then: $\{x_\mu + y_\mu\}_{\mu=1}^\infty \sim \{x'_\mu + y'_\mu\}_{\mu=1}^\infty$

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G. A. Mendoza (Temple University)
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If \( \{x_\nu \}_{\nu=1}^\infty \) is a Cauchy sequence in \( \mathbb{N} \), then using a diagonalization argument one constructs an equivalence class \( x \in \mathbb{N} \) such that \( \lim_{\nu \to \infty} x_\nu = x \). So \( \mathbb{N} \) is complete.

\( \mathbb{N} \) is contained in \( \mathbb{N} \):

The map \( \iota \) that sends \( x \in \mathbb{N} \) to the class of the constant sequence \( \{x_\mu \}_{\mu=1}^\infty \) is injective and continuous (of norm 1), with dense image.

If \( \mathbb{N} \) is a pre-Hilbert space, then \( \mathbb{N} \) is a Hilbert space:

Suppose the norm of \( \mathbb{N} \) comes from an inner product. Again suppose \( \{x_\mu \}_{\mu=1}^\infty \sim \{x'_\mu \}_{\mu=1}^\infty \) and \( \{y_\mu \}_{\mu=1}^\infty \sim \{y'_\mu \}_{\mu=1}^\infty \). Then both limits \( \lim_{\mu \to \infty} (x_\mu, y_\mu) \), \( \lim_{\mu \to \infty} (x'_\mu, y'_\mu) \) exist and are equal. So \( \mathbb{N} \) gets a bilinear form induced from that of \( \mathbb{N} \), which one verifies is an inner product.

End of crash course on completions
If \( \{x_\nu\}_{\nu=1}^\infty \) is a Cauchy sequence in \( \bar{\mathcal{N}} \), then using a diagonalization argument one constructs an equivalence class \( x \in \bar{\mathcal{N}} \) such that \( \lim x_\nu = x \). So \( \bar{\mathcal{N}} \) is complete.
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If \( N \) is a pre-Hilbert space, then \( \overline{N} \) is a Hilbert space: Suppose the norm of \( N \) comes from an inner product. Again suppose \( \{ x_\mu \}_{\mu=1}^\infty \sim \{ x_\mu' \}_{\mu=1}^\infty \) and \( \{ y_\mu \}_{\mu=1}^\infty \sim \{ y_\mu' \}_{\mu=1}^\infty \). Then both limits

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\lim_{\mu \to \infty} (x_\mu, y_\mu), \quad \lim_{\mu \to \infty} (x_\mu', y_\mu')
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**$\overline{N}$ is complete:** If $\{x_\nu\}_{\nu=1}^\infty$ is a Cauchy sequence in $\overline{N}$, then using a diagonalization argument one constructs an equivalence class $x \in \overline{N}$ such that $\lim_{\nu \to \infty} x_\nu = x$. So $\overline{N}$ is complete.

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End of crash course on completions
Friedrichs extension


Suppose \( A \) is symmetric semibounded: There is \( C \in \mathbb{R} \) such that
\[
(A \phi, \phi)_{L^2} \geq C \| \phi \|_{L^2}^2
\]
for all \( \phi \in C^\infty_0(\mathring{M}, \mathbb{C}^n) \).

(this statement includes the assertion that \((A \phi, \phi)_{L^2}\) is a real number).

Then
\[
((A - C + 1) \phi, \phi)_{L^2} \geq \| \phi \|_{L^2}^2
\]
for all \( \phi \in C^\infty_0(\mathring{M}, \mathbb{C}^n) \).

So we may assume (for the time being, at least) that \( C > 0 \), in which case we say that \( A \) is a positive operator.

The construction of the Friedrichs extension takes advantage of the positivity of \( A \) through the definition
\[
(\phi, \psi)_{F} = (A \phi, \psi)_{L^2}, \phi, \psi \in C^\infty_0(\mathring{M}, \mathbb{C}^n).
\]

The positivity of \( A \) implies that \((\cdot, \cdot)_F\) defines a norm on \( C^\infty_0(\mathring{M}, \mathbb{C}^n) \).

This inner product makes \( C^\infty_0(\mathring{M}, \mathbb{C}^n) \) into a pre-Hilbert space. We let \( H \) be its completion.
Friedrichs extension

Suppose $A$ is symmetric semibounded: There is $C \in \mathbb{R}$ such that

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$$( \phi, \psi )_F = (A\phi, \psi)_{L^2}, \quad \phi, \psi \in C_c^\infty(\mathcal{M}, \mathbb{C}^n).$$

The positivity of $A$ implies that $(\cdot, \cdot)_F$ defines a norm on $C_c^\infty(\mathcal{M}, \mathbb{C}^n)$. This inner product makes $C_c^\infty(\mathcal{M}, \mathbb{C}^n)$ into a pre-Hilbert space. We let $H$ be its completion.
Briefly, let $H$ be the completion of $C_c^\infty$ with respect to $\| \cdot \|_F$. Then $C_c^\infty \subset H \subset L^2$. There is $B : L^2 \to H$ such that $(u, f)_{L^2} = (u, Bf)_{L^2}$ for all $u \in H$, $f \in L^2$.

One defines $D_F = \text{rg} B$ and shows:

- $D_{\text{min}} \subset D_F \subset D_{\text{max}}$.
- The operator $A_{D_F}$ is selfadjoint.

If you follow the construction with $A = -d^2/dx^2$ on $M = [0, 1]$, you get $H = H_1^1(M)$, $D_F = H_1^0(M) \cap H_2^2(M)$. For instance, if $\varphi, \psi \in C_c^\infty(\circ M)$, then $(\varphi, \psi)_F = -\left(\frac{d^2\varphi}{dx^2}, \psi\right)$.

Integration by parts gives $\| \varphi \|_2^2_F = \left(\frac{d\varphi}{dx}, \frac{d\varphi}{dx}\right)$ if $\varphi \in C_c^\infty$, the reason why $H = H_1^1$. Why is $\| \varphi \|_2^2_F \geq c \| \varphi \|_2^2_{L^2}$?
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\[ \text{G. A. Mendoza (Temple University) } \]
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For instance, if $\phi, \psi \in C_c^\infty(\mathcal{M})$, then
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The inclusion $\iota : C_c^\infty(\hat{\mathcal{M}}, \mathbb{C}^n) \to L^2(\mathcal{M}, \mathbb{C}^n)$ extends to a continuous map $\iota : H \to L^2(\mathcal{M}, \mathbb{C}^n)$:
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Suppose $\{\phi_\mu\}_{\mu=1}^\infty$ is a Cauchy sequence in $C_c^\infty(\hat{\mathcal{M}}, \mathbb{C}^n)$ (with respect to $\|\cdot\|_F$) representing an element $\phi \in H$. Then

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$$\| \phi_\mu - \phi_\nu \|_{L^2} \leq \| \phi_\mu - \phi_\nu \|_F \to 0 \text{ as } \mu, \nu \to \infty$$

So $\{ \phi_\mu \}_{\mu=1}^\infty$ is a Cauchy sequence in $C_c^\infty (\mathcal{M}, \mathbb{C}^n)$ with respect to the $L^2$ inner product: its class also represents an element of $L^2$. 
We may view \( C_c^\infty(\tilde{M}, \mathbb{C}^n) \) as a subset (a dense subset) of \( H \).

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\[
\|\phi_\mu - \phi_\nu\|_2 \leq \|\phi_\mu - \phi_\nu\|_F \rightarrow 0 \text{ as } \mu, \nu \rightarrow \infty
\]

So \( \{\phi_\mu\}_{\mu=1}^\infty \) is a Cauchy sequence in \( C_c^\infty(\tilde{M}, \mathbb{C}^n) \) with respect to the \( L^2 \) inner product: its class also represents an element of \( L^2 \). This element is independent of the representative of \( \phi \), so there is a well defined element \( \iota \phi \) associated to \( \phi \). The map thus defined is continuous.
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The map $\iota$ is injective: To say that $\iota \phi = 0$ is to say that there is a Cauchy sequence $\{\phi_\mu\}$ in $C_c^\infty$ (with respect to $\|\cdot\|_F$) which is equivalent to the zero sequence in the $L^2$-norm, that is, $\|\phi_\mu - 0\|_{L^2} \to 0$ as $\mu \to \infty$. 

G. A. Mendoza (Temple University)
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$$(A\psi, \phi_\nu)_{L^2}, (\psi, \phi_\nu)_{L^2} \rightarrow 0 \text{ as } \nu \rightarrow \infty$$
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$$\langle A\psi, \phi_\nu \rangle_{L^2}, \langle \psi, \phi_\nu \rangle_{L^2} \rightarrow 0 \text{ as } \nu \rightarrow \infty$$

so $\langle \psi, \phi_\mu \rangle_F \rightarrow 0$, hence $\langle \psi, \phi \rangle_F = 0$. By continuity $\langle \phi, \phi \rangle_F = 0$, so $\phi = 0$. 

G. A. Mendoza (Temple University)
Fix \( f \in L^2 \), let \( u \in H \). Then

\[
| (u, f)_{L^2} | \leq \| u \|_{L^2} \| f \|_{L^2} \leq \| f \|_{L^2} \| u \|_F
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so the map \( H \ni u \mapsto (u, f)_{L^2} \in \mathbb{C} \) is continuous. By the Riesz Representation Theorem, there is a unique \( Bf \in H \) such that

\[
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\[\text{G. A. Mendoza (Temple University)}\]

\[\text{State College, August 2010} \quad 8 / 18\]
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The orthogonal of the range of $B : L^2 \to L^2$ is the kernel of $B^*$. But $B^* = B$, which is injective. So $B$ has dense range, that is $D_F \subset H$ is a dense subspace of $L^2$. 


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Fix $f \in L^2$, let $\psi \in C^\infty_c$ be arbitray. Then
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so $Bf \in \mathcal{D}_{\text{max}}(A)$ and $ABf = f$ by definition. Therefore $Bf \in \mathcal{D}_{\text{max}}(A)$.

Thus $\mathcal{D}_F \subset \mathcal{D}_{\text{max}}(A)$.
Also $\mathcal{D}_{\min} \subset \mathcal{D}_F$ (thus $A$ with domain $\mathcal{D}_F$ is truly an extension of $A_{\min}$):
Also $D_{\text{min}} \subset D_F$ (thus $A$ with domain $D_F$ is truly an extension of $A_{\text{min}}$):

Fix $\phi \in C_c^\infty$, take $\psi \in C_c^\infty$ arbitrary.

\[(B(A\phi), \psi)_{L^2} = (A\phi, B\psi)_{L^2} = (\phi, B\psi)_F = (\phi, \psi)_{L^2}\]
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The consequence of this is that $BA\phi = \phi$, so $\phi \in \text{rg } B = \mathcal{D}_F$. Thus $C_c^\infty \subset \mathcal{D}_F$. 


Also $D_{\text{min}} \subset D_F$ (thus $A$ with domain $D_F$ is truly an extension of $A_{\text{min}}$):

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G. A. Mendoza (Temple University)
Also $\mathcal{D}_{\text{min}} \subset \mathcal{D}_F$ (thus $A$ with domain $\mathcal{D}_F$ is truly an extension of $A_{\text{min}}$):

Fix $\phi \in \mathcal{C}_c^\infty$, take $\psi \in \mathcal{C}_c^\infty$ arbitrary.

$$\langle B(A\phi), \psi \rangle_{L^2} = \langle A\phi, B\psi \rangle_{L^2} = \langle \phi, B\psi \rangle_F = \langle \phi, \psi \rangle_{L^2}$$

The consequence of this is that $BA\phi = \phi$, so $\phi \in \text{rg } B = \mathcal{D}_F$. Thus $\mathcal{C}_c^\infty \subset \mathcal{D}_F$. Suppose $\{\phi_\mu\}_{\mu=1}^\infty \subset \mathcal{C}_c^\infty$ converges in $\mathcal{D}_{\text{max}}$ to $\phi$ (so $\phi \in \mathcal{D}_{\text{min}}$). This means that $\{\phi_\mu\}_{\mu=1}^\infty$ and $\{A\phi_\mu\}_{\mu=1}^\infty$ converge in $L^2$, hence $\{\phi_\mu\}_{\mu=1}^\infty$ is a Cauchy sequence with respect to $\| \cdot \|_F$:

$$\| \phi_\mu - \phi_\nu \|_F^2 = \langle \phi_\mu - \phi_\nu, \phi_\mu - \phi_\nu \rangle_F = \langle A\phi_\mu - A\phi_\nu, \phi_\mu - \phi_\nu \rangle_{L^2}^2$$

$$\leq \| A\phi_\mu - A\phi_\nu \|_{L^2} \| \phi_\mu - \phi_\nu \|_{L^2} \to 0 \text{ as } \mu, \nu \to \infty.$$
Finally, $A$ with domain $D_F$ is selfadjoint:
Finally, $A$ with domain $D_F$ is selfadjoint: Let $u, v \in D_F$. Then

$$(Au, v)_{L^2} = (u, v)_F = (v, u)_F = (Av, u)_F = (u, Av)_{L^2}$$

gives that $A$ with domin $D_F$ is symmetric.
Finally, $A$ with domain $\mathcal{D}_F$ is selfadjoint: Let $u, v \in \mathcal{D}_F$. Then

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gives that $A$ with domain $D_F$ is symmetric.

Suppose now that $u \in \mathcal{D}^*_F$. Thus $Av \in L^2$ and

$$(Au, v)_{L^2} = (u, Av)_{L^2} \quad \forall u \in \mathcal{D}_F.$$
Finally, $A$ with domain $\mathcal{D}_F$ is selfadjoint: Let $u, v \in \mathcal{D}_F$. Then

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Suppose now that $u \in \mathcal{D}_F^*$. Thus $Av \in L^2$ and

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When $u \in \mathcal{D}_F$,

$$(u, Av)_{L^2} = (u, BAv)_F = (Au, BAv)_{L^2}$$

Since $A$ is surjective when given the domain $\mathcal{D}_F$, $(Au, v)_{L^2} = (Au, BAv)_{L^2}$ for all $u \in \mathcal{D}_F$ implies $v = BAv$, with the consequence that $v \in \text{rg } B = D_F$. So $\mathcal{D}_F^* \subset \mathcal{D}_F$, hence $A_{\mathcal{D}_F}$ is selfadjoint.

Since $\mathcal{D}_{\text{min}} \subset \mathcal{D}_F \subset \mathcal{D}_{\text{max}}$, we have

$$\mathcal{D}_F = D_F + \mathcal{D}_{\text{min}}, \quad D_F \subset \mathcal{E}.$$
Finally, $A$ with domain $\mathcal{D}_F$ is selfadjoint: Let $u, \nu \in \mathcal{D}_F$. Then

$$(Au, \nu)_{L^2} = (u, \nu)_F = (\nu, u)_F = (Av, u)_F = (u, Av)_{L^2}$$

gives that $A$ with domain $\mathcal{D}_F$ is symmetric.

Suppose now that $u \in \mathcal{D}_F^\ast$. Thus $Av \in L^2$ and

$$(Au, \nu)_{L^2} = (u, Av)_{L^2} \quad \forall u \in \mathcal{D}_F.$$ 

When $u \in \mathcal{D}_F$,

$$(u, Av)_{L^2} = (u, BAv)_F = (Au, BAv)_{L^2}$$

Since $A$ is surjective when given the domain $\mathcal{D}_F$, $(Au, \nu)_{L^2} = (Au, BAv)_{L^2}$ for all $u \in \mathcal{D}_F$ implies $\nu = BAv$, with the consequence that $\nu \in \text{rg } B = \mathcal{D}_F$. So $\mathcal{D}_F^\ast \subset \mathcal{D}_F$, hence $A_{\mathcal{D}_F}$ is selfadjoint.

Since $\mathcal{D}_{\text{min}} \subset \mathcal{D}_F \subset \mathcal{D}_{\text{max}}$, we have

$$\mathcal{D}_F = \mathcal{D}_F + \mathcal{D}_{\text{min}}, \quad \mathcal{D}_F \subset \mathcal{E}.$$ 

$\mathcal{E}$ is the orthogonal of $\mathcal{D}_{\text{min}}$ in $\mathcal{D}_{\text{max}}$. 

G. A. Mendoza (Temple University)  
Domains of extensions  
State College, August 2010  
10 / 18
The spectrum of the Friedrichs extension

Suppose $A$ is semibounded on $C_c^\infty$: There is $C$ such that

$$(A\phi, \phi) \geq C\|\phi\|^2 \text{ if } \phi \in C_c^\infty.$$
The spectrum of the Friedrichs extension

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So $\{(A\phi, \phi) : \phi \in C_c^\infty, \|\phi\| = 1\}$ is bounded below.
The spectrum of the Friedrichs extension

Suppose \( A \) is semibounded on \( C_\infty^c \): There is \( C \) such that

\[
(A\phi, \phi) \geq C\|\phi\|^2 \text{ if } \phi \in C_\infty^c.
\]

So \( \{(A\phi, \phi) : \phi \in C_\infty^c, \|\phi\| = 1\} \) is bounded below. Let \( c_0 \) be the infimum. Then

\[
(A\phi, \phi) \geq c_0\|\phi\|^2 \text{ if } \phi \in C_\infty^c.
\]
The spectrum of the Friedrichs extension

Suppose $A$ is semi-bounded on $C_c^\infty$: There is $C$ such that

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$$(A\phi, \phi) \geq c_0\|\phi\|^2 \text{ if } \phi \in C_c^\infty.$$ 

Let $\mathcal{D}_F$ be the domain of the Friedrichs extension of $A - c_0 + 1$. 

G. A. Mendoza (Temple University) 

Domains of extensions 

State College, August 2010 11 / 18
The spectrum of the Friedrichs extension

Suppose $A$ is semibounded on $C_c^\infty$: There is $C$ such that

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Let $\mathcal{D}_F$ be the domain of the Friedrichs extension of $A - c_0 + 1$. It is easy to see that $A$ with domain $\mathcal{D}_F$ is selfadjoint.
The spectrum of the Friedrichs extension

Suppose $A$ is semibounded on $C_c^\infty$: There is $C$ such that

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So $\{(A\phi, \phi) : \phi \in C_c^\infty, \|\phi\| = 1\}$ is bounded below. Let $c_0$ be the infimum. Then

$$(A\phi, \phi) \geq c_0\|\phi\|^2 \text{ if } \phi \in C_c^\infty.$$ 

Let $\mathcal{D}_F$ be the domain of the Friedrichs extension of $A - c_0 + 1$. It is easy to see that $A$ with domain $\mathcal{D}_F$ is selfadjoint. Further, if $u \in \mathcal{D}_F$, then

$$((A - c_0 + 1)u, u)_F \geq \|u\|^2,$$

hence

$$(Au, u) \geq c_0\|u\|^2.$$
The spectrum of the Friedrichs extension

Suppose \( A \) is semibounded on \( C_c^\infty \): There is \( C \) such that

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\]

So \( \{(A\phi, \phi) : \phi \in C_c^\infty, \|\phi\| = 1\} \) is bounded below. Let \( c_0 \) be the infimum. Then

\[
(A\phi, \phi) \geq c_0\|\phi\|^2 \text{ if } \phi \in C_c^\infty.
\]

Let \( \mathcal{D}_F \) be the domain of the Friedrichs extension of \( A - c_0 + 1 \). It is easy to see that \( A \) with domain \( \mathcal{D}_F \) is selfadjoint. Further, if \( u \in \mathcal{D}_F \), then

\[
((A - c_0 + 1)u, u)_F \geq \|u\|^2,
\]

hence

\[
(Au, u) \geq c_0\|u\|^2.
\]

So the estimate (†) is preserved by passing to the Friedrichs extension.
The spectrum of $A_{DF}$ is a subset of the real numbers.

$$\text{spec}(A_{DF}) \text{ is bounded from below by } c_0.$$
The spectrum of $A_{DF}$ is a subset of the real numbers.

\[ \text{spec}(A_{DF}) \text{ is bounded from below by } c_0. \]

**Proof**: Each point of $\text{spec}(A_{DF})$ is an eigenvalue. Let $\lambda \in \text{spec}(A_{DF})$, $\psi \in DF$, $\psi \neq 0$, such that $A\psi = \lambda \psi$. 
The spectrum of $A_{DF}$ is a subset of the real numbers.

\[ \text{spec}(A_{DF}) \text{ is bounded from below by } c_0. \]

**Proof:** Each point of $\text{spec}(A_{DF})$ is an eigenvalue. Let $\lambda \in \text{spec}(A_{DF})$, $\psi \in D_F$, $\psi \neq 0$, such that $A\psi - \lambda\psi$. Then

\[ c_0\|\psi\|^2 \leq (A\psi, \psi) \]
The spectrum of $A_{DF}$ is a subset of the real numbers.

\textit{spec}(A_{DF}) \textit{is bounded from below by} c_0.

\textbf{Proof}: Each point of \textit{spec}(A_{DF}) is an eigenvalue. Let $\lambda \in \textit{spec}(A_{DF})$, $\psi \in D_F$, $\psi \neq 0$, such that $A\psi - \lambda \psi$. Then

$$c_0 \|\psi\|^2 \leq (A\psi, \psi) = (\lambda \psi, \psi) = \lambda \|\psi\|^2$$
The spectrum of $A_{DF}$ is a subset of the real numbers.

spec($A_{DF}$) is bounded from below by $c_0$.

**Proof**: Each point of spec($A_{DF}$) is an eigenvalue. Let $\lambda \in \text{spec}(A_{DF})$, $\psi \in D_F$, $\psi \neq 0$, such that $A\psi - \lambda \psi$. Then

$$c_0 \|\psi\|^2 \leq (A\psi, \psi) = (\lambda \psi, \psi) = \lambda \|\psi\|^2$$

so $c_0 \leq \lambda$. □
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Let $E_\lambda = \ker(A_{DF} - \lambda I)$. So $E_\lambda$ is $\neq 0$ or $= 0$ depending on whether $\lambda \in$ spec$(A_{DF})$ or not. As in finite dimensional linear algebra

$E_\lambda \perp E_{\lambda'}$ if $\lambda \neq \lambda'$ in the $L^2$ sense
The spectrum of $A_{DF}$ is a subset of the real numbers.

\[ \text{spec}(A_{DF}) \text{ is bounded from below by } c_0. \]

**Proof:** Each point of $\text{spec}(A_{DF})$ is an eigenvalue. Let $\lambda \in \text{spec}(A_{DF})$, $\psi \in \mathcal{D}_F$, $\psi \neq 0$, such that $A\psi - \lambda \psi$. Then

\[ c_0 \|\psi\|^2 \leq (A\psi, \psi) = (\lambda \psi, \psi) = \lambda \|\psi\|^2 \]

so $c_0 \leq \lambda$. \[ \square \]

Let $\mathcal{E}_\lambda = \ker(A_{DF} - \lambda I)$. So $\mathcal{E}_\lambda$ is \( \neq 0 \) or \( = 0 \) depending on whether $\lambda \in \text{spec}(A_{DF})$ or not. As in finite dimensional linear algebra

\[ \mathcal{E}_\lambda \perp \mathcal{E}_{\lambda^\prime} \text{ if } \lambda \neq \lambda' \text{ in the } L^2 \text{ sense} \]

Let $\lambda_k$ be a listing of the elements of $\text{spec}(A_{DF})$, $\{\lambda_k\}_{k=0}^{\infty}$ monotonically increasing, with a given element $\lambda$ of $\text{spec}(A_{DF})$ repeated as many times as $\dim \mathcal{E}_\lambda$. 
The spectrum of $A_{DF}$ is a subset of the real numbers.

$$\text{spec}(A_{DF}) \text{ is bounded from below by } c_0.$$  

**Proof:** Each point of $\text{spec}(A_{DF})$ is an eigenvalue. Let $\lambda \in \text{spec}(A_{DF})$, $\psi \in D_F$, $\psi \neq 0$, such that $A\psi - \lambda \psi$. Then

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Let $E_\lambda = \ker(A_{DF} - \lambda I)$. So $E_\lambda$ is $\neq 0$ or $= 0$ depending on whether $\lambda \in \text{spec}(A_{DF})$ or not. As in finite dimensional linear algebra

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Let $\lambda_k$ be a listing of the elements of $\text{spec}(A_{DF})$, $\{\lambda_k\}_{k=0}^\infty$ monotonically increasing, with a given element $\lambda$ of $\text{spec}(A_{DF})$ repeated as many times as $\dim E_\lambda$. Associated to this listing there is an $L^2$-orthonormal system $\psi_k \in D_F$ of eigenfunctions, $A\psi_k = \lambda_k \psi_k$. 
The spectrum of $A_{DF}$ is a subset of the real numbers.

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By a general theorem,

**the $\psi_k$ form a complete orthonormal system for $L^2$.**
If \( u \in \mathcal{D}_F \) then

\[
    u = \sum_{k=0}^{\infty} (u, \psi_k) \psi_k, \quad Au = \sum_{k=0}^{\infty} \lambda_k (u, \psi_k) \psi_k
\]

both series with convergence in \( L^2 \).
If \( u \in D_F \) then

\[
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\]

both series with convergence in \( L^2 \). We have

\[
    (Au, u) = \sum_{k=0}^{\infty} \lambda_k |(u, \psi_k)|^2 \geq \lambda_0 \sum_{k=0}^{\infty} |(u, \psi_k)|^2 = \lambda_0 \|u\|^2
\]

So \( \lambda_0 \leq c_0 \).
If \( u \in \mathcal{D}_F \) then

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\]

both series with convergence in \( L^2 \). We have

\[
c_0 = \inf \{(Au, u) : u \in \mathcal{D}_F, \|u\| = 1\}
\]

\[
(Au, u) = \sum_{k=0}^{\infty} \lambda_k |(u, \psi_k)|^2 \geq \lambda_0 \sum_{k=0}^{\infty} |(u, \psi_k)|^2 = \lambda_0 \|u\|^2
\]

So \( \lambda_0 \leq c_0 \).
If $u \in \mathcal{D}_F$ then

$$u = \sum_{k=0}^{\infty} (u, \psi_k) \psi_k, \quad Au = \sum_{k=0}^{\infty} \lambda_k (u, \psi_k) \psi_k$$

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$$c_0 = \inf \{(Au, u) : u \in \mathcal{D}_F, \|u\| = 1\}$$

$$\sum_{k=0}^{\infty} \lambda_k |(u, \psi_k)|^2 \geq \lambda_0 \sum_{k=0}^{\infty} |(u, \psi_k)|^2 = \lambda_0 \|u\|^2$$

So $\lambda_0 \leq c_0$. Therefore $c_0 = \lambda_0$. 
Resolvent family

Suppose \( \lambda \not\in \text{spec}(A_D) \), so \( \lambda \in \text{bg-res}(A) \), since \( \text{bg-spec}(A) \subset \text{spec}(A_D) \).

Suppose \( \lambda \not\in \text{spec}(A_D) \), \( u \in D_F \) and \( (A - \lambda I)u = f \). Then

\[
u = \sum_k u_k \psi_k, \quad (A - \lambda I)u = \sum_k (\lambda_k - \lambda)u_k \psi_k = \sum_k f_k \psi_k\]

Thus \( u_k = f_k / (\lambda_k - \lambda) \) and \( u = \sum_k (f_k, \psi_k) \lambda_k - \lambda \psi_k \).

Define \( B_F(\lambda) : L^2 \rightarrow D_F \),

\[
B_F(\lambda)f = \sum_{k=0}^{\infty} (f, \psi_k) \lambda_k - \lambda \psi_k.
\]

The family \( B_F(\lambda) \) is the resolvent family of \( A_D \).
Resolvent family

so $\lambda \in \text{bg-res}(A)$, since $\text{bg-spec}(A) \subset \text{spec}(A_{DF})$

Suppose $\lambda \notin \text{spec}(A_{DF})$, $u \in D_F$ and $(A - \lambda I)u = f$. Then

$$u = \sum_k u_k \psi_k, \quad (A - \lambda I)u = \sum_k (\lambda_k - \lambda) u_k \psi_k = \sum_k f_k \psi_k$$

Thus $u_k = f_k / (\lambda_k - \lambda)$ and

$$u = \sum_k \frac{(f, \psi_k)}{\lambda_k - \lambda} \psi_k.$$
Resolvent family

Suppose \( \lambda \notin \text{spec}(A_{D_F}) \), \( u \in D_F \) and \((A - \lambda I)u = f\). Then

\[
u = \sum_k u_k \psi_k, \quad (A - \lambda I)u = \sum_k (\lambda_k - \lambda) u_k \psi_k = \sum_k f_k \psi_k
\]

Thus \( u_k = f_k / (\lambda_k - \lambda) \) and

\[
u = \sum_k \frac{(f, \psi_k)}{\lambda_k - \lambda} \psi_k.
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Define

\[
B_F(\lambda) : L^2 \rightarrow D_F, \quad B_F(\lambda)f = \sum_{k=0}^{\infty} \frac{(f, \psi_k)}{\lambda_k - \lambda} \psi_k.
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The family \( B_F(\lambda) \) is the resolvent family of \( A_{D_F} \).
The spaces $\mathcal{K}_\lambda$

$$B_F(\lambda)f = \sum_{k=0}^{\infty} \frac{(f, \psi_k)}{\lambda_k - \lambda} \psi_k$$

$$\mathcal{D}_F = D_F + \mathcal{D}_{\min}, \ D_F \subset \mathcal{E}$$

$$D_{aF} = \text{orthogonal of } D_F \text{ in } \mathcal{E}$$

$$\pi_{\mathcal{D}_F} : \mathcal{D}_{\max} \rightarrow \mathcal{D}_{\max} \text{ is the orthogonal projection on } \mathcal{D}_F, \ \pi_{D_{aF}} = I - \pi_{\mathcal{D}_F}$$

is the ortho-projection on $D_{aF}$
The spaces $\mathcal{K}_\lambda$

Suppose $\lambda \in \text{bg-res}(A)$ and $\phi \in \mathcal{K}_\lambda$: $(A - \lambda I)\phi = 0$. Then

$$B_F(\lambda)f = \sum_{k=0}^{\infty} \frac{(f, \psi_k)}{\lambda_k - \lambda} \psi_k$$

$$\mathcal{D}_F = D_F + \mathcal{D}_{\text{min}}, D_F \subset \mathcal{E}$$
$$D_{aF} = \text{orthogonal of } D_F \text{ in } \mathcal{E}$$
$$\pi_{\mathcal{D}_F} : \mathcal{D}_{\text{max}} \to \mathcal{D}_{\text{max}} \text{ is the orthogonal projection on } \mathcal{D}_F,$$
$$\pi_{D_{aF}} = I - \pi_{\mathcal{D}_F}$$

is the ortho-projection on $D_{aF}$.
The spaces $\mathcal{K}_\lambda$

Suppose $\lambda \in \text{bg-res}(A)$ and $\phi \in \mathcal{K}_\lambda$: $(A - \lambda I)\phi = 0$. Then

$$0 = (A - \lambda I)(\pi_{D_{af}} \phi + \pi_{D_F} \phi) = (A - \lambda I)\pi_{D_{af}} \phi + (A - \lambda I)\pi_{D_F} \phi$$

gives

$B_F(\lambda)f = \sum_{k=0}^{\infty} \frac{(f, \psi_k)}{\lambda_k - \lambda} \psi_k$
The spaces $\mathcal{K}_\lambda$

Suppose $\lambda \in \text{bg-res}(A)$ and $\phi \in \mathcal{K}_\lambda$: $(A - \lambda I)\phi = 0$. Then

$$0 = (A - \lambda I)(\pi_{D_{aF}} \phi + \pi_{D_F} \phi) = (A - \lambda I)\pi_{D_{aF}} \phi + (A - \lambda I)\pi_{D_F} \phi$$

gives

$$0 = B_F(\lambda)(A - \lambda I)\pi_{D_{aF}} \phi + B_F(\lambda)(A - \lambda I)\pi_{D_F} \phi.$$

$D_F = D_F + D_{\text{min}}$, $D_F \subseteq \mathcal{E}$

$D_{aF}$ = orthogonal of $D_F$ in $\mathcal{E}$

$\pi_{D_F} : D_{\text{max}} \rightarrow D_{\text{max}}$ is the orthogonal projection on $D_F$, $\pi_{D_{aF}} = I - \pi_{D_F}$ is the ortho-projection on $D_{aF}$
The spaces $\mathcal{K}_\lambda$

Suppose $\lambda \in \text{bg-res}(A)$ and $\phi \in \mathcal{K}_\lambda$: $(A - \lambda I)\phi = 0$. Then

$$0 = (A - \lambda I)(\pi_{D_{aF}} \phi + \pi_D \phi) = (A - \lambda I)\pi_{D_{aF}} \phi + (A - \lambda I)\pi_D \phi$$

gives

$$0 = B_F(\lambda)(A - \lambda I)\pi_{D_{aF}} \phi + B_F(\lambda)(A - \lambda I)\pi_D \phi.$$ 

Since $\pi_D \phi \in D_F$,

$$B_F(\lambda)(A - \lambda I)\pi_D \phi = \pi_D \phi.$$
The spaces $K_\lambda$

Suppose $\lambda \in \text{bg-res}(A)$ and $\phi \in K_\lambda$: $(A - \lambda I)\phi = 0$. Then

$$0 = (A - \lambda I)(\pi_{D_{aF}} \phi + \pi_D \phi) = (A - \lambda I)\pi_{D_{aF}} \phi + (A - \lambda I)\pi_D \phi$$

gives

$$0 = B_F(\lambda)(A - \lambda I)\pi_{D_{aF}} \phi + B_F(\lambda)(A - \lambda I)\pi_D \phi.$$ 

Since $\pi_D \phi \in D$, 

$$B_F(\lambda)(A - \lambda I)\pi_D \phi = \pi_D \phi.$$ 

So

$$\pi_D \phi = -B_F(\lambda)(A - \lambda I)\pi_{D_{aF}} \phi,$$

$B_F(\lambda)f = \sum_{k=0}^{\infty} \frac{(f, \psi_k)}{\lambda_k - \lambda} \psi_k$
The spaces $\mathcal{K}_\lambda$

Suppose $\lambda \in \text{bg\text{-}res}(A)$ and $\phi \in \mathcal{K}_\lambda$: $(A - \lambda I)\phi = 0$. Then

$$0 = (A - \lambda I)(\pi_{D_{aF}} \phi + \pi_D \phi) = (A - \lambda I)\pi_{D_{aF}} \phi + (A - \lambda I)\pi_D \phi$$

gives

$$0 = B_F(\lambda)(A - \lambda I)\pi_{D_{aF}} \phi + B_F(\lambda)(A - \lambda I)\pi_D \phi.$$ 

Since $\pi_D \phi \in \mathcal{D}_F$,

$$B_F(\lambda)(A - \lambda I)\pi_D \phi = \pi_D \phi.$$ 

So

$$\pi_D \phi = -B_F(\lambda)(A - \lambda I)\pi_{D_{aF}} \phi, \quad \phi = \pi_{D_{aF}} \phi - B_F(\lambda)(A - \lambda I)\pi_{D_{sF}} \phi.$$ 

$\mathcal{D}_F = \mathcal{D}_F + \mathcal{D}_{\text{min}}, \mathcal{D}_F \subset \mathcal{E}$

$\mathcal{D}_{aF}$ = orthogonal of $\mathcal{D}_F$ in $\mathcal{E}$

$\pi_{\mathcal{D}_F} : \mathcal{D}_{\text{max}} \to \mathcal{D}_{\text{max}}$ is the orthogonal projection on $\mathcal{D}_F$, $\pi_{D_{aF}} = I - \pi_D$ is the ortho-projection on $D_{aF}$
The spaces $\mathcal{K}_\lambda$

Suppose $\lambda \in \text{bg-res}(A)$ and $\phi \in \mathcal{K}_\lambda$: $(A - \lambda I)\phi = 0$. Then

$$0 = (A - \lambda I)(\pi_{D_{aF}}\phi + \pi_{D_F}\phi) = (A - \lambda I)\pi_{D_{aF}}\phi + (A - \lambda I)\pi_{D_F}\phi$$

gives

$$0 = B_F(\lambda)(A - \lambda I)\pi_{D_{aF}}\phi + B_F(\lambda)(A - \lambda I)\pi_{D_F}\phi.$$

Since $\pi_{D_F}\phi \in D_F$,

$$B_F(\lambda)(A - \lambda I)\pi_{D_F}\phi = \pi_{D_F}\phi.$$

So

$$\pi_{D_F}\phi = -B_F(\lambda)(A - \lambda I)\pi_{D_{aF}}\phi, \quad \phi = \pi_{D_{aF}}\phi - B_F(\lambda)(A - \lambda I)\pi_{D_{sF}}\phi$$

Since $\dim D_{aF} = \dim \mathcal{K}_\lambda = \dim \pi_{\text{max}}\mathcal{K}_\lambda$, the elements of $\mathcal{K}_\lambda$ are all of the form $u - B_F(\lambda)(A - \lambda I)u$ with $u \in D_{aF}$:

$$\mathcal{K}_\lambda = \{u - B_F(\lambda)(A - \lambda I)u : u \in D_{aF}\}, \quad \lambda \notin \text{spec}(A_{D_F}).$$
Let \( S : D_{aF} \to D_{aF} \) be selfadjoint (with respect to the \( A \)-inner product), let \( T = -AS : D_{aF} \to D_{F} \), and let

\[
D_T = \{ u + Tu : u \in D_{aF} \},
\]

an element of \( \mathcal{A} : A \) with domain \( \mathcal{D}_T = D_T + \mathcal{D}_{\text{min}} \) is selfadjoint.
Let $S : D_{aF} \to D_{aF}$ be selfadjoint (with respect to the $A$-inner product), let $T = -AS : D_{aF} \to D_F$, and let

$$D_T = \{ u + Tu : u \in D_{aF} \},$$

an element of $\mathfrak{A}$: $A$ with domain $\mathcal{D}_T = D_T + \mathcal{D}_{\text{min}}$ is selfadjoint.

For $\lambda \in \text{res}(A_{DF})$

$$\mathcal{K}_\lambda = \{ u - B_F(\lambda)(A - \lambda I)u : u \in D_{aF} \}$$

Note that $\pi_{\text{max}}B_F(\lambda)(A - \lambda I)u \in D_F$

$$\pi_{\text{max}}\mathcal{K}_\lambda = \{ u - \pi_{\text{max}}B_F(\lambda)(A - \lambda)u : u \in D_{aF} \}.$$
Let $S : D_{aF} \to D_{aF}$ be selfadjoint (with respect to the $A$-inner product), let $T = -AS : D_{aF} \to D_F$, and let

$$D_T = \{u + Tu : u \in D_{aF}\},$$

an element of $\mathcal{S}A$: $A$ with domain $\mathcal{D}_T = D_T + D_{\text{min}}$ is selfadjoint.

For $\lambda \in \text{res}(A_{DF})$

$$\pi_{\max} \mathcal{K}_\lambda = \{u - \pi_{\max} B_F(\lambda)(A - \lambda I)u : u \in D_{aF}\},$$

Note that $\pi_{\max} B_F(\lambda)(A - \lambda I)u \in D_F$

so, $D_T \cap \pi_{\max} \mathcal{K}_\lambda \neq 0$ if and only if there is $u \in D_{aF}$, $u \neq 0$, such that

$$-\pi_{\max} B_D(\lambda)(A - \lambda I)u = -ASu.$$
Let $S : D_{aF} \to D_{aF}$ be selfadjoint (with respect to the $A$-inner product), let $T = -AS : D_{aF} \to D_{F}$, and let

$$D_T = \{u + Tu : u \in D_{aF}\},$$

an element of $\mathfrak{A} : A$ with domain $\mathcal{D}_T = D_T + \mathcal{D}_{\text{min}}$ is selfadjoint.

For $\lambda \in \text{res}(A_D)$

$$\pi_{\text{max}}\mathcal{K}_\lambda = \{u - \pi_{\text{max}}B_{F}(\lambda)(A - \lambda I)u : u \in D_{aF}\}.$$  

Note that $\pi_{\text{max}}B_{F}(\lambda)(A - \lambda I)u \in D_F$.

so, $D_T \cap \pi_{\text{max}}\mathcal{K}_\lambda \neq 0$ if and only if there is $u \in D_{aF}$, $u \neq 0$, such that $-\pi_{\text{max}}B_D(\lambda)(A - \lambda I)u = -ASu$. Setting

$$F_{D_F}(\lambda) = -A\pi_{\text{max}}B_{F}(\lambda)(A - \lambda I)|_{aF}, \quad \lambda \in \text{res}(A_{D_F})$$

an operator $D_{aF} \to D_{aF}$
Let \( S : D_{aF} \to D_{aF} \) be selfadjoint (with respect to the \( A \)-inner product), let \( T = -AS : D_{aF} \to D_{F} \), and let

\[
D_T = \{ u + Tu : u \in D_{aF} \},
\]

an element of \( \mathcal{A} : A \) with domain \( D_T = D_T + D_{\min} \) is selfadjoint.

For \( \lambda \in \text{res}(A_{DF}) \)

\[
\pi_{\max} \mathcal{K}_\lambda = \{ u - \pi_{\max} B_{F}(\lambda)(A - \lambda I)u : u \in D_{aF} \}.
\]

so, \( D_T \cap \pi_{\max} \mathcal{K}_\lambda \neq 0 \) if and only if there is \( u \in D_{aF}, u \neq 0 \), such that

\[
-\pi_{\max} B_{D}(\lambda)(A - \lambda I)u = -ASu.
\]

Setting

\[
F_{DF}(\lambda) = -A\pi_{\max} B_{F}(\lambda)(A - \lambda I)|_{aF}, \quad \lambda \in \text{res}(A_{DF})
\]

an operator \( D_{aF} \to D_{aF} \) we thus have

\[
\lambda \in \text{spec}(A_{DT}) \cap \text{res}(A_{DF}) \iff F_{D}(\lambda) - S \text{ has nontrivial kernel}.
\]
The map $F_{DF}(\lambda)$ satisfies

$$F_{DF}(\lambda)^* = F_{DF}(\bar{\lambda}), \quad \lambda \in \text{res}(A_{DF}).$$
The map $F_{DF}(\lambda)$ satisfies

$$F_{DF}(\lambda)^* = F_{DF}(\bar{\lambda}), \quad \lambda \in \text{res}(A_{DF}).$$

In addition, for any $\lambda \in \text{res}(A_D)$,

$$(F_{DF}(\lambda)u, u')_A = \sum_{k=0}^{\infty} \frac{\langle \delta u, \psi_k \rangle \langle \delta u', \psi_k \rangle}{1 + \lambda^2_k} \frac{1 + \lambda \lambda_k}{\lambda_k - \lambda}, \quad u, u' \in D_{aF}.$$
The map $F_{D_F}(\lambda)$ satisfies

$$F_{D_F}(\lambda)^* = F_{D_F}(\bar{\lambda}), \quad \lambda \in \text{res}(A_{D_F}).$$

In addition, for any $\lambda \in \text{res}(A_D)$,

$$(F_{D_F}(\lambda)u, u')_A = \sum_{k=0}^{\infty} \frac{\langle \delta_u, \psi_k \rangle \langle \delta_{u'}, \psi_k \rangle}{1 + \lambda_k^2} \frac{1 + \lambda \lambda_k}{\lambda_k - \lambda}, \quad u, u' \in D_{aF}.$$

Thus

$$Q_{D_F,\lambda}(u, u') = (F_{D_F}(\lambda)u, u')_A, \quad u, u' \in D_{aF}$$

is Hermitian when $\lambda \in \text{res}(A_{D_F}) \cap \mathbb{R}$. 
The map $F_{DF}(\lambda)$ satisfies

$$F_{DF}(\lambda)^* = F_{DF}(\bar{\lambda}), \quad \lambda \in \text{res}(A_{DF}).$$

In addition, for any $\lambda \in \text{res}(A_D)$,

$$(F_{DF}(\lambda)u, u')_A = \sum_{k=0}^{\infty} \frac{\langle \delta u, \psi_k \rangle \langle \delta u', \psi_k \rangle}{1 + \lambda^2_k} \frac{1 + \lambda \lambda_k}{\lambda_k - \lambda}, \quad u, u' \in D_{aF}.$$

Thus

$$Q_{DF,\lambda}(u, u') = (F_{DF}(\lambda)u, u')_A, \quad u, u' \in D_{aF}$$

is Hermitian when $\lambda \in \text{res}(A_{DF}) \cap \mathbb{R}$. We have

$$\frac{\partial Q_{DF,\lambda}(u, u)}{\partial \lambda} = \sum_{k=0}^{\infty} \frac{|\langle \delta u, \psi_k \rangle|^2}{(\lambda_k - \lambda)^2}$$

so $\partial Q_{DF,\lambda}/\partial \lambda$ is positive definite when $\lambda \in \mathbb{R} \setminus \text{spec}(A_{DF})$. 
Setting $\lambda_{-1} = -\infty$, it follows that if $Q_{D_F,\tilde{\lambda}}$ is positive indefinite (negative indefinite) for some $\tilde{\lambda} \in (\lambda_{k-1}, \lambda_k)$, $k = 0, 1, \ldots$, then $Q_{D_F,\lambda}$ is positive (negative) definite for $\lambda \in (\tilde{\lambda}, \lambda_k)$ ($\lambda \in (\lambda_{k-1}, \tilde{\lambda})$).

These observations, plus the fact that

$$Q_{D_F,\lambda}(u, u) \to -\infty \text{ as } \lambda \to -\infty \quad \text{for all } u \in D_{aF}, u \neq 0,$$

gives very precise information about the spectrum of $A$ with domains $D + D_{\text{min}}$, $D \in \mathfrak{S}A$. 