Domains of closed extensions of ODEs

Part IV

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Semibounded means symmetric on $C_c^\infty(\mathring{\mathcal{M}}, \mathbb{C}^n)$, plus there is $C \in \mathbb{R}$ such that

$$(A\phi, \phi) \geq C\|\phi\|^2 \quad \text{for all } \phi \in C_c^\infty(\mathring{\mathcal{M}})$$

or the opposite inequality. We will assume the inequality as stated.

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$\overline{\mathcal{N}}$ has a norm: The limits

$$\lim_{\mu\to\infty} \|x_\mu\|, \quad \lim_{\mu\to\infty} \|x'_\mu\|$$

exist and are equal. This defines a norm on $\overline{\mathcal{N}}$.

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End of crash course on completions

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Suppose A is symmetric semibounded: There is $C \in \mathbb{R}$ such that

$$(A\phi, \phi)_{L^2} \geq C\|\phi\|^2_{L^2} \quad \text{for all } \phi \in C_c^\infty(\mathring{\mathcal{M}}, \mathbb{C}^n).$$

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$$\big((A - C + 1)\phi, \phi\big)_{L^2} \geq \|\phi\|^2_{L^2} \quad \text{for all } \phi \in C_c^\infty(\mathring{\mathcal{M}}, \mathbb{C}^n).$$

So we may assume (for the time being, at least) that $C > 0$, in which case we say that A is a positive operator.

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The construction of the Friedrichs extension takes advantage of the positivity of A through the definition

$$(\phi, \psi)_F = (A\phi, \psi)_{L^2}, \quad \phi, \psi \in C_c^\infty(\mathring{\mathcal{M}}, \mathbb{C}^n).$$

The positivity of A implies that $(\cdot, \cdot\cdot)_F$ defines a norm on $C_c^\infty(\mathring{\mathcal{M}}, \mathbb{C}^n)$. This inner product makes $C_c^\infty(\mathring{\mathcal{M}}, \mathbb{C}^n)$ into a pre-Hilbert space. We let H be its completion.

Briefly, let H be the completion of C_c^∞ with respect to $\|\cdot\|_F$. Then

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One defines $\mathcal{D}_F = \operatorname{rg} B$ and shows:

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If you follow the construction with $A = -d^2/dx^2$ on $\mathcal{M} = [0, 1]$, you get

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Integration by parts gives

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$$(A\psi, \phi_\nu)_{L^2},\ (\psi, \phi_\nu)_{L^2} \to 0 \text{ as } \nu \to \infty$$

We may view $C_c^\infty(\mathring{\mathcal{M}}, \mathbb{C}^n)$ as a subset (a dense subset) of H.

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Suppose $\{\phi_\mu\}_{\mu=1}^\infty$ is a Cauchy sequence in $C_c^\infty(\mathring{\mathcal{M}}, \mathbb{C}^n)$ (with respect to $\|\cdot\|_F$) representing an element $\phi \in H$. Then

$$\|\phi_\mu - \phi_\nu\|_{L^2} \leq \|\phi_\mu - \phi_\nu\|_F \to 0 \text{ as } \mu, \nu \to \infty$$

So $\{\phi_\mu\}_{\mu=1}^\infty$ is a Cauchy sequence in $C_c^\infty(\mathring{\mathcal{M}}, \mathbb{C}^n)$ with respect to the L^2 inner product: its class also represents an element of L^2. This element is independent of the representative of ϕ, so there is a well defined element $\iota\phi$ associated to ϕ. The map thus defined is continuous.

The map ι is injective: To say that $\iota\phi = 0$ is to say that there is a Cauchy sequence $\{\phi_\mu\}$ in C_c^∞ (with respect to $\|\cdot\|_F$) which is equivalent to the zero sequence in the L^2-norm, that is, $\|\phi_\mu - 0\|_{L^2} \to 0$ as $\mu \to \infty$. Fix any $\psi \in C_c^\infty$. Then

$$(A\psi, \phi_\nu)_{L^2},\ (\psi, \phi_\nu)_{L^2} \to 0 \text{ as } \nu \to \infty$$

so $(\psi, \phi_\mu)_F \to 0$, hence $(\psi, \phi)_F = 0$. By continuity $(\phi, \phi)_F = 0$, so $\phi = 0$.

Fix $f \in L^2$, let $u \in H$. Then

$$|(u, f)_{L^2}| \leq \|u\|_{L^2} \|f\|_{L^2} \leq \|f\|_{L^2} \|u\|_F$$

so the map $H \ni u \mapsto (u, f)_{L^2} \in \mathbb{C}$ is continuous. By the Riesz Representation Theorem, there is a unique $Bf \in H$ such that

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so $Bf \in \mathcal{D}_{\max}(A)$ and $ABf = f$ by definition. Therefore $Bf \in \mathcal{D}_{\max}(A)$. Thus $\mathcal{D}_F \subset \mathcal{D}_{\max}(A)$.

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$$\begin{aligned}\|\phi_\mu - \phi_\nu\|_F^2 = (\phi_\mu - \phi_\nu, \phi_\mu - \phi_\nu)_F = (A\phi_\mu - A\phi_\nu, \phi_\mu - \phi_\nu)_L^2 \\ \leq \|A\phi_\mu - A\phi_\nu\|_{L^2} \|\phi_\mu - \phi_\nu\|_{L^2} \to 0 \text{ as } \mu, \nu \to \infty.\end{aligned}$$

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Since A is surjective when given the domain $\mathcal{D}_F$, $(Au, v)_{L^2} = (Au, BAv)_{L^2}$ for all $u \in \mathcal{D}_F$ implies $v = BAv$, with the consequence that $v \in \operatorname{rg} B = D_F$. So $\mathcal{D}_F^* \subset \mathcal{D}_F$, hence $A_{\mathcal{D}_F}$ is selfadjoint.
Since $\mathcal{D}_{\min} \subset \mathcal{D}_F \subset \mathcal{D}_{\max}$, we have

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So the estimate (†) is preserved by passing to the Friedrics extension.

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By a general theorem,

the ψ_k form a complete orthonormal system for L^2.

If $u \in \mathcal{D}_F$ then

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$$(Au, u) = \sum_{k=0}^{\infty} \lambda_k |(u, \psi_k)|^2 \geq \lambda_0 \sum_{k=0}^{\infty} |(u, \psi_k)|^2 = \lambda_0 \|u\|^2$$

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Resolvent family

so $\lambda \in$ bg-res(A), since bg-spec$(A) \subset$ spec$(A_{\mathcal{D}_F})$

Suppose $\lambda \notin$ spec$(A_{\mathcal{D}_F})$, $u \in \mathcal{D}_F$ and $(A - \lambda \mathrm{I})u = f$. Then

$$u = \sum_k u_k \psi_k, \quad (A - \lambda \mathrm{I})u = \sum_k (\lambda_k - \lambda) u_k \psi_k = \sum_k f_k \psi_k$$

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Define

$$\mathcal{B}_F(\lambda) : L^2 \to \mathcal{D}_F, \quad B_F(\lambda) f = \sum_{k=0}^{\infty} \frac{(f, \psi_k)}{\lambda_k - \lambda} \psi_k.$$

The family $B_F(\lambda)$ is the resolvent family of $A_{\mathcal{D}_F}$.

The spaces $\mathcal{K}_\lambda$

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Since $\dim D_{aF} = \dim \mathcal{K}_\lambda = \dim \pi_{\max}\mathcal{K}_\lambda$, the elements of $\mathcal{K}_\lambda$ are all of the form $u - B_F(\lambda)(A - \lambda\mathrm{I})u$ with $u \in D_{aF}$:

$$\mathcal{K}_\lambda = \{u - B_F(\lambda)(A - \lambda\mathrm{I})u : u \in D_{aF}\}, \quad \lambda \notin \mathrm{spec}(A_{\mathcal{D}_F}).$$

Let $S : D_{aF} \to D_{aF}$ be selfadjoint (with respect to the A-inner product), let $T = -AS : D_{aF} \to D_F$, and let

$$D_T = \{u + Tu : u \in D_{aF}\},$$

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so, $D_T \cap \pi_{\max}\mathcal{K}_\lambda \neq 0$ if and only if there is $u \in D_{aF}$, $u \neq 0$, such that $-\pi_{\max} B_{\mathcal{D}}(\lambda)(A - \lambda \mathrm{I})u = -ASu$.

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$$F_{\mathcal{D}_F}(\lambda) = -A\pi_{\max} B_F(\lambda)(A - \lambda \mathrm{I})|_{aF}, \quad \lambda \in \operatorname{res}(A_{\mathcal{D}_F})$$

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$$\lambda \in \operatorname{spec}(A_{\mathcal{D}_T}) \cap \operatorname{res}(A_{\mathcal{D}_F}) \iff F_{\mathcal{D}}(\lambda) - S \text{ has nontrivial kernel.}$$

The map $F_{\mathcal{D}_F}(\lambda)$ satisfies

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$$(F_{\mathcal{D}_F}(\lambda)u, u')_A = \sum_{k=0}^{\infty} \frac{\overline{\langle \delta_u, \psi_k \rangle} \langle \delta_{u'}, \psi_k \rangle}{1+\lambda_k^2} \frac{1+\lambda\lambda_k}{\lambda_k - \lambda}, \quad u, u' \in D_{aF}.$$

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Thus

$$Q_{\mathcal{D}_F,\lambda}(u, u') = (F_{\mathcal{D}_F}(\lambda)u, u')_A, \quad u, u' \in D_{aF}$$

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is Hermitian when $\lambda \in \operatorname{res}(A_{\mathcal{D}_F}) \cap \mathbb{R}$. We have

$$\frac{\partial Q_{\mathcal{D}_F,\lambda}(u,u)}{\partial \lambda} = \sum_{k=0}^{\infty} \frac{|\langle \delta_u, \psi_k \rangle|^2}{(\lambda_k - \lambda)^2}$$

so $\partial Q_{\mathcal{D}_F,\lambda}/\partial\lambda$ is positive definite when $\lambda \in \mathbb{R} \setminus \operatorname{spec}(A_{\mathcal{D}_F})$.

Setting $\lambda_{-1} = -\infty$, it follows that if $Q_{\mathcal{D}_F,\tilde\lambda}$ is positive indefinite (negative indefinite) for some $\tilde\lambda \in (\lambda_{k-1}, \lambda_k)$, $k = 0, 1, \dots$, then $Q_{\mathcal{D}_F,\lambda}$ is positive (negative) definite for $\lambda \in (\tilde\lambda, \lambda_k)$ ($\lambda \in (\lambda_{k-1}, \tilde\lambda)$).
These observations, plus the fact that

$$Q_{\mathcal{D}_F,\lambda}(u,u) \to -\infty \text{ as } \lambda \to -\infty \quad \text{for all } u \in D_{aF}, u \neq 0,$$

gives very precise information about the spectrum of A with domains $D + \mathcal{D}_{\min}$, $D \in \mathfrak{S}\mathfrak{A}$.