Domains of closed extensions of ODEs

Part III

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Semibounded means symmetric, plus there is $C \in \mathbb{R}$ such that

$$(A\phi, \phi) \geq C\|\phi\|^2 \quad \text{for all } \phi \in C_c^\infty(\mathcal{M})$$

(or the opposite inequality).

[Note: There was no time for the part on symmetric semibounded operators]

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If there is $\lambda_0 \in \mathbb{C}$ such that $A_{\mathsf{min}} - \lambda_0 \mathrm{I}$ is injective, then $\{\lambda : \mathcal{K}_\lambda \cap \mathcal{D}_{\mathsf{min}} \neq 0\}$ is a closed discrete subset of $\mathbb{C}$.

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The proof will illustrate some aspects of analytic Fredholm theory.

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The operator $\mathcal{A}(\lambda) = \wp_{\mathcal{R}_{\lambda_0}} (A_{\min} - \lambda \mathrm{I}) : \mathcal{D}_{\min} \to \mathcal{R}_{\lambda_0}$ is invertible for all λ in some neighborhood of λ_0.

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$(\mathrm{I} - (\lambda - \lambda_0)\Theta)^{-1} = \sum_{k=0}^\infty (\lambda - \lambda_0)^k \Theta^k$. The series converges because $|\lambda - \lambda_0| \|\Theta\| < 1$. The expresion is analytic in λ

It follows that $(\dagger)$ has an analytic inverse $Q(\lambda) : \mathcal{R}_{\lambda_0} \to \mathcal{R}_{\lambda_0}$.
Let $\mathcal{B}(\lambda) = \mathcal{B}(\lambda_0) Q(\lambda)$. This $\mathcal{B}(\lambda)$ is analytic, inverts $\mathcal{A}(\lambda)$. □

[1] by the open mapping theorem. See Rudin, *Functional Analysis*, 2nd Ed., McGraw-Hill, 1991.

From previous slide

- $\mathcal{R}_{\lambda_0} = \operatorname{rg}(A_{\min} - \lambda_0 \mathrm{I})$
- $\wp_{\mathcal{R}_{\lambda_0}} : L^2 \to L^2$ is the orthogonal projection on $\mathcal{R}_{\lambda_0}$.
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$\{\lambda : \mathcal{K}_\lambda \cap \mathcal{D}_{\min} \neq 0\}$ is a closed discrete subset of $\mathbb{C}$.

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$A_{\min} - \lambda \mathrm{I} = \wp_{\mathcal{R}_{\lambda_0}}(A_{\min} - \lambda \mathrm{I}) + \wp_{\mathcal{R}_{\lambda_0}^{\perp}}(A_{\min} - \lambda \mathrm{I})$,

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The set $\mathcal{J} = \{\lambda : A_{\min} - \lambda \mathrm{I} \text{ is injective}\}$ is open.

From previous slide

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The set $\mathcal{J} = \{\lambda : A_{\text{min}} - \lambda \mathrm{I} \text{ is injective}\}$ is open.

Note that $\mathcal{J} \neq \emptyset$ since $\lambda_0 \in \mathcal{J}$. We now show that the complement of $\mathcal{J}$ is discrete.

Let $\lambda_0 \in \mathbb{C}$, let $K = \ker(A_{\min} - \lambda_0 \mathrm{I})$, let $R = \operatorname{rg}(A_{\min} - \lambda_0 \mathrm{I})$. Since $A_{\min} - \lambda_0 \mathrm{I}$ is Fredholm, K is finite-dimensional, $R \subset L^2$ is closed, and $R^\perp$ has finite dimension.

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$$\begin{bmatrix} \mathcal{A}_{11}(\lambda) & \mathcal{A}_{12}(\lambda) \\ \mathcal{A}_{21}(\lambda) & \mathcal{A}_{22}(\lambda) \end{bmatrix} : \begin{matrix} K \\ \oplus \\ K^\perp \end{matrix} \to \begin{matrix} R^\perp \\ \oplus \\ R \end{matrix}$$

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Then $\psi = -\mathcal{B}(\lambda)\mathcal{A}_{21}(\lambda)\phi$, hence $\mathcal{A}_{11}(\lambda)\phi - \mathcal{A}_{12}(\lambda)\mathcal{B}(\lambda)\mathcal{A}_{21}(\lambda)\phi = 0$.

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Suppose from now on that there is λ_0 such that $A_{\mathsf{min}} - \lambda_0 \mathrm{I}$ is injective. So $A_{\mathsf{min}} - \lambda \mathrm{I}$ is injective for all λ in the complement of a closed discrete set.

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$$\begin{aligned}\text{bg-spec}(A) = \{\lambda \in \mathbb{C} : A_{\mathsf{min}} - \lambda \mathrm{I} \ &\textit{is not injective} \\ &\textit{or } A_{\mathsf{max}} - \lambda \mathrm{I} \ \textit{is not surjective} \ \}\end{aligned}$$

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The map bg-res$(A) \ni \lambda \mapsto \pi_{\max}\mathcal{K}_\lambda \in \mathrm{Gr}_{d'}(\mathcal{E})$ is analytic.

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Let $\mathcal{D} = D + \mathcal{D}_{\min}$ be a domain for A, $D \subset \mathcal{E}$.

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But if $\lambda \in$ bg-res(A), $\mathrm{Ind}(A_{\max}) = \mathrm{Ind}(A_{\max} - \lambda I) =$
$\dim\ker(A_{\max} - \lambda I) - \dim\mathrm{coker}(A_{\max} - \lambda I) = \dim \mathcal{K}_\lambda = d'$.

$$d' = \dim \mathcal{E} - \dim(D) = \dim \mathcal{E} - d'', \textit{ that is, } \dim \mathcal{E} = d' + d''.$$

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Let $\lambda \in$ bg-res(A)*. Then* $\lambda \in \mathrm{spec}(A_{\mathcal{D}})$ *iff* $\mathcal{K}_\lambda \cap \mathcal{D} \neq 0$*.*

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- *$\mathcal{K}_\lambda \cap \mathcal{D} = 0 \iff \pi_{\max}\mathcal{K}_\lambda \cap D = 0$.*
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$$\pi_{\max}\mathcal{K}_\lambda \cap D = 0 \iff \pi_{\max}\mathcal{K}_\lambda + D = \mathcal{E}.$$

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Given $D \in \mathrm{Gr}_{d''}(\mathcal{E})$, let

$$\mathscr{V}_D = \{K \in \mathrm{Gr}_{d'}(\mathcal{E}) : K \cap D \neq 0\}.$$

Theorem:
$\text{spec}(A_{\mathcal{D}}) = \text{bg-spec}(A) \cup \{\lambda \in \text{bg-res}(A) : \pi_{\max}\mathcal{K}_\lambda \in \mathscr{V}_D\}$.

The set $\mathfrak{V}_D$ is a closed subspace of $\mathrm{Gr}_{d'}(\mathcal{E})$ with empty interior. More precisely, $\mathfrak{V}_D$ is closed and every $K_0 \in \mathrm{Gr}_{d'}(\mathcal{E})$ has a neighborhood W in $\mathrm{Gr}_{d'}(\mathcal{E})$ on which there is an analytic function $f : W \to \mathbb{C}$ such that $\mathfrak{V}_D \cap W = \{K \in W : f(K) = 0\}$. In other words, $\mathfrak{V}_D$ is an analytic variety of codimension 1.

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Recall that $\text{bg-spec}(A) \ni \lambda \mapsto \mathfrak{K}(\lambda) = \pi_{\max}\mathcal{K}_\lambda$ is analytic. The theorem in the previous slide says that if $\mathcal{D} = D + \mathcal{D}_{\min}$ with $D \in \mathrm{Gr}_{d''}(\mathcal{E})$, then

$$\mathrm{spec}(A_{\mathcal{D}}) \cap \text{bg-res}(A) = \mathfrak{K}^{-1}(\mathscr{V}_D).$$