Elliptic operators on compact manifolds with simple strata

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Overview

1. I'll discuss some aspects of the general asymptotics of the resolvent of an elliptic operator in the realm of conical singularities assuming existence of rays of minimal for the principal symbols.

2. Recent results on the nature of domains of elliptic complexes.

3. The bundle of traces (the bundle of Cauchy data) of an elliptic "wedge" operator on a compact manifold with fibered boundary and some aspects of the analysis on these.

Work in collaboration with Juan Gil or Thomas Krainer or both.

Background

$\mathcal{M}$ is a compact manifold with boundary $\mathcal{N}$, the latter fibers over another manifold $\mathcal{Y}$, with fibers $\mathcal{Z}$:

$$\begin{array}{ccc} \mathcal{Z} \hookrightarrow & \mathcal{N} & \subset \mathcal{M} \\ & \downarrow & \\ & \mathcal{Y} & \end{array}$$

Operators: $A = x^{-m}P$ with $P \in \mathrm{Diff}_e^m(\mathcal{M}; E, F)$. This means, $P \in \mathrm{Diff}^m(\mathcal{M}; E, F)$ and in coordinates x, $y_1, \dots, y_q$, $z_1, \dots, z_{n-q-1}$ near $p_0 \in \mathcal{N}$:

$$A = \frac{1}{x^m} \sum_{k+|\alpha|+|\beta| \le m} a_{k,\alpha,\beta}(x, y, z)(xD_x)^k (xD_y)^\alpha D_z^\beta.$$

Ellipticity: ellipticity in the interior and

$$\sum_{k+|\alpha|+|\beta|=m} a_{k,\alpha,\beta}(x, y, z)\xi^k \eta^\alpha \zeta^\beta \text{ is invertible if } (\xi, \eta, \zeta) \ne 0$$

$$\mathcal{Z} \hookrightarrow \mathcal{N} \subset \mathcal{M}, \quad \mathcal{N} \downarrow \mathcal{Y}$$

Fancy definition:

Let $\mathcal{R} = \{f \in C^\infty(\mathcal{M}) : f|_{\mathcal{N}} \in \wp^* C^\infty(\mathcal{Y})\}$.

$$P \in \operatorname{Diff}^m_e(\mathcal{M}; E, F) \iff P \in \operatorname{Diff}^m(\mathcal{M}; E, F)$$
$$\text{and for all } f \in \mathcal{R} : \ [P, f] \in x \operatorname{Diff}^{m-1}_e(\mathcal{M}; E, F)$$

w-cotangent bundle: ${}^w\pi : {}^wT^*\mathcal{M} \to \mathcal{M}$, the vector bundle whose sections are

$$\{\alpha \in C^\infty(\mathcal{M}; T^*\mathcal{M}) : \iota^*_{\mathcal{Z}_y}\alpha = 0, \ \text{all } y \in \mathcal{Y}\}$$

Proposition:

$A \in x^{-m} \operatorname{Diff}^m_e(\mathcal{M}; E, F)$ has a principal symbol

$\mathcal{Z}_y = \wp^{-1}(y)$

$${}^w\boldsymbol{\sigma}(A) \in C^\infty({}^wT^*\mathcal{M}; \operatorname{Hom}({}^w\pi^* E, {}^w\pi^* F)).$$

Domains

E and F have Hermitian metrics, $\mathfrak{m}$ is a smooth positive density on $\mathcal{M}$, $\mathfrak{m}_b = \dfrac{1}{x}\mathfrak{m}$.
A is always elliptic. Initially

$$A : C_c^\infty(\mathcal{M}; E) \subset x^{-m/2} L_b^2(\mathcal{M}; E) \to x^{-m/2} L_b^2(\mathcal{M}; F) \qquad (*)$$

Then

$$\mathcal{D}_{\min} = \text{domain of closure of } (*),$$
$$\mathcal{D}_{\max} = \{u \in x^{-m/2} L_b^2(\mathcal{M}; E) : Au \in x^{-m/2} L_b^2(\mathcal{M}; F)\}$$

Inner product:

$$(u, v)_A = (u, v) + (Au, Av), \quad u, v \in \mathcal{D}_{\max}$$

Orthogonal of $\mathcal{D}_{\min}$ in $\mathcal{D}_{\max}$: $\mathcal{E}$.
Orthoprojection of $\mathcal{D}_{\max}$ onto $\mathcal{E}$: $\pi_{\max}$

Conic case ($\dim \mathcal{Y} = 0$)

$$\begin{array}{c} \mathcal{Z} \hookrightarrow \mathcal{N} \subset \mathcal{M} \\ \downarrow \\ \mathcal{Y} \end{array}$$

Proposition:

A with domain $\mathcal{D}_{\min}$ or $\mathcal{D}_{\max}$ is Fredholm. Consequently

$$\mathcal{D}_{\min} \text{ has finite codimension in } \mathcal{D}_{\max},$$

Every subspace $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$ is closed of finite codimension and

$$\operatorname{Ind}(A_{\mathcal{D}}) = \operatorname{Ind} A_{\min} + \dim(\mathcal{D}/\mathcal{D}_{\min}).$$

If $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$, then

$$\mathcal{D} = \mathcal{D}_{\min} + D, \quad D = \pi_{\max}(\mathcal{D}) \subset \mathcal{E}.$$

For spectral analysis work with $D \in \mathrm{Gr}_{d_0}(\mathcal{E})$, $\operatorname{Ind}(A_{\min}) + d_0 = 0$.

Proposition:

1. $\mathcal{D}_{\min} = \bigcap_{\varepsilon > 0} x^{m/2-\varepsilon} H_b^m(\mathcal{M}; E)$.
2. $\mathcal{E} = \mathcal{D}_{\max} \cap \ker(A^\star A + Id)$.

Definition:

1. bg-res(A):

$$\{\lambda \in \mathbb{C} : A_{\mathsf{min}} - \lambda \text{ is injective and } A_{\mathsf{max}} - \lambda \text{ is surjective}\}$$

2. bg-spec(A) $= \mathbb{C}\setminus$ bg-res.
3. $\mathcal{K}_\lambda = \ker(A_{\mathsf{max}} - \lambda)$, $\lambda \in$ bg-res(A).

If $\mathcal{K}_\lambda \cap \mathcal{D} = 0$, then $\mathcal{D}_{\mathsf{max}} = \mathcal{K}_\lambda \oplus \mathcal{D}$

For $\lambda \in$ bg-res A let $B_{\mathsf{max}}(\lambda)$ be the right inverse of $A_{\mathsf{max}} - \lambda$ with range is orthogonal to $\mathcal{K}_\lambda$ with respect to

$$(u, v)_A = (Au, Av) + (u, v),$$

and let $B_{\mathsf{min}}(\lambda)$ be the left inverse of $A_{\mathsf{min}} - \lambda$ with kernel the orthogonal of rg$(A_{\mathsf{min}} - \lambda)$ in $x^{-m/2}L^2_b$.

Proposition:

If $\lambda \in$ res $A_{\mathcal{D}}$, then $B_{\mathcal{D}}(\lambda) = (A_{\mathcal{D}} - \lambda)^{-1}$ is

$$B_{\mathcal{D}}(\lambda) = B_{\mathsf{max}}(\lambda) - \big(\mathrm{I} - B_{\mathsf{min}}(\lambda)(A-\lambda)\big)\pi_{\mathcal{K}_\lambda,\mathcal{D}}B_{\mathsf{max}}(\lambda)$$

For $\mathcal{K}_\lambda \cap \mathcal{D} = 0$: $\pi_{\mathcal{K}_\lambda,\mathcal{D}}$ is the projection on $\mathcal{K}_\lambda$.

In principle both $B_{\mathsf{min}}(\lambda)$ and $B_{\mathsf{max}}(\lambda)$ can be written as pseudodifferential operators, a purely analytic problem

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Both $I - B_{\mathsf{min}}(\lambda)(A-\lambda)$ and $\pi_{\mathcal{K}_\lambda,\mathcal{D}}$ vanish on $\mathcal{D}_{\mathsf{min}}$, so

$$B_{\mathcal{D}}(\lambda) = B_{\mathsf{max}}(\lambda) - \big(\mathrm{I} - B_{\mathsf{min}}(\lambda)(A-\lambda)\big)\pi_{\mathsf{max}}\,\pi_{\mathcal{K}_\lambda,\mathcal{D}}\,\pi_{\mathsf{max}} B_{\mathsf{max}}(\lambda).$$

On the other hand,

$$\lambda \in \operatorname{res} A_{\mathcal{D}} \iff \lambda \in \text{bg-res}\, A \text{ and } \pi_{\mathsf{max}}\mathcal{K}_\lambda \cap \pi_{\mathsf{max}}\mathcal{D} = 0,$$

The map $\pi_{\mathsf{max}}\,\pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\mathsf{max}}}$ is the projection on $K_\lambda = \pi_{\mathsf{max}}\mathcal{K}_\lambda$ according to this decomposition of $\mathcal{E}_{\mathsf{max}}$

$$B_{\mathcal{D}}(\lambda) = B_{\mathsf{max}}(\lambda) - \big(\mathrm{I} - B_{\mathsf{min}}(\lambda)(A-\lambda)\big)\pi_{K_\lambda,D}\pi_{\mathsf{max}} B_{\mathsf{max}}(\lambda)$$

The dimension of $K_\lambda = \pi_{\max}\mathcal{K}_\lambda$ is $d_1 = \operatorname{Ind}(A_{\max} - \lambda)$ if $\lambda \in$ bg-res(A).
For such λ, $\dim \mathcal{E} = d_0 + d_1$. Let $\mathcal{D} = D + \mathcal{D}_{\min}$, $D \in \operatorname{Gr}_{d_0}(\mathcal{E})$

$$\mathfrak{V}_D = \{\mathcal{V} \in \operatorname{Gr}_{d_1}(\mathcal{E}_{\max}) : \mathcal{V} \cap \pi_{\max}(\mathcal{D}) \neq 0\}.$$

(a codimension 1 complex subvariety of $\operatorname{Gr}_{d_1}(\mathcal{E}_{\max})$.

Lemma:

The condition that $\lambda \in \text{bg-res}\, A \cap \operatorname{spec} A_{\mathcal{D}}$ is $\pi_{\max}\mathcal{K}_\lambda \in \mathfrak{V}_D$ belongs to the set

$$\mathfrak{V}_{\mathcal{D}} = \{\mathcal{V} \in \operatorname{Gr}_{d'}(\mathcal{E}_{\max}) : \mathcal{V} \cap \pi_{\max}(\mathcal{D}) \neq 0\}.$$

For $\lambda_0 \in \text{bg-res}\, A$, the ray $\{r\lambda_0 : r > R\}$ contains no point of $\operatorname{spec} A_{\mathcal{D}}$ iff the curve in $\operatorname{Gr}_{d'}(\mathcal{E}_{\max})$ given by $r \mapsto \pi_{\max}\mathcal{K}_{r\lambda_0}$ has no point in $\mathfrak{V}_{\mathcal{D}}$ when $r > R$.

If $\mathcal{V} \in \operatorname{Gr}_{d'}(\mathcal{E}_{\max})\backslash\mathfrak{V}_{\mathcal{D}}$, then the norm of the projection on $\mathcal{V}$ using $\mathcal{E}_{\max} = \mathcal{V} \oplus D$ can be estimated in simple terms. This is used to estimate the norm of the resolvent of $A_{\mathcal{D}}$ near a point in $\operatorname{spec} A_{\mathcal{D}} \cap \text{bg-res}\, A$.

The elements of $\mathcal{E}$ are in one to one correspondence with certain elements of the form

$$\sum_{\sigma_0 \in \Sigma \cap \operatorname{spec}_b(P)} \sum_{\ell=0}^{N_{\sigma_0}} u_{\sigma_0,\ell} x^{i\sigma_0} \log^\ell x \quad a_{\sigma_0,\ell} \in C^\infty(\mathcal{N}; E_{\mathcal{N}}),\ x > 0.$$

$(\Sigma = \{-m/2 < \Im\sigma < m/2\})$
On such elements we have an action κ_ϱ, $\varrho \in R^+$:

$$(\kappa_\varrho u)(x,z) = u(\varrho x, z)$$

The condition $\mathcal{K}_\lambda \notin \mathfrak{V}_D$ can be expressed as the condition

$$\kappa_{|\lambda|^{-1/m}} D \notin \mathfrak{V}_{\mathcal{K}_{\lambda/|\lambda|}} \subset \operatorname{Gr}_{d_0}(\mathcal{E})$$

Theorem:
The ray through λ_0 is a ray of minimal growth for $A_{\mathcal{D}}$ if it is a ray of minimal growth for $\boldsymbol{\sigma}(A)$ and the limit set of $r \mapsto \kappa_{|r\lambda_0|^{-1/m}} D$ is disjoint from $\mathfrak{V}_{\mathcal{K}_{\lambda_0/|\lambda_0|}}$. The asymptotics of the resolvent can be obtained from those of $B_{\min}(\lambda)$, $B_{\max}(\lambda)$ and $\pi_{K_{\lambda/|\lambda|}, \kappa_{|\lambda|^{-1/m}} D}$

Complexes

Problem: Given the complex

$$0 \to C_c(\mathring{\mathcal{M}}; E^0) \xrightarrow{A_0} C_c^\infty(\mathring{\mathcal{M}}; E^1) \to \cdots$$
$$\cdots \to C_c^\infty(\mathring{\mathcal{M}}; E^{m-1}) \xrightarrow{A_{m-1}} C_c^\infty(\mathring{\mathcal{M}}; E^m) \to 0$$

with first order cone operators, elliptic (the sequence of principal symbols is exact), determine the spaces

$$\mathcal{E}^q = \text{orthogonal of } \mathcal{D}^q_{\min} \text{ in } \mathcal{D}^q_{\max}.$$

relative to $x^{-1/2}L^2_b(\mathcal{M}; E^q)$.
Let A^0_q be the normal operator of A_q. Let

$A^0_q = x^{-1}P^0_q$

$$S^q_\sigma = \{\sum_{\ell=0}^{N_{\sigma_0}} u_{\sigma_0,\ell} x^{i\sigma_0} \log^\ell x \quad u_{\sigma_0,\ell} \in C^\infty(\mathcal{N}; E^q_{\mathcal{N}})\}.$$

Get a complex

$$\cdots \to S^{q-1}_{\sigma-iq} \xrightarrow{A^0_{q-1}} S^q_\sigma \xrightarrow{A^0_q} S^{q+1}_{\sigma-iq+i} \to \ldots$$

With the complex

$$\cdots \to S^{q-1}_{\sigma-iq} \xrightarrow{A^0_{q1}} S^q_\sigma \xrightarrow{A^0_q} S^{q+1}_{\sigma-iq+i} \to \cdots$$

we have

Theorem:

$$\mathcal{E}^q \approx \bigoplus_{\sigma\in\Sigma} H^q_{A^0_*}.$$

Notes:

1. Take σ with $-3/2 < \Im\sigma < 1/2$. This gives $S^{q-1}_{\sigma-iq} \subset \mathcal{D}^{q-1}_{\min}$, so $A_{q-1}S^{q-1}_{\sigma-iq} \subset \mathcal{D}^q_{\min}$. So there are elements in the minimal domain of A_q with pole in $-1/2 < \Im\sigma < 1/2$.

2. Even worse: if $\chi(\sigma)$ is supported in $\Im\sigma < -1/2$ and $a \in H^1(\mathcal{N}; E^{q-1}$, then

$$v = \omega(x)\int_{\mathbb{C}} x^{i\sigma}\chi(\sigma)a(z)\,d\sigma \in \mathcal{D}^{q-1}_{\min}$$

so $v \in \mathcal{D}^{q-1}_{\min}$ and $u = A_{q-1}v \in \mathcal{D}^q_{\min}$ but the Mellin transform of u is not meromorphic in $\Im\sigma > -1/2$.

On the minimal domains

Proposition:

$$\mathcal{D}^q_{\max} \cap \bigcap_{\varepsilon>0} x^{1/2-\varepsilon} H_b^{-\infty} \subset \mathcal{D}^q_{\min}$$

Theorem: Let $u \in \mathcal{D}^q_{\min}$. There is $v \in \mathcal{D}^{q-1}_{\min}$ such that

$$u - A_{q-1} v \in \bigcap_{\varepsilon>0} x^{1/2-\varepsilon} H_b^{-\infty} \subset \mathcal{D}^q_{\min}$$

Back to fibered boundary

$$\begin{array}{ccc} \mathcal{Z} \hookrightarrow & \mathcal{N} & \subset \mathcal{M} \\ & \downarrow & \\ & \mathcal{Y} & \end{array}$$

The issue is that the boundary spectrum of A varies with $y \in \mathcal{Y}$.

If $\mathrm{spec}_b(A)$ does not intersect $\{\Im\sigma = \pm m/2\}$, then one can make a vector bundle, $\mathscr{T} \to \mathcal{Y}$ out of the boundary spectrum information. The vector field $x\partial_x$ is the infinitesimal generator of the action κ.

The coefficients of the terms $x^{i\sigma} \log^\ell x$ are like the Taylor coefficients of the Taylor polynomial at the boundary of solutions, and their regularity is conditioned by $\Im\sigma$.

Because of this one ends up working with Sobolev spaces with anisotropically varying regularity (and a pseudodifferential calculus reflecting this, of type $1, \delta$ for arbitrarily small $\delta > 0$).
All together, we can set up boundary value problems.

The simplest case, already resolved, is for first order "wedge"-elliptic operators; for these we have the exact analogue of the Lopatinskii-Shapiro condition of classical elliptic differential operators on a compact manifold with boundary.

The End (of this talk)

Happy Birthday, Victor!