Elliptic operators on manifolds with conical singularities, I

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Cylindrical ends

For the purposes of this talk, a manifold with cylindrical ends is a smooth manifold $\mathcal{M}$ in which there is a compact submanifold $\mathcal{K}$ with smooth boundary such that $\mathcal{M} \setminus \mathcal{K}$ is diffeomorphic to a disjoint union of cylinders

$$\mathcal{M} \setminus \mathcal{K} \approx \bigsqcup_{j} (-\infty, 0) \times \mathcal{N}_j$$

where the $\mathcal{N}_j$ are the components of $\partial \mathcal{K}$. 
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The cylindrical compactification of $\mathcal{M}$ is the manifold with boundary obtained by

1. attaching each one of the $\mathcal{N}_j$ at $-\infty$ to the corresponding component of $\mathcal{M} \setminus \mathcal{K}$ in the natural way,
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This is a topological construction.
Cylindrical ends

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\[
\mathcal{M} \setminus \mathcal{K} \cong \bigsqcup_j (\mathcal{M} \cap \mathcal{K})_{\sim} \times N_j
\]

where the \( N_j \) are the components of \( \partial \mathcal{K} \).

The cylindrical compactification of \( \mathcal{M} \) is the manifold with boundary obtained by

1. attaching each one of the \( N_j \) at \(-\infty\) to the corresponding component of \( \mathcal{M} \setminus \mathcal{K} \) in the natural way,

2. while at the same time regarding the functions

\[
(\mathcal{M} \setminus \mathcal{K})_{\sim} \ni (t, p) \mapsto x = e^t, \quad j = 1, 2, \ldots
\]

as smooth defining functions of the new boundary component.
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For the purposes of this talk, a manifold with cylindrical ends is a smooth manifold $M$ in which there is a compact submanifold $K$ with smooth boundary such that $M \setminus K$ is diffeomorphic to a disjoint union of cylinders

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$$( -\infty, 0 ) \times \mathcal{N}_{j} \ni ( t, p ) \mapsto x = e^{t}, \quad j = 1, 2, \ldots$$

as smooth defining functions of the new boundary component.

*The compactification is diffeomorphic to $\mathcal{K}$.***
\(b\)-metrics

The cylindrical metric \(dt^2 + h_j\) on \((-\infty, 0) \times N_j\), where \(h_j\) is a metric on \(N_j\), is transformed to

\[
\frac{dx^2}{x^2} + h_j
\]

under the change \(t = \log x\).
**b-metrics**

The cylindrical metric \( dt^2 + h_j \) on \( (−∞, 0) \times N_j \), where \( h_j \) is a metric on \( N_j \), is transformed to

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Let \( M \) be a compact manifold with boundary, let \( x \) be a defining function for its boundary, positive in \( \partial M \).

A *b-metric* on a compact manifold \( M \) with boundary together is a smooth Riemannian metric on \( \partial M \) which is of the form

\[
g = a \frac{dx^2}{x^2} + h
\]

near \( \partial M \), where \( h \) is such smooth and whose restriction to \( \{x = ε\} \) is a Riemannian metric for each \( ε ≥ 0 \).
A \( b \)-Laplacian is, naturally, the Laplace operator of a \( b \)-metric. Near \( \partial M \) (on functions) it has the general form

\[
a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j} xD_x D_y + \sum_{i,j=1}^{n} a_{ij} D_y D_y + a_0 xD_x + \sum_{j=1}^{n} a_j D_y \quad (*)
\]
A $b$-Laplacian is, naturally, the Laplace operator of a $b$-metric. Near $\partial M$ (on functions) it has the general form:

\[
\Delta = \frac{-1}{|g|^{1/2}} \partial_x \left( \frac{1}{|g|^{1/2}} g_{ij} \partial_x \right)
\]

where $(x, y)$ are coordinates near $\partial M$.
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$$a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j} xD_x D_yj + \sum_{i,j=1}^{n} a_{ij} D_yi D_yj + a_0 xD_x + \sum_{j=1}^{n} a_j D_yj$$

(*)

$(x, y)$ are coordinates near $\partial M$

For example, if $g = \frac{dx^2}{x^2}$ on $[0, \infty)$, then $|g| = \det[g_{ij}] = \frac{1}{x^2}$ and $[g^{ij}] = x^2$, 
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$$\Delta = -\frac{1}{|g|^{1/2}} \sum \frac{\partial}{\partial x_i} |g|^{1/2} g^{ij} \frac{\partial}{\partial x_j}$$
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The operator (*) is constructed using vector the fields $x \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y^j}$:
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For example, if $g = \frac{dx^2}{x^2}$ on $[0, \infty)$, then $|g| = \det[g_{ij}] = \frac{1}{x^2}$ and $[g_{ij}] = x^2$, so

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The operator \text{(**)} is constructed using vector the fields $x\partial_x$ and $\partial_y$: these are tangential vector vector fields.
**b-Manifolds**

A **b-manifold** is a manifold in which the role tangent bundle is replaced by that of the **b-tangent bundle**:
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A *b-manifold* is a manifold in which the role tangent bundle is replaced by that of the *b-tangent bundle*:

*The b-tangent bundle is the vector bundle* \( bT\mathcal{M} \to \mathcal{M} \) *whose smooth sections are the vector fields of* \( \mathcal{M} \) *which along* \( \partial \mathcal{M} \) *are tangential.*
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Let

$$C_{\text{tan}}(\mathcal{M}; TM) = \{X \in C^\infty(\mathcal{M}; TM) : X|_{\partial \mathcal{M}} \text{ is tangent to } \partial \mathcal{M}\}$$

There is

$$\text{ev} : bT\mathcal{M} \to TM$$

(a bundle homomorphism)

such that

$$\text{ev}_* : C^\infty(\mathcal{M}; bT\mathcal{M}) \xrightarrow{\sim} C_{\text{tan}}(\mathcal{M}; TM)$$
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$$C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M}) = \{ X \in C^\infty(\mathcal{M}; T\mathcal{M}) : X|_{\partial \mathcal{M}} \text{ is tangent to } \partial \mathcal{M} \}$$

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$C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M})$ is a finitely generated projective module over $C(\mathcal{M})$. By a theorem of Swan there is a vector bundle $bT\mathcal{M} \to \mathcal{M}$ whose space of continuous sections is $C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M})$. 

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\( C_{\tan}^{}(\mathcal{M}; T\mathcal{M}) \) is a finitely generated projective module over \( C(\mathcal{M}) \).

By a theorem of Swan there is a vector bundle \( bT\mathcal{M} \to \mathcal{M} \) whose space of continuous sections is \( C_{\tan}^{}(\mathcal{M}; T\mathcal{M}) \).

If \( x, y_1, \ldots, y_n \) is a local chart of \( \mathcal{M} \) near \( p \in \partial \mathcal{M} \), then

\[
x\partial_x, \partial_{y_1}, \ldots, \partial_{y_n}
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are the image elements of a frame of \( bT\mathcal{M} \).
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$C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M})$ *is a finitely generated projective module over* $C(\mathcal{M})$.

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If $x, y_1, \ldots, y_n$ is a local chart of $\mathcal{M}$ near $p \in \partial \mathcal{M}$, then

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are the image elements of a frame of $^bT \mathcal{M}$.

$\text{ev}$ is a bundle isomorphism over $\mathcal{M}$, has 1-dimensional kernel over $\partial \mathcal{M}$.

$x \partial_x$ is a canonical section of $^bT \mathcal{M}$ along $\partial \mathcal{M}$. 
The $b$-cotangent bundle; $b$-metrics

The $b$-cotangent bundle is the dual of the $b$-tangent bundle:

$$bT^*M = (bTM)^*$$

If $x\partial_x, \partial_{y_1}, \ldots, \partial_{y_n}$ is a frame of $bTM$ near $p_0 \in \partial M$, then

$$\frac{dx}{x}, dy_1, \ldots, dy_n$$

is a frame of $bT^*M$ near $p_0$. The bundle map $ev : bTM \rightarrow TM$ gives

$$ev^* : T^*M \rightarrow bT^*M$$
The \( b \)-cotangent bundle; \( b \)-metrics

The \( b \)-cotangent bundle is the dual of the \( b \)-tangent bundle:

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\[
\frac{dx}{x}, dy_1, \ldots, dy_n
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is a frame of \( bT^*M \) near \( p_0 \). The bundle map \( \text{ev} : bT^*M \to TM \) gives

\[
\text{ev}^* : T^*M \to bT^*M
\]

\( \text{ev}^* \) is a bundle isomorphism over \( \tilde{M} \), has 1-dimensional kernel over \( \partial M \).

A \( b \)-metric is a Riemannian metric on \( bT^*M \). Locally such metrics have the form

\[
g_{00} \left( \frac{dx}{x} \otimes \frac{dx}{x} \right) + \sum_j g_{0j} \left( \frac{dx}{x} \otimes dy_j + dy_j \otimes \frac{dx}{x} \right) + \sum_{i,j} a_{ij} dy_i \otimes dy_j.
\]
**b-Differential operators**

A linear $b$-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial \mathcal{M}$ has the form

$$
\sum_{\alpha+k \leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial \mathcal{M}.
$$

More generally:

*Let $E, F \to \mathcal{M}$ be vector bundles. A differential operator

$$
P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)
$$

is a $b$-differential operator of order $m$ if

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x^{-\nu}Px^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.
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* $x$ is a defining function for $\partial \mathcal{M}$ with $x > 0$ in $\mathcal{M}$.  

Diff$^m(\mathcal{M}; E, F)$ is the space of linear differential operators of order $m$ with smooth coefficients.*
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Example: Let $P = a \partial_x$. Then

$$Pxu = a \partial_x xu = x a \partial_x u + au$$
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Example: Let \( P = a\partial_x \). Then

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x^{-1}Pxu = x^{-1}a\partial_x xu = xa\partial_x u + x^{-1}au.
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If this has smooth coefficients, then $a = xa'$. 

$x$ is a defining function for $\partial \mathcal{M}$ with $x > 0$ in $\mathcal{M}$.
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If this has smooth coefficients, then $a = xa'$. So $P = a' x \partial_x$. 

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A linear $b$-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial \mathcal{M}$ has the form

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$$x^{-1}Pxu = x^{-1}a \partial_x xu = xa \partial_x u + x^{-1}au.$$ 

If this has smooth coefficients, then $a = xa'$. So $P = a'x \partial_x$. 

$\text{Diff}^m_b(\mathcal{M}; E, F)$ is the space of linear $b$-differential operators on $\mathcal{M}$ of order $m$ with smooth coefficients.
**b-Symbol**

The vector bundle map \( \text{ev} : bTM \to TM \) is an isomorphism over \( \mathring{M} \), so the dual map \( \text{ev}^* : T^*M \to bT^*M \) is also an isomorphism over \( \mathring{M} \).

Let \( P \in \text{Diff}_b^m(M; E, F) \). Its principal symbol is a homomorphism

\[
\sigma(P) : \pi^* E \to \pi^* F
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$b$-Symbol

The vector bundle map $\text{ev} : bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\hat{\mathcal{M}}$, so the dual map $\text{ev}^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\hat{\mathcal{M}}$.

Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \to \pi^*F$$

$\sigma(P)$ is a section of $\pi^*\text{Hom}(E, F)$
The vector bundle map \( \text{ev} : bT\mathcal{M} \to T\mathcal{M} \) is an isomorphism over \( \mathcal{M} \), so the dual map \( \text{ev}^* : T^*\mathcal{M} \to bT^*\mathcal{M} \) is also an isomorphism over \( \mathcal{M} \).

Let \( P \in \text{Diff}^m_b(\mathcal{M}; E, F) \). Its principal symbol is a homomorphism \( \sigma(P) : \pi^*E \to \pi^*F \), \( \sigma(P) \) is a section of \( \pi^*\text{Hom}(E, F) \). 

---

**b-Symbol**

\[ \pi : T^*\mathcal{M} \to \mathcal{M} \]
The vector bundle map $ev : bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $ev^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \to \pi^*F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^*E \to b\pi^*F$$

by way of the following argument:
$b$-Symbol

The vector bundle map $ev : bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $ev^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \to \pi^*F$$

$\sigma(P)$ is a section of $\pi^* \text{Hom}(E, F)$

This induces a homomorphism

$$b\sigma(P) : b\pi^*E \to b\pi^*F$$

by way of the following argument:

$$\pi^* \text{Hom}(E, F)$$

$$\pi$$

$$T^*\mathcal{M}$$
The vector bundle map $ev : bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $ev^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}^m_\mathcal{B}(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^* E \to \pi^* F$$

$\sigma(P)$ is a section of $\pi^* \text{Hom}(E, F)$

This induces a homomorphism

$$b\sigma(P) : b\pi^* E \to b\pi^* F$$

by way of the following argument:

\[
\begin{array}{ccc}
\pi^* \text{Hom}(E, F) & \xrightarrow{\pi} & \sigma(P) \\
\downarrow & & \downarrow \\
T^*\mathcal{M} & & T^*\mathcal{M}
\end{array}
\]
The vector bundle map $ev : bT \mathcal{M} \to T \mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $ev^* : T^* \mathcal{M} \to bT^* \mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^* E \to \pi^* F$$

$\sigma(P)$ is a section of $\pi^* \text{Hom}(E, F)$

This induces a homomorphism

$$b\sigma(P) : b\pi^* E \to b\pi^* F$$

by way of the following argument:

$$\begin{array}{ccc}
\pi^* \text{Hom}(E, F) & \xrightarrow{\pi} & \pi^* \text{Hom}(E, F)\\
b\pi & \downarrow & \pi\\
bT^* \mathcal{M} & \xrightarrow{\sigma(P)} & T^* \mathcal{M}
\end{array}$$
$b$-Symbol

The vector bundle map $\text{ev} : bTM \to TM$ is an isomorphism over $\overset{\circ}{\mathcal{M}}$, so the dual map $\text{ev}^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\overset{\circ}{\mathcal{M}}$.

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^* E \to \pi^* F$$

$\sigma(P)$ is a section of $\pi^* \text{Hom}(E, F)$

This induces a homomorphism

$$b\sigma(P) : b\pi^* E \to b\pi^* F$$

by way of the following argument:

$$\begin{array}{ccc}
\pi^* \text{Hom}(E, F) & \pi \downarrow \\
\pi^* \text{Hom}(E, F) & \sigma(P) \\
T^*\mathcal{M} & T^*\mathcal{M} & \\
bT^*\mathcal{M} & bT^*\mathcal{M} \leftarrow \\
b\pi^* \text{Hom}(E, F) & b\pi^* E \\
\text{ev}^* & \text{ev}^* \leftarrow \\
isomorphism \text{ over } \overset{\circ}{\mathcal{M}}, & \text{image of codimension 1 over } \partial \mathcal{M}
\end{array}$$
The vector bundle map $\text{ev} : bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $\text{ev}^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^* E \to \pi^* F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^* E \to b\pi^* F$$

by way of the following argument:

$$\begin{array}{ccc}
b\pi^* \text{Hom}(E, F) & \xleftarrow{\text{ev}^*} & \pi^* \text{Hom}(E, F) \\
b\pi & \downarrow & \pi \\
bT^*\mathcal{M} & \xrightarrow{\text{ev}^*} & T^*\mathcal{M}
\end{array}$$

$\sigma(P)$ is a section of $\pi^* \text{Hom}(E, F)$.
The vector bundle map \( \text{ev} : bT\mathcal{M} \rightarrow T\mathcal{M} \) is an isomorphism over \( \mathring{\mathcal{M}} \), so the dual map \( \text{ev}^* : T^*\mathcal{M} \rightarrow bT^*\mathcal{M} \) is also an isomorphism over \( \mathring{\mathcal{M}} \).

Let \( P \in \text{Diff}^m_b(\mathcal{M}; E, F) \). Its principal symbol is a homomorphism \( \sigma(P) : \pi^*E \rightarrow \pi^*F \)

This induces a homomorphism \( b\sigma(P) : b\pi^*E \rightarrow b\pi^*F \)

by way of the following argument:

\[
\begin{array}{ccc}
 b\pi^*\text{Hom}(E, F) & \xleftarrow{\text{ev}^*} & \pi^*\text{Hom}(E, F) \\
b\pi \downarrow & & \pi \downarrow \\
bT^*\mathcal{M} & \xleftarrow{\text{isomorphism over } \mathring{\mathcal{M}}, \text{image of codimension 1 over } \partial\mathcal{M}} & T^*\mathcal{M} \\
\end{array}
\]

\( \sigma(P) \) is a section of \( \pi^*\text{Hom}(E, F) \)
**b-Symbol**

The vector bundle map $ev : bT\mathcal{M} \rightarrow T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $ev^* : T^*\mathcal{M} \rightarrow bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \rightarrow \pi^*F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^*E \rightarrow b\pi^*F$$

by way of the following argument:

\[
\begin{array}{ccc}
\pi^* \text{Hom}(E, F) & \xrightarrow{\pi} & \pi^* \text{Hom}(E, F) \\
b\pi & \downarrow & b\pi \\
bT^*\mathcal{M} & \xleftarrow{ev^*} & T^*\mathcal{M}
\end{array}
\]

With frame $[x\partial_x], [\partial_y], \text{dual frame } [\frac{dx}{x}, [dy_j], \text{the dual of}$

$$ev : a_0[x\partial_x] + \sum a_j[\partial_y] \mapsto a_0x\partial_x + \sum a_j\partial_y$$

is

$$\nu = \xi dx + \sum \eta_j dy_j \mapsto$$

$$ev^*(\nu) = \xi x\left[\frac{dx}{x}\right] + \sum \eta_j[dy_j].$$
The vector bundle map \( \text{ev} : \mathcal{T} \rightarrow \mathcal{M} \) is an isomorphism over \( \mathcal{M} \), so the dual map \( \text{ev}^* : \mathcal{F} \rightarrow \mathcal{T} \) is also an isomorphism over \( \mathcal{M} \).

Let \( \mathcal{P} \in \text{Diff}^m(\mathcal{M}; E, F) \). Its principal symbol is a homomorphism

\[
\sigma(\mathcal{P}) : \pi^* E \rightarrow \pi^* F
\]

This induces a homomorphism

\[
b\sigma(\mathcal{P}) : b\pi^* E \rightarrow b\pi^* F
\]

by way of the following argument:

\[
\begin{array}{ccc}
b\pi^* \text{Hom}(E, F) & \xrightarrow{b\pi} & \pi^* \text{Hom}(E, F) \\
\xrightarrow{b\sigma(\mathcal{P})} & & \xrightarrow{\pi}\sigma(\mathcal{P}) \\
\mathcal{T}^* \mathcal{M} & \xrightarrow{\text{ev}^*} & \mathcal{F}
\end{array}
\]

\[
\text{isomorphism over } \mathcal{M},
\]

\[
\text{image of codimension 1 over } \partial \mathcal{M}
\]

so if \( \tilde{\nu} = \text{ev}^* \nu = \tilde{\xi} \left[ \frac{dx}{x} \right] + \sum \eta_j [dy_j] \)

With frame \( [x\partial_x], [\partial_y] \), dual frame \( \left[ \frac{dx}{x} \right], [dy_j] \), the dual of

\[
\text{ev} : a_0[x\partial_x] + \sum a_j [\partial_y] \mapsto a_0 x\partial_x + \sum a_j \partial_y
\]

is

\[
\nu = \xi dx + \sum \eta_j dy_j \mapsto
\]

\[
\text{ev}^*(\nu) = \xi x \left[ \frac{dx}{x} \right] + \sum \eta_j [dy_j].
\]

If \( \mathcal{P} = \sum_{k+|\alpha| \leq m} a_{k\alpha} (xD_x)^k D_y^\alpha \)

then

\[
\sigma(\mathcal{P})(\nu) = \sum_{k+|\alpha| = m} a_{k\alpha} (x\xi)^k \eta^\alpha
\]
$b$-Symbol

The vector bundle map $\text{ev} : bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $\text{ev}^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}^m_b(M; E, F)$. Its principal symbol is a homomorphism $\sigma(P) : \pi^*E \to \pi^*F$. This induces a homomorphism $b\sigma(P) : b\pi^*E \to b\pi^*F$ by way of the following argument:

$$
\begin{array}{ccc}
\pi^*\text{Hom}(E, F) & \xleftarrow{b\pi} & \pi^*\text{Hom}(E, F) \\
\downarrow & & \downarrow \\
bT^*\mathcal{M} & \xleftarrow{\text{ev}^*} & T^*\mathcal{M} \\
\xrightarrow{\text{isomorphism over } \mathcal{M}} & & \xrightarrow{\text{image of codimension 1 over } \partial\mathcal{M} }
\end{array}
$$

so if $\tilde{\nu} = \text{ev}^*\nu = \left(\xi\frac{dx}{x}\right) + \sum \eta_j[dy_j]]$

With frame $[x\partial_x], [\partial_y]$, dual frame $\left[\frac{dx}{x}\right], [dy_j]$, the dual of $\text{ev} : a_0[x\partial_x] + \sum a_j[\partial_y] \mapsto a_0x\partial_x + \sum a_j\partial_yj$

is $\nu = \xi dx + \sum \eta_jdy_j \mapsto$

$$
\text{ev}^*(\nu) = \left(\xi\frac{dx}{x}\right) + \sum \eta_j[dy_j].
$$

If $P = \sum_{k+|\alpha|\leq m} a_{k\alpha}(xD_x^kD_y^\alpha$ then

$$
\sigma(P)(\nu) = \sum_{k+|\alpha|=m} a_{k\alpha}(x\xi)^k\eta^\alpha
$$
The vector bundle map \( \text{ev} : bT\mathcal{M} \to T\mathcal{M} \) is an isomorphism over \( \mathcal{M} \), so the dual map \( \text{ev}^* : T^*\mathcal{M} \to bT^*\mathcal{M} \) is also an isomorphism over \( \mathcal{M} \).

Let \( P \in \text{Diff}^m_b(\mathcal{M}; E, F) \). Its principal symbol is a homomorphism
\[
\sigma(P) : \pi^*E \to \pi^*F
\]
This induces a homomorphism
\[
b\sigma(P) : b\pi^*E \to b\pi^*F
\]
by way of the following argument:
\[
b\pi^* \text{Hom}(E, F) \xleftarrow{\pi} \pi^* \text{Hom}(E, F)
\]

So if \( \tilde{\nu} = \text{ev}^*\nu = (\xi) \frac{dx}{x} + \sum \eta_j dy_j \) then \( \sigma(P)((\text{ev}^*)^{-1}\tilde{\nu}) = \sum_{k+|\alpha|=m} a_{k\alpha} (x\xi)^k \eta^\alpha \).
$b$-Symbol

The vector bundle map $\text{ev} : bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $\text{ev}^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \to \pi^*F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^*E \to b\pi^*F$$

by way of the following argument:

$$b\pi^*\text{Hom}(E, F) \xrightarrow{b\pi} b\pi^*\text{Hom}(E, F) \xrightarrow{\pi} \pi^*\text{Hom}(E, F)$$

$$\xrightarrow{\pi^*\text{Hom}(E, F)}$$

$$bT^*\mathcal{M} \xrightarrow{\text{ev}^*} T^*\mathcal{M}$$

isomorphism over $\mathcal{M}$, image of codimension 1 over $\partial\mathcal{M}$

so if $\tilde{\nu} = \text{ev}^*\nu = \begin{pmatrix} \xi \end{pmatrix} \frac{dx}{x} + \sum \eta_j[dy_j]$ then $\sigma(P)((\text{ev}^*)^{-1}\tilde{\nu}) = \sum_{k+|\alpha|=m} a_{k\alpha}(x\xi)^k \eta^\alpha$.
**b-ellipticity**

Naturally,

*The operator $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$ is b-elliptic if $^b\sigma(P)$ is invertible on $^bT^*\mathcal{M}\setminus0$.***

**Example:** Let $g = \frac{dx^2}{x^2} + dy^2$ on $[0, \infty) \times S^1$. Then

$$\Delta = -(x\partial_x)^2 - \partial_y^2$$
**b-ellipticity**

Naturally,

The operator \( P \in \text{Diff}_b^m(\mathcal{M}; E, F) \) is \( b \)-elliptic if \( b\sigma(P) \) is invertible on \( bT^*\mathcal{M}\setminus 0 \).

**Example:** Let \( g = \frac{dx^2}{x^2} + dy^2 \) on \([0, \infty) \times S^1\). Then

\[
\Delta = -(x\partial_x)^2 - \partial_y^2, \quad \sigma(\Delta) = (x\xi)^2 + \eta^2
\]

and so

\[
b\sigma(\Delta) = \xi^2 + \eta^2.
\]

Thus \( \Delta \) is \( b \)-elliptic.
**b-ellipticity**

Naturally,

The operator $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$ is $b$-elliptic if $^b\sigma(P)$ is invertible on $^bT^*\mathcal{M}\setminus 0$.

**Example**: Let $g = \frac{dx^2}{x^2} + dy^2$ on $[0, \infty) \times S^1$. Then

$$
\Delta = -(x\partial_x)^2 - \partial_y^2, \quad \sigma(\Delta) = (x\xi)^2 + \eta^2
$$

and so

$$
^b\sigma(\Delta) = \xi^2 + \eta^2.
$$

Thus $\Delta$ is $b$-elliptic.

In general, if $P = \sum_{k+|\alpha| \leq m} a_{k\alpha} (xD_x)^k D_y^\alpha$, then

$$
^b\sigma(P) = \sum_{k+|\alpha| = m} a_{k\alpha} \xi^k \eta^\alpha
$$
Mellin transform

If $\mathcal{M}$ is a manifold with boundary, then

$$\dot{\mathcal{C}}^\infty(\mathcal{M}) = \{ u \in C^\infty(\mathcal{M}) : u \text{ vanishes to infinite order on } \partial\mathcal{M} \}$$

Let $u \in \dot{\mathcal{C}}^\infty[0, \infty)$ be compactly supported. The Mellin transform of $u$ is

$$\mathcal{M}(u)(\sigma) = \int_0^\infty x^{-i\sigma} u(x) \frac{dx}{x}$$

The inverse Mellin transform is

$$\mathcal{M}^{-1}(v) = \frac{1}{2\pi} \int_{\text{Re } \sigma = 0} x^{i\sigma} v(\sigma) \, d\sigma.$$ 

The Mellin transform is the Fourier transform with $e^t$ replaced by $x$. 

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If \( u \in L^2([0, \infty); \frac{dx}{x}) \) then \( \mathcal{M}(u) \in L^2(\mathbb{R}) \).

Further, if \( u \in L^2([0, \infty); \frac{dx}{x}) \) has compact support, then \( \mathcal{M}(u) \) is holomorphic in \( \text{Im} \sigma > 0 \): Since \( |x^{-i \sigma}| = x^{\text{Im} \sigma} \),

\[
\int_0^\infty x^{-i \sigma} u(x) \frac{dx}{x}
\]

is integrable if \( \text{Im} \sigma > 0 \) and defines a holomorphic function of \( \sigma \) there.
If \( u \in L^2([0, \infty); \frac{dx}{x}) \) then
\[
\mathcal{M}(u) \in L^2(\mathbb{R}).
\]

Further, if \( u \in L^2([0, \infty); \frac{dx}{x}) \) has compact support, then \( \mathcal{M}(u) \) is holomorphic in \( \text{Im} \, \sigma > 0 \): Since \( |x^{-i\sigma}| = x^{\text{Im} \, \sigma} \),
\[
\int_{0}^{\infty} x^{-i\sigma} u(x) \frac{dx}{x}
\]
is integrable if \( \text{Im} \, \sigma > 0 \) and defines a holomorphic function of \( \sigma \) there.

A (smooth) \( b \)-density is a density on \( \mathcal{M} \) of the form \( m_b = \frac{1}{x} m \) where \( m \) is a smooth positive density on \( M \). Define
\[
L^2_b(M) = L^2(M, m_b)
\]

Let \( N = \partial M \), let \([0, \varepsilon) \times N\) be a tubular neighborhood of \( N \) in \( M \). Let \( \omega \in C^\infty(M) \), \( \omega = 1 \) near \( N \), \( \omega \) compactly supported in \([0, \varepsilon) \times N\).

\[
\mathcal{M}(u)(\sigma, p) = \int_{0}^{\infty} x^{-i\sigma} \omega(x, p) u(x, p) \frac{dx}{x}, \quad p \in N, \sigma \in \mathbb{C}, \text{Im} \, \sigma \geq 0.
\]
If $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u}|_{\mathcal{N}}$. 

If $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u}|_\mathcal{N}$.

$P\tilde{u}|_\mathcal{N}$ depends only on $u$. The difference between two extensions of $u$ is $xv$, $v \in C^\infty(\mathcal{M}; E)$. By definition of $b$ operator, $x^{-1}Pxv$ is smooth up to $\mathcal{N}$ if $v$ is. So $P(xv)|_\mathcal{N} = 0$. 
If $P \in \text{Diff}_b^m(M; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial M$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u}\big|_\mathcal{N}$.

$P\tilde{u}\big|_\mathcal{N}$ depends only on $u$. The difference between two extensions of $u$ is $xv, \; v \in C^\infty(M; E)$. By definition of $b$ operator, $x^{-1}Pxv$ is smooth up to $\mathcal{N}$ if $v$ is. So $P(xv)\big|_\mathcal{N} = 0$.

Note that $P \in \text{Diff}_b^m \implies x^{-i\sigma}Px^i\sigma \in \text{Diff}_b^m$ for all $\sigma \in \mathbb{C}$:

$$(xD_x)^k(x^i\sigma u) = x^i\sigma (xD_x + \sigma)^k u$$
If $P \in \text{Diff}_b^m (\mathcal{M}; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

Given $u \in C^\infty (\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u}|_{\mathcal{N}}$.

$P\tilde{u}|_{\mathcal{N}}$ depends only on $u$. The difference between two extensions of $u$ is $xv$, $v \in C^\infty (\mathcal{M}; E)$. By definition of $b$ operator, $x^{-1}Pxv$ is smooth up to $\mathcal{N}$ if $v$ is. So $P(xv)|_{\mathcal{N}} = 0$.

Note that $P \in \text{Diff}_b^m \implies x^{-i\sigma}Px^{i\sigma} \in \text{Diff}_b^m$ for all $\sigma \in \mathbb{C}$:

$$(xD_x)^k (x^{i\sigma} u) = x^{i\sigma} (xD_x + \sigma)^k u$$

$P = P(x, y, xD_x, D_y) \implies x^{-i\sigma}Px^{i\sigma} = P(x, y, xD_x + \sigma, D_y)$.
If \( P \in \text{Diff}^m_b(\mathcal{M}; E, F) \), then \( P \) defines an operator \( P_b \) on \( \mathcal{N} = \partial \mathcal{M} \):

Given \( u \in C^\infty(\mathcal{N}; E) \), let \( \tilde{u} \) be a smooth extension of \( u \), let \( P_bu = P\tilde{u} \big|_{\mathcal{N}} \).

\( P\tilde{u} \big|_{\mathcal{N}} \) depends only on \( u \). The difference between two extensions of \( u \) is \( xv, v \in C^\infty(\mathcal{M}; E) \). By definition of \( b \) operator, \( x^{-1}Pxv \) is smooth up to \( \mathcal{N} \) if \( v \) is. So \( P(xv) \big|_{\mathcal{N}} = 0 \).

Note that \( P \in \text{Diff}^m_b \implies x^{-i\sigma}Px^{i\sigma} \in \text{Diff}^m_b \) for all \( \sigma \in \mathbb{C} \):

\[
(xD_x)^k(x^{i\sigma}u) = x^{i\sigma}(xD_x + \sigma)^k u
\]

\[
P = P(x, y, xD_x, D_y) \implies x^{-i\sigma}Px^{i\sigma} = P(x, y, xD_x + \sigma, D_y).
\]

The indicial operator of \( P \in \text{Diff}^m_b(\mathcal{M}; E, F) \) is

\[
\hat{P}(\sigma) = (x^{-i\sigma}Px^{i\sigma})_b
\]

\( \hat{P}(\sigma) \) is a polynomial in \( \sigma \in \mathbb{C} \) with values in \( \text{Diff}(\mathcal{N}; E, F) \).
Let $P \in \text{Diff}_b^m(M; E, F)$, so

$$P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D^\alpha_y$$

locally. Then

$$x^{i\sigma} P x^{-i\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D^\alpha_y$$

and

$$x^{i\sigma} P x^{-i\sigma} \bigg|_\mathcal{N} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D^\alpha_y.$$
Let $P \in \text{Diff}^m_b (\mathcal{M}; E, F)$, so

$$P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha$$

locally. Then

$$x^{i\sigma} P x^{-i\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha$$

and

$$\hat{P}(\sigma) = x^{i\sigma} P x^{-i\sigma} \big|_{\mathcal{N}} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.$$
Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$, so

$$P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha$$

locally. Then

$$x^{i\sigma} P x^{-i\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha$$

and

$$\hat{P}(\sigma) = x^{i\sigma} P x^{-i\sigma} \bigg|_{\tilde{\mathcal{N}}} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.$$
Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$, so

$$P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha$$

locally. Then

$$x^{i\sigma} P x^{-i\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha$$

and

$$\hat{P}(\sigma) = x^{i\sigma} P x^{-i\sigma} \bigg|_{\mathcal{N}} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.$$ 

The principal symbol of $\hat{P}(\sigma)$ is

$$\sigma(\hat{P}(\sigma)) = \sum_{|\alpha|=m} a_{0\alpha}(0, y)\eta^\alpha = b_{\sigma}(P)(0, y, 0, \eta)$$

if $|\Im \sigma| / |\Re \sigma| < c$ with small enough $c > 0$. 

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Let $P \in \text{Diff}_b^m(M; E, F)$, so

$$P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha$$

locally. Then

$$x^{i\sigma} P x^{-i\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha$$

and

$$\hat{P}(\sigma) = x^{i\sigma} P x^{-i\sigma} \bigg|_{\mathcal{N}} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.$$ 

The principal symbol of $\hat{P}(\sigma)$ is

$$\sigma(\hat{P}(\sigma)) = \sum_{|\alpha| = m} a_{0\alpha}(0, y)\eta^{\alpha} = b_\sigma(P)(0, y, 0, \eta)$$

If $P$ is elliptic, then $\hat{P}(\sigma)$ is elliptic.
Let $P \in \text{Diff}^m_b(M; E, F)$, so

$$P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha$$

locally. Then

$$x^i \sigma P x^{-i \sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha$$

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$$\hat{P}(\sigma) = x^i \sigma P x^{-i \sigma} \bigg|_{\mathcal{N}} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y) \sigma^k D_y^\alpha.$$ 

The principal symbol of $\hat{P}(\sigma)$ is

$$\sigma(\hat{P}(\sigma)) = \sum_{|\alpha|=m} a_{0\alpha}(0, y) \eta^\alpha = b_\sigma(P)(0, y, 0, \eta)$$

If $P$ is elliptic, then $\hat{P}(\sigma)$ is elliptic.

More is true because $\sum_{k+|\alpha| = m} a_{k\alpha}(x, y) \sigma^k \eta^\alpha \neq 0$ if $\Re \sigma$ is large relative to $|\Im \sigma|$.
Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$, so

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locally. Then

$$x^i\sigma P x^{-i\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha$$

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$$\hat{P}(\sigma) = x^i\sigma P x^{-i\sigma} \big|_\mathcal{N} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.$$  

The principal symbol of $\hat{P}(\sigma)$ is

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More is true because $\sum_{k+|\alpha| = m} a_{k\alpha}(x, y)\sigma^k \eta^\alpha \neq 0$ if $\text{Re} \sigma$ is large relative to $|\text{Im} \sigma|$.

$$\sum_{k+|\alpha| = m} a_{k\alpha}(0, y)\sigma^k \eta^\alpha = \sum_{k+|\alpha| = m} a_{k\alpha}(0, y)(\text{Re} \sigma + i\text{Im} \sigma)^k \eta^\alpha$$

$$= \sum_{k+|\alpha| = m} a_{k\alpha}(0, y)(\text{Re} \sigma)^k (1 + i\text{Im} \sigma / \text{Re} \sigma)^k \eta^\alpha$$

$$= b_\sigma(P)(0, y, \text{Re} \sigma, \eta)(I + \mathcal{O}(|\text{Re} \sigma|^{-1}))$$

if $|\text{Im} \sigma / \text{Re} \sigma| < c$ with small enough $c > 0$.  

Using the invertibility of

$$\sum_{k+|\alpha|=m} a_{k\alpha}(0, y) \sigma^k \eta^\alpha$$

(with $|\text{Im} \sigma| < c|\text{Re} \sigma|$) one finds a family of pseudodifferential operators $Q(\sigma)$ on $\mathcal{N}$ such that

$$Q(\sigma) \hat{P}(\sigma) = I - R(\sigma)$$

with $R(\sigma)$ of order $-1$ and $\|R(\sigma)\| \leq C/|\text{Re} \sigma|$.
Using the invertibility of
\[ \sum_{k + |\alpha| = m} a_{k\alpha}(0, y) \sigma^k \eta^\alpha \]
(with \(|\text{Im } \sigma| < c|\text{Re } \sigma| \)) one finds a family of pseudodifferential operators \(Q(\sigma)\) on \(N\) such that
\[ Q(\sigma) \hat{P}(\sigma) = I - R(\sigma) \]
with \(R(\sigma)\) of order \(-1\) and \(||R(\sigma)|| \leq C/|\text{Re } \sigma|\). Consequently:
\[ \hat{P}(\sigma) \text{ is invertible in } |\text{Im } \sigma| < c|\text{Re } \sigma|. \]
Using the invertibility of
\[
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(with \(|\text{Im} \sigma| < c|\text{Re} \sigma|\)) one finds a family of pseudodifferential operators \(Q(\sigma)\) on \(\mathcal{N}\) such that
\[
Q(\sigma)\hat{P}(\sigma) = I - R(\sigma)
\]
with \(R(\sigma)\) of order \(-1\) and \(\|R(\sigma)\| \leq C/|\text{Re} \sigma|\). Consequently:

\(\hat{P}(\sigma)\) is invertible in \(|\text{Im} \sigma| < c|\text{Re} \sigma|\). Since
\[
\hat{P}(\sigma) : H^m(\mathcal{N}) \subset L^2(\mathcal{N}) \to L^2(\mathcal{N})
\]
is a holomorphic Fredholm family which is invertible at some point, the set
\[
\text{spec}_b(P) = \{ \sigma \in \mathbb{C} : \hat{P}(\sigma) \text{ is not invertible} \}
\]
is discrete without points of accumulation.
Using the invertibility of
\[
\sum_{k+|\alpha|=m} a_{k\alpha}(0, y) \sigma^k \eta^\alpha
\]
(with $|\text{Im}\, \sigma| < c|\text{Re}\, \sigma|$) one finds a family of pseudodifferential operators $Q(\sigma)$ on $\mathcal{N}$ such that
\[
Q(\sigma)\hat{P}(\sigma) = I - R(\sigma)
\]
with $R(\sigma)$ of order $-1$ and $\|R(\sigma)\| \leq C/|\text{Re}\, \sigma|$. Consequently:

\[\hat{P}(\sigma)\] is invertible in $|\text{Im}\, \sigma| < c|\text{Re}\, \sigma|$. Since
\[
\hat{P}(\sigma) : H^m(\mathcal{N}) \subset L^2(\mathcal{N}) \to L^2(\mathcal{N})
\]
is a holomorphic Fredholm family which is invertible at some point, the set
\[
\text{spec}_b(P) = \{\sigma \in \mathbb{C} : \hat{P}(\sigma) \text{ is not invertible}\}
\]
is discrete without points of accumulation. For any $a > 0$, the set
\[
\text{spec}_b(P) \cap \{\sigma : |\text{Im}\, \sigma| < a\}
\]
is finite.
Sobolev spaces

1. If $s$ is a nonnegative integer, then $H^s_b(M)$ consists of all $u \in L^2_b(M)$ such that

$$X_1 \ldots X_k u \in L^2_b(M) \text{ for all } X_1, \ldots, X_k \in C^\infty(M; bT^*M), k \leq s$$

2. The space $H^{-s}_b(M)$ is the dual of $H^s_b(M)$

3. If $s$ is not an integer, then $H^s_b(M)$ is defined by interpolation.

4. If $s$ and $\nu$ are real numbers, then $x^{\nu} H^s_b(M) = \{x^{\nu} u : u \in H^s_b(M)\}$.

If $P \in \text{Diff}^m_b(M; E, F)$, then $P : x^{\nu} H^s_b(M; E) \to x^{\nu} H^{-m-s}_b(M; F)$ is continuous.
Elliptic regularity

Let $P \in \text{Diff}^m_b(M; E, F)$ be $b$-elliptic. If $u \in x^\nu H^s_b(M; E)$ and $Pu \in x^\nu H^s_b(M; F)$, then $u \in x^\nu H^{s+m}_b(M; E)$.

The proof is by construction of an operator $Q$ such that

$$QP = I - R$$

with $Q: x^\nu H^t_b \to x^\nu H^{t+m}_b$ and $R: x^\nu H^t_b \to x^\nu H^\infty_b$ for any $t$. 
The operator $Q$ is obtained by constructing its parametrix. Suppose for the moment that

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$$

is an operator on an open set $U \subset \mathbb{R}^n$. If $u \in C_c^\infty(U)$ then

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left[ \int_U e^{-i x' \cdot \xi} u(x') \, dx' \right] \, d\xi$$

so

$$Pu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \left[ \int_U e^{-i x' \cdot \xi} u(x') \, dx' \right] \, d\xi$$

Let $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$. If $P$ is elliptic, that is,

$$\sigma(P)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

is invertible, then $p(x, \xi)$ is invertible for large $\xi$. 
Parametrices

The operator $Q$ is obtained by constructing its parametrix. Suppose for the moment that

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$$

is an operator on an open set $U \subset \mathbb{R}^n$. If $u \in C^\infty_c(U)$ then

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left[ \int_U e^{-i x' \cdot \xi} u(x') \, dx' \right] \, d\xi$$

so

$$Pu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \left[ \int_U e^{-i x' \cdot \xi} u(x') \, dx' \right] \, d\xi$$

Let $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$. If $P$ is elliptic, that is,

$$\sigma(P)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

is invertible, then $p(x, \xi)$ is invertible for large $\xi$. 
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. 
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. 
Suppose \( P = p(x, D_x) \) is elliptic. Then \( p(x, \xi) \) is invertible for large \( |\xi| \), say if \( |\xi| > C \). Let \( \chi \in C^\infty(\mathbb{R}^n) \) be such that \( \chi(\xi) = 0 \) if \( |\xi| < 2C \) and \( \chi(\xi) = 1 \) if \( |\xi| > 3C \). Let

\[
q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}
\]
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let
\[ q(x, \xi) = \chi(\xi)p(x, \xi)^{-1} \]
So if
\[ Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} q(x, \xi) u(x') \, dx' \, d\xi \]
then
\[ PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi \]
\[ + \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \binom{\alpha}{\beta} a_\alpha(x) \xi^\beta D_\alpha^{\alpha-\beta} q(x, \xi) u(x') \, dx' \, d\xi \]
Suppose \( P = p(x, D_x) \) is elliptic. Then \( p(x, \xi) \) is invertible for large \( |\xi| \), say if \( |\xi| > C \). Let \( \chi \in C^\infty(\mathbb{R}^n) \) be such that \( \chi(\xi) = 0 \) if \( |\xi| < 2C \) and \( \chi(\xi) = 1 \) if \( |\xi| > 3C \). Let

\[
q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}
\]

So if

\[
Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} q(x, \xi) u(x') \, dx' \, d\xi
\]

then

\[
PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi + e^{i(x-x') \cdot \xi} \sum_{\beta < \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) a_\alpha(x) \xi^\beta D_x^{\alpha-\beta} q(x, \xi) u(x') \, dx' \, d\xi
\]
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}$$

So if

$$Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot \xi} q(x, \xi)u(x') \, dx' \, d\xi$$

then

$$PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot \xi} \chi(\xi)u(x') \, dx' \, d\xi$$

$$+ \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot \xi} \sum_{|\alpha| \leq m} \sum_{\beta} \left( \frac{\alpha}{\beta} \right) a_\alpha(x) \xi^\beta D_x^{\alpha-\beta} q(x, \xi)u(x') \, dx' \, d\xi$$

$$+ \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot \xi} \chi(\xi)u(x') \, dx' \, d\xi = u(x) + \int \tilde{\chi}(x-x')u(x') \, dx' \, d\xi = u(x) + (\tilde{\chi} * u)(x)$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

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$$= u(x) + \int \tilde{\chi}(x-x') u(x') \, dx' \, d\xi = u(x) + (\tilde{\chi} * u)(x)$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

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$$= \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} \chi(\xi)u(x') \, dx' \, d\xi = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} u(x') \, dx' \, d\xi + \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} (\chi(\xi) - 1)u(x') \, dx' \, d\xi$$

$$= u(x) + \int \chi(x - x')u(x') \, dx' \, d\xi = u(x) + (\check{\chi} \ast u)(x)$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

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So if

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$$= u(x) + \int \tilde{\chi}(x - x') u(x') \, dx' \, d\xi = u(x) + (\tilde{\chi} * u)(x)$$

$$|q(x, \xi)| \leq C(1 + |\xi|)^{-m}$$

$$|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C(1 + |\xi|)^{-m-|\beta|}$$

$$|\sum_\alpha \sum_{\beta < \alpha} \left( \frac{\alpha}{\beta} \right) a_\alpha(x) \xi^\beta D_x^{\alpha-\beta} q(x, \xi)| \leq C(1 + |\xi|)^{-1}$$
Suppose \( P = p(x, D_x) \) is elliptic. Then \( p(x, \xi) \) is invertible for large \( |\xi| \), say if \( |\xi| > C \). Let \( \chi \in C^\infty(\mathbb{R}^n) \) be such that \( \chi(\xi) = 0 \) if \( |\xi| < 2C \) and \( \chi(\xi) = 1 \) if \( |\xi| > 3C \). Let

\[
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So if

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Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} q(x, \xi) u(x') \, dx' \, d\xi
\]

then

\[
PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi
\]

\[
+ \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) a_\alpha(x) \xi^\beta D_x^{\alpha-\beta} q(x, \xi) u(x') \, dx' \, d\xi
\]

\[
= \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi + \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} (\chi(\xi) - 1) u(x') \, dx' \, d\xi
\]

\[
= u(x) + \int \tilde{\chi}(x - x') u(x') \, dx' \, d\xi = u(x) + (\tilde{\chi} * u)(x)
\]

\[
PQ = I - R \text{ where } R \text{ improves Sobolev regularity by 1.}
\]
Suppose $q(x, \xi)$ is smooth and

$$|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C(1 + |\xi|)^{m-|\beta|}$$

for some $m$. Then

$$\int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi$$

has singularities only on $x = x'$:

$$(x - x')^\beta e^{i(x-x') \cdot \xi} = D_\xi^\beta e^{i(x-x') \cdot \xi}$$

so

$$K_\beta(x, x') = (x - x')^\beta \int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi$$
Suppose $q(x, \xi)$ is smooth and
\[ |\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C(1 + |\xi|)^{m-|\beta|} \]
for some $m$. Then
\[ \int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi \]
has singularities only on $x = x'$:
\[ (x - x')^\beta e^{i(x-x') \cdot \xi} = D_\xi^\beta e^{i(x-x') \cdot \xi} \]
so
\[ K_\beta(x, x') = (x - x')^\beta \int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi \]
\[ = (-1)^{|\beta|} \int e^{i(x-x') \cdot \xi} D_\xi^\beta q(x, \xi) \, d\xi \]
If $m - |\beta| < -n$ then $D_\xi^\beta q(x, \xi)$ is integrable, so $K_\beta$ is continuous.
Suppose \( q(x, \xi) \) is smooth and

\[
| \partial_x^\alpha \partial_\xi^\beta q(x, \xi) | \leq C(1 + |\xi|)^{m - |\beta|}
\]

for some \( m \). Then

\[
\int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi
\]

has singularities only on \( x = x' \):

\[
(x - x')^\beta e^{i(x-x') \cdot \xi} = D_\xi^\beta e^{i(x-x') \cdot \xi}
\]

so

\[
K_\beta(x, x') = (x - x')^\beta \int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi
\]

\[
= (-1)^{|\beta|} \int e^{i(x-x') \cdot \xi} D_\xi^\beta q(x, \xi) \, d\xi
\]

If \( m - |\beta| < -n \) then \( D_\xi^\beta q(x, \xi) \) is integrable, so \( K_\beta \) is continuous.

If \( m - |\beta| < -n - k \) then \( K_\beta \in C^k \).
Suppose $q(x, \xi)$ is smooth and
\[
|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C(1 + |\xi|)^{m-|\beta|}
\]
for some $m$. Then
\[
\int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi
\]
has singularities only on $x = x'$:
\[
(x - x')^\beta e^{i(x-x') \cdot \xi} = D_\xi^\beta e^{i(x-x') \cdot \xi}
\]
so
\[
K_\beta(x, x') = (x - x')^\beta \int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi
= (-1)^{|\beta|} \int e^{i(x-x') \cdot \xi} D_\xi^\beta q(x, \xi) \, d\xi
\]
If $m - |\beta| < -n$ then $D_\xi^\beta q(x, \xi)$ is integrable, so $K_\beta$ is continuous.

If $m - |\beta| < -n - k$ then $K_\beta \in C^k$. 
Suppose $P \in \text{Diff}^b_m$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p')\,dm(p')$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p')) \cdot \xi} q(p, p', \xi)\,d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. 
Suppose \( P \in \text{Diff}^m_b \) is \( b \)-elliptic. To construct \( Q \) such that \( PQ = I - R \) with “good” \( R \) we try to find \( K_Q \) defined on \( \mathcal{M} \times \mathcal{M} \) such that

\[
Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')
\]

In \( \mathcal{M} \times \mathcal{M} \) we know that \( K_Q \) is locally given by

\[
\int e^{i(z(p) - z(p')) \cdot \xi} q(p, p', \xi) \, d\xi.
\]

This has singularities on the diagonal in \( \mathcal{M} \times \mathcal{M} \).
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p')) \cdot \xi} q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. 

\[\begin{array}{c}
\mathcal{M} \\
\Delta \\
\mathcal{M}
\end{array}\]
Suppose $P \in \text{Diff}^b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with "good" $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p'))} \xi q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. Resolve the singularity.
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

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Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

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In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p')) \cdot \xi} q(p, p', \xi) \, d\xi.$$ 

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In \( \mathcal{M} \times \mathcal{M} \) we know that \( K_Q \) is locally given by

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Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

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Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p').$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p'))} \cdot \xi \cdot q(p, p', \xi) \, d\xi.$$ 

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This results in a new smooth manifold with corners, $\mathcal{M} \tilde{\times} \mathcal{M}$. It has a left face, a right face, and a “new” face, the “front face” $r = 0$. 
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

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This results in a new smooth manifold with corners, $\mathcal{M} \tilde{\times} \mathcal{M}$. It has a left face, a right face, and a “new” face, the “front face” $r = 0$. The lifting of the diagonal in $\mathcal{M} \times \mathcal{M}$ has closure

$$\tilde{\Delta} = \{s = 0, \; y = y'\}.$$
Suppose \( P \in \text{Diff}^b \) is \( b \)-elliptic. To construct \( Q \) such that \( PQ = I - R \) with “good” \( R \) we try to find \( K_Q \) defined on \( \mathcal{M} \times \mathcal{M} \) such that

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Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')
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In \( \check{\mathcal{M}} \times \check{\mathcal{M}} \) we know that \( K_Q \) is locally given by

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\int e^{i(z(p) - z(p'))} \xi q(p, p', \xi) \, d\xi.
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\[
\check{\Delta} = \{ s = 0, \ y = y' \}.
\]

It intersects \( \partial_{ff} \check{\mathcal{M}} \times \check{\mathcal{M}} \) transversally.
Let $\varphi : M \times M \to M \times M$ be the blow-down map. View the operator $P$ as acting in the first factor in $M \times M$. 
Let $\varphi : \tilde{\mathcal{M}} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ be the blow-down map. View the operator $P$ as acting in the first factor in $\mathcal{M} \times \mathcal{M}$. Let

$$\pi_L : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

$$\mathcal{M} \times \mathcal{M} \xrightarrow{\pi_L} \mathcal{M}$$
Let \( \varphi : M \tilde{\times} M \rightarrow M \times M \) be the blow-down map. View the operator \( P \) as acting in the first factor in \( M \times M \). Let

\[
\begin{array}{c}
\pi_L & : & M \\
\downarrow & & \downarrow \\
M \times M & \rightarrow & M \\
\pi_R & : & M
\end{array}
\]
Let $\varphi : M \times M \rightarrow M \times M$ be the blow-down map. View the operator $P$ as acting in the first factor in $M \times M$. Let

$$
M \times M \xrightarrow{\varphi} M \times M
$$

with

$$
\pi_L \quad \text{and} \quad \pi_R
$$
Let $\varphi : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ be the blow-down map. View the operator $P$ as acting in the first factor in $\mathcal{M} \times \mathcal{M}$. Let

\[
\begin{array}{c}
\mathcal{M} \times \mathcal{M} \\
\xrightarrow{\varphi} \mathcal{M} \times \mathcal{M} \\
\xleftarrow{\pi_R} \mathcal{M} \\
\xrightarrow{\pi_L} \mathcal{M}
\end{array}
\]

Lift $P$ through the left factor. Since

\[
x = \frac{r(1 + s)}{2}, \quad x' = \frac{r(1 - s)}{2},
\]

we have

\[
x \partial_x = \varphi_*(\frac{1}{2}[(1 + s)r \partial_r + (1 - s^2)\partial_s]),
\]

\[
x' \partial_{x'} = \varphi_*(\frac{1}{2}[(1 - s)r \partial_r - (1 - s^2)\partial_s])
\]

and $\varphi_*(\partial y^j) = \partial y^j$, etc.
Let \( \varphi : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M} \) be the blow-down map. View the operator \( P \) as acting in the first factor in \( \mathcal{M} \times \mathcal{M} \). Let

\[
\begin{array}{c}
\mathcal{M} \times \mathcal{M} \\
\xrightarrow{\phi} \mathcal{M} \times \mathcal{M} \\
\xrightarrow{\pi_L} \mathcal{M} \\
\xleftarrow{\pi_R} \mathcal{M}
\end{array}
\]

Lift \( P \) through the left factor. Since

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\[
x \partial_x = \varphi_* \left( \frac{1}{2} [(1 + s) r \partial_r + (1 - s^2) \partial_s] \right),
\]

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x' \partial_{x'} = \varphi_* \left( \frac{1}{2} [(1 - s) r \partial_r - (1 - s^2) \partial_s] \right)
\]

and \( \varphi_*(\partial y^j) = \partial y^j \), etc. The resulting lifted operator is

\[
\tilde{\pi}_L^* P = \sum_{k + |\alpha| \leq m} a_{k,\alpha} \left( (1 + s) r, y \right) \left( \frac{1}{2} [(1 + s) r D_r + (1 - s^2) D_s] \right)^k D_y^{\alpha}
\]
From previous slide,

\[
\tilde{\pi}_L^* P = \sum_{k + |\alpha| \leq m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} \left[ (1 + s)rD_r + (1 - s^2)D_s \right] \right)^k D_y^\alpha
\]
From previous slide,

\[ \tilde{\pi}_L^* P = \sum_{k+|\alpha| \leq m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} \left[ (1 + s)rD_r + (1 - s^2)D_s \right] \right)^k D_y^\alpha \]

The principal symbol of \( \tilde{\pi}_L^* P \) is

\[ \sum_{k+|\alpha| = m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} \left[ (1 + s)\rho + (1 - s^2)\sigma \right] \right)^k \eta^\alpha \]

Setting \( s = 0 \) and \( \rho = 0 \) (to see what happens on the conormal bundle of the lifted diagonal) we get:

\[ \sum_{k+|\alpha| = m} a_{k\alpha}(r, y) \left( \frac{1}{2} \sigma \right)^k \eta^\alpha \]
From previous slide,

\[ \hat{\pi}_L^*P = \sum_{k+|\alpha|\leq m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} \left[ (1 + s)rD_r + (1 - s^2)D_s \right] \right)^k D_y^\alpha \]

The principal symbol of \( \hat{\pi}_L^*P \) is

\[ \sum_{k+|\alpha|=m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} \left[ (1 + s)\rho + (1 - s^2)\sigma \right] \right)^k \eta^\alpha \]

Setting \( s = 0 \) and \( \rho = 0 \) (to see what happens on the conormal bundle of the lifted diagonal) we get:

\[ \sum_{k+|\alpha|=m} a_{k\alpha}(r, y) \left( \frac{1}{2} \sigma \right)^k \eta^\alpha \]

This is

\[ b_\sigma(P)(r, y, \frac{1}{2} \sigma, \eta) \]

which is invertible (\( b \)-ellipticity).
From previous slide,

\[ \tilde{\pi}_L^* P = \sum_{k + |\alpha| \leq m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} [(1 + s)rD_r + (1 - s^2)D_s] \right)^k D_y^\alpha \]

The principal symbol of \( \tilde{\pi}_L^* P \) is

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\[ \sum_{k + |\alpha| = m} a_{k\alpha}(r, y) \left( \frac{1}{2} \sigma \right)^k \eta^\alpha \]

This is

\[ b_\sigma(P)(r, y, \frac{1}{2} \sigma, \eta) \]

which is invertible (\( b \)-ellipticity).
Further,

\[
\tilde{\pi}^*_L P = \sum_{k+|\alpha| \leq m} a_{k\alpha}((1+s)r, y) \left( \frac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D_y^\alpha
\]

extends smoothly across \( s = 0 \). So there is a distribution \( \tilde{\mathcal{K}}_Q \) on \( \tilde{\mathcal{M}} \times \tilde{\mathcal{M}} \) which is the restriction of a distribution on an extension of \( \tilde{\mathcal{M}} \times \tilde{\mathcal{M}} \) across \( \partial_{ff} \tilde{\mathcal{M}} \times \tilde{\mathcal{M}} \) supported near \( \tilde{\Delta} \) such that

\[
\tilde{\pi}^*_L \tilde{\mathcal{K}}_Q = \delta(s)\delta(y - y') - \tilde{\mathcal{K}}_R
\]

where \( \tilde{\mathcal{K}}_R \) is smooth. The distribution \( \tilde{\mathcal{K}}_Q \) and function \( \tilde{\mathcal{K}}_R \) descend form the interior of \( \mathcal{M} \times \mathcal{M} \) to \( \mathcal{M} \times \mathcal{M} \), give a distribution and function \( \mathcal{K}_Q, \mathcal{K}_R \) such that

\[
\pi^*_L P \mathcal{K}_Q = \delta_\Delta - \mathcal{K}_R.
\]

This gives

\[
PQ = I - R
\]

likewise \( R \) with \( \mathcal{K}_R \).
Further,
\[ \tilde{\pi}_L^* P = \sum_{k+|\alpha| \leq m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} [(1 + s)rD_r + (1 - s^2)D_s] \right)^k D_{\alpha}^y \]

extends smoothly across \( s = 0 \). So there is a distribution \( \tilde{K}_Q \) on \( \tilde{M} \times \tilde{M} \) which is the restriction of a distribution on an extension of \( \tilde{M} \times \tilde{M} \) across \( \partial_{ff} \tilde{M} \times \tilde{M} \) supported near \( \tilde{\Delta} \) such that
\[ \tilde{\pi}_L^* P \tilde{K}_Q = \delta(s) \delta(y - y') - \tilde{K}_R \]

where \( \tilde{K}_R \) is smooth. The distribution \( \tilde{K}_Q \) and function \( \tilde{K}_R \) descend form the interior of \( \tilde{M} \times \tilde{M} \) to \( \hat{M} \times \hat{M} \), give a distribution and function \( K_Q, K_R \) such that
\[ \pi_L^* PK_Q = \delta_\Delta - K_R. \]

This gives
\[ PQ = I - R \]

The operator \( x^\nu Qx^{-\nu} \) maps \( x^\nu H^t_b \) to \( x^\nu H^{t+m}_b \) for arbitrary \( t \) and \( x^\nu Rx^{-\nu} \) maps \( x^\nu H^t_b \) to \( x^\nu H_\infty^b \).
Asymptotics

Suppose $P$ is $b$-elliptic and $u \in x^\nu H^s_b$ is such that $Pu = 0$ (or $Pu \in x^\infty H^\infty_b$). Then $u \in x^\nu H^\infty_b$: With $Q$ and $R$ as just constructed,

$$Pu = f,$$
Suppose $P$ is $b$-elliptic and $u \in \mathcal{X}^\nu H^s_b$ is such that $Pu = 0$ (or $Pu \in \mathcal{X}^\infty H^\infty_b$). Then $u \in \mathcal{X}^\nu H^\infty_b$: With $Q$ and $R$ as just constructed,

$$Pu = f, \quad Qf = QPu = u - Ru$$
Asymptotics

Suppose $P$ is $b$-elliptic and $u \in x^\nu H^s_b$ is such that $Pu = 0$ (or $Pu \in x^\infty H^\infty_b$). Then $u \in x^\nu H^\infty_b$: With $Q$ and $R$ as just constructed,

$$Pu = f, \quad Qf = QPu = u - Ru$$

The operator $x^{-\mu} Q x^\mu$ is of the same kind as $Q$, so

$$x^{-\nu} Qf = x^{-\nu} Q x^\nu (x^{-\nu} f) \in H^\infty_b \quad \text{since} \quad x^{-\nu} f \in H^\infty_b$$

Since $Ru \in x^\nu H^\infty_b$,

$$u = Qf + Ru \in x^\nu H^\infty_b.$$
Suppose $u \in x^\nu L^2_b$ and $Pu \in x^\infty H^\infty_b$. Then

$\mathcal{M}(u)(\sigma, y) = \int x^{-i\sigma} u(x, y) \frac{dx}{x}$

is holomorphic in $\text{Im} \sigma > -\nu$. (assume $u$ supported near $\partial \mathcal{M}$)
Suppose \( u \in x^{\nu} L^2_b \) and \( Pu \in x^\infty H^\infty_b \). Then

\[
\mathcal{M}(u)(\sigma, y) = \int x^{-i\sigma} u(x, y) \frac{dx}{x}
\]

is holomorphic in \( \text{Im} \sigma > -\nu \). Near \( \partial \mathcal{M} \),

\[
u(x, y) = \frac{1}{2\pi} \int_{\text{Im} \sigma = t} x^{i\sigma} \mathcal{M}(u)(\sigma, y) d\sigma
\]

with \( t \geq -\nu \).
Suppose $u \in x^\nu L^2_b$ and $Pu \in x^\infty H^\infty_b$. Then

$$\mathcal{M}(u)(\sigma, y) = \int x^{-i\sigma} u(x, y) \frac{dx}{x}$$

is holomorphic in $\text{Im} \sigma > -\nu$. Near $\partial \mathcal{M}$,

$$u(x, y) = \frac{1}{2\pi} \int_{\text{Im} \sigma = t} x^{i\sigma} \mathcal{M}(u)(\sigma, y) d\sigma$$

with $t \geq -\nu$. We have $P(u) \in \dot{C}^\infty(\mathcal{M})$ so $\mathcal{M}(Pu)$ is entire. Using the Taylor expansion of $P$ at $x = 0$:

$$P = \sum_{\ell=0}^{N} x^\ell P_\ell(y, xD_x, D_y) + x^{N+1} \tilde{P}_{N+1}(x, y, xD_x, D_y)$$

we get

$$\mathcal{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} \int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x, y) \frac{dx}{x}$$

$$+ \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x}$$
The left hand side of

\[
\mathcal{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} \int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x, y) \frac{dx}{x} \\
+ \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x}
\]

is entire. Since \( P_\ell(y, xD_x, D_y) u \in x^\nu H^\infty_b \), the term

\[
\int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x, y) \frac{dx}{x} = P_\ell(y, \sigma, D_y) \int x^{-i(\sigma+i\ell)} u(x, y) \frac{dx}{x}
\]

is holomorphic in \( \text{Im} (\sigma + i\ell) > -\nu \), that is, \( \text{Im} \sigma > -\nu - \ell \). Likewise the reminder gives a term holomorphic in \( \text{Im} \sigma > -\nu - N - 1 \).
The left hand side of

\[ \mathcal{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} \int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x, y) \frac{dx}{x} \]

\[ + \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x} \]

is entire. Since \( P_\ell(y, xD_x, D_y) u \in x^\nu H_b^\infty \), the term

\[ \int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x, y) \frac{dx}{x} = P_\ell(y, \sigma, D_y) \int x^{-i(\sigma+i\ell)} u(x, y) \frac{dx}{x} \]

is holomorphic in \( \text{Im} (\sigma + i\ell) > -\nu \), that is, \( \text{Im} \sigma > -\nu - \ell \). Likewise the reminder gives a term holomorphic in \( \text{Im} \sigma > -\nu - N - 1 \). So (with \( N = 0 \)):

\[ P_0(y, \sigma, D_y) \mathcal{M}(u)(\sigma) = \mathcal{M}(Pu)(\sigma) - \int x^{-i(\sigma+i)} \tilde{P}_1(x, y, xD_x, D_y) u(x, y) \frac{dx}{x} \]

is holomorphic in \( \text{Im} \sigma > -\nu - 1 \). But \( P_0(y, \sigma, D_y) = \hat{P}(\sigma) \).
From previous slide:

\[ P_0(y, \sigma, D_y) \mathcal{M}(u)(\sigma) = \mathcal{M}(Pu)(\sigma) - \int x^{-i(\sigma+i)} \tilde{P}_1(x, y, xD_x, D_y) u(x, y) \frac{dx}{x} \]

is holomorphic in \( \text{Im} \sigma > -\nu - 1 \) and \( P_0(y, \sigma, D_y) = \tilde{P}(\sigma) \). So

\[ \mathcal{M}(u)(\sigma) = \tilde{P}(\sigma)^{-1} \left[ \mathcal{M}(Pu)(\sigma) - \int x^{-i(\sigma+i)} \tilde{P}_1(x, y, xD_x, D_y) u(x, y) \frac{dx}{x} \right] \]

which is holomorphic in \( \text{Im} \sigma > -\nu \), extends as a meromorphic function to \( \text{Im} \sigma > -\nu - 1 \) with poles in \( \text{spec}_b(P) \cap \{-\nu - 1 < \text{Im} \sigma < \nu\} \).
In general,

\[ M(Pu)(\sigma) = \sum_{\ell=0}^{N} P_\ell(y, \sigma, D_y) M(u)(\sigma + i\ell) \]

\[ + \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x} \]

gives

\[ \hat{P}(\sigma) M(u)(\sigma) = M(Pu)(\sigma) - \sum_{\ell=1}^{N} P_\ell(y, \sigma, D_y) M(u)(\sigma + i\ell) \]

\[ - \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x} \]

Once one has shown that \( M(u)(\sigma) \) has a meromorphic extension to \( \text{Im} \sigma > -\nu - N \), the right hand side is meromorphic in \( \text{Im} \sigma > -\nu - N - 1 \), and so \( M(u)(\sigma) \) has a meromorphic extension to \( \text{Im} \sigma > -\nu - N - 1 \).
The poles of $\mathcal{M}(u)$ are contained in

$$\{\sigma - i\ell : \sigma \in \text{spec}_b(P), \text{Im} \sigma < -\nu, \ k = 0, 1, \ldots \}$$

Using the Mellin inversion formula:

$$u(x, y) = \frac{1}{2\pi} \int_{\text{Im} \sigma = t} x^{i\sigma} \mathcal{M}(u)(\sigma, y) \, d\sigma$$

(with some $t > -\nu$) one gets

$$u(x, y) = \sum_{\sigma \in \text{spec}_b(P)} \sum_{k=0}^{N_{\sigma, \ell}} x^{i\sigma + \ell} u_{\sigma, \ell, k}(y) \log^k x + \frac{1}{2\pi} \int_{\text{Im} \sigma = t-s} x^{i\sigma} \mathcal{M}(u)(\sigma, y) \, d\sigma$$

($t - s \notin \text{spec}_b(P)$ and $\ell$ runs through nonnegative integers). The reminder is $O(x^{-\nu-s+\varepsilon})$ and the $u_{\sigma, \ell, k}(y)$ are smooth.
Thus:

Suppose $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$ is $b$-elliptic and $u \in x^\nu H^s_b(\mathcal{M}; E)$ is such that $Pu \in \dot{C}^\infty(\mathcal{M}; F)$. Then $u \in x^\nu H^\infty_b(\mathcal{M}; E)$ and

$$u(x, y) \sim \sum_{\sigma \in \text{spec}_b(P)} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} x^{i\sigma + \ell} u_{\sigma, \ell, k}(y) \log^k x$$

with $u_{\sigma, \ell, k} \in C^\infty(\mathcal{N}'; E)$. 


Elliptic operators on manifolds with conical singularities, II

Gerardo A. Mendoza

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Paderborn, May 26, 2011
$b$-Differential operators, $b$-symbol, $b$-ellipticity

Let $E, F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

is a $b$-differential operator of order $m$ if

$$x^{-\nu} Px^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.$$
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b-Differential operators, b-symbol, b-ellipticity

Let $E, F \rightarrow M$ be vector bundles. A differential operator

$$P : C^\infty(M; E) \rightarrow C^\infty(M; F)$$

is a b-differential operator of order $m$ if

$$x^{-\nu} P x^\nu \in \text{Diff}^m(M; E, F), \quad \nu = 1, \ldots, m.$$ 

Near $\partial M$, $P$ has the form

$$\sum_{\alpha + k \leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial M.$$
Let $E, F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

is a $b$-differential operator of order $m$ if

$$x^{-\nu}Px^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.$$

Near $\partial \mathcal{M}$, $P$ has the form

$$\sum_{\alpha+k\leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial \mathcal{M}.$$

The $b$-symbol of $P$ is

$$b\sigma(P) = \sum_{\alpha+k\leq m} a_{k\alpha} \xi^k \eta^\alpha.$$

$x$ is a defining function for $\partial \mathcal{M}$ with $x > 0$ in $\mathring{\mathcal{M}}$.

$\text{Diff}^m(\mathcal{M}; E, F)$ is the space of linear differential operators of order $m$ with smooth coefficients.

$\text{Diff}^m_b(\mathcal{M}; E, F)$ is the space of linear $b$-differential operators on $\mathcal{M}$ of order $m$ with smooth coefficients.

$b\sigma(P)$ is a section of $b\pi^* \text{Hom}(E, F)$.

$b\pi : bT^* \mathcal{M} \to \mathcal{M}$.
Let $E, F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

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$$x^{-\nu}Px^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.$$

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The $b$-symbol of $P$ is

$$b\sigma(P) = \sum_{\alpha + k \leq m} a_{k\alpha} \xi^k \eta^\alpha.$$

$P$ is $b$-elliptic if $b\sigma(P)$ is invertible for $(\xi, \eta) \neq 0$. 

$x$ is a defining function for $\partial \mathcal{M}$ with $x > 0$ in $\mathcal{M}$

$\text{Diff}^m(\mathcal{M}; E, F)$ is the space of linear differential operators of order $m$ with smooth coefficients.

$\text{Diff}_b^m(\mathcal{M}; E, F)$ is the space of linear $b$-differential operators on $\mathcal{M}$ of order $m$ with smooth coefficients.

$b\sigma(P)$ is a section of $b\pi^* \text{Hom}(E, F)$

$b\pi : bT^* \mathcal{M} \to \mathcal{M}$

$\pi : T^* \mathcal{M} \to \mathcal{M}$
Mellin transform

If $\mathcal{M}$ is a manifold with boundary, then

$$\dot{C}^\infty(\mathcal{M}) = \{ u \in C^\infty(\mathcal{M}) : u \text{ vanishes to infinite order on } \partial\mathcal{M} \}$$

**Let $u \in \dot{C}^\infty[0, \infty)$ be compactly supported. The Mellin transform of $u$ is**

$$\mathcal{M}(u)(\sigma) = \int_0^\infty x^{-i\sigma} u(x) \frac{dx}{x}$$

The inverse Mellin transform is

$$\mathcal{M}^{-1}(v) = \frac{1}{2\pi} \int_{\text{Re } \sigma = 0} x^{i\sigma} v(\sigma) \, d\sigma.$$ 

The Mellin transform is the Fourier transform with $e^t$ replaced by $x$. 

If $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u}|_{\mathcal{N}}$. 
If $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u}|_{\mathcal{N}}$. The indicial operator of $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$ is

$$\hat{P}(\sigma) = (x^{-i\sigma} P x^{i\sigma})_b$$
If $P \in \text{Diff}^m_b(M; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial M$:

*Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_bu = P\tilde{u}\big|_{\mathcal{N}}$. The indicial operator of $P \in \text{Diff}^m_b(M; E, F)$ is*

$$\hat{P}(\sigma) = (x^{-i\sigma}Px^{i\sigma})_b$$

$\hat{P}(\sigma)$ is a polynomial in $\sigma \in \mathbb{C}$ with values in $\text{Diff}^m(\mathcal{N}; E, F)$. Locally,

$$\hat{P}(\sigma) = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_\alpha y.$$
If $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u}\big|_\mathcal{N}$. The indicial operator of $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$ is

$$\hat{P}(\sigma) = (x^{-i\sigma} P x^{i\sigma})_b$$

$\hat{P}(\sigma)$ is a polynomial in $\sigma \in \mathbb{C}$ with values in $\text{Diff}_b^m(\mathcal{N}; E, F)$. Locally,

$$\hat{P}(\sigma) = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D^\alpha_y.$$ 

If $P$ is elliptic, then $\hat{P}(\sigma)$ is elliptic. In this case, $\hat{P}(\sigma)$ is a holomorphic Fredholm family of index 0. The set

$$\text{spec}_b(P) = \{\sigma \in \mathbb{C} : \hat{P}(\sigma) \text{ is not injective}\}$$

is discrete,

$$\text{spec}_b(P) \cap \{\sigma : |\sigma| < a\}$$

is finite for every $a > 0$. 

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Sobolev spaces

1. If $s$ is a nonnegative integer, then $H^s_b(M)$ consists of all $u \in L^2_b(M)$ such that

$$X_1 \ldots X_k u \in L^2_b(M) \text{ for all } X_1, \ldots, X_k \in C^\infty(M; bT^*M), k \leq s$$

2. The space $H^{-s}_b(M)$ is the dual of $H^s_b(M)$

3. If $s$ is not an integer, then $H^s_b(M)$ is defined by interpolation.

4. If $s$ and $\nu$ are real numbers, then $x^\nu H^s_b(M) = \{x^\nu u : u \in H^s_b(M)\}$.

**If** $P \in \text{Diff}^m_b(M; E, F)$, **then** $P : x^\nu H^s_b(M; E) \to x^\nu H^{s-m}_b(M; F)$ **is continuous.**
Elliptic regularity

Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$ be $b$-elliptic. If $u \in x^\nu H^s_b(\mathcal{M}; E)$ and $Pu \in x^\nu H^s_b(\mathcal{M}; F)$, then $u \in x^\nu H^{s+m}_b(\mathcal{M}; E)$.

The proof is by construction of an operator $Q$ such that

$$QP = I - R$$

with $Q : x^\nu H^t_b \to x^\nu H^{t+m}_b$ and $R : x^\nu H^t_b \to x^\nu H^\infty_b$ for any $t$. 
Elliptic regularity

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$ be $b$-elliptic. If $u \in \mathcal{X}^\nu H_b^s(\mathcal{M}; E)$ and $Pu \in \mathcal{X}^\nu H_b^s(\mathcal{M}; F)$, then $u \in \mathcal{X}^\nu H_b^{s+m}(\mathcal{M}; E)$.

The proof is by construction of an operator $Q$ such that

$$QP = I - R$$

with $Q : \mathcal{X}^\nu H_b^t \to \mathcal{X}^\nu H_b^{t+m}$ and $R : \mathcal{X}^\nu H_b^t \to \mathcal{X}^\nu H_b^\infty$ for any $t$.

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$ be $b$-elliptic. The map

$P : \mathcal{X}^\nu H_b^s(\mathcal{M}; E) \to \mathcal{X}^\nu H_b^s(\mathcal{M}; F)$ is Fredholm if and only if

$-\nu \notin \{\text{Im } \sigma : \sigma \in \text{spec}_b(P)\}$

$P : \mathcal{X}^\nu H_b^{s+m}(\mathcal{M}; E) \to \mathcal{X}^\nu H_b^s(\mathcal{M}; F)$ is Fredholm

$\iff -\nu \notin \{\text{Im } \sigma : \sigma \in \text{spec}_b(P)\}$
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic and $-\nu \notin \text{Im} \, \text{spec}_b(P)$. To prove

$$P : \mathcal{H}^s_b(M; E) \to \mathcal{H}^s_b(M; F)$$

is Fredholm.
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic and $-\nu \notin \text{Im \ spec}_b(P)$. To prove

$$P : x^\nu \mathcal{H}_b^s(\mathcal{M}; E) \to x^\nu \mathcal{H}_b^s(\mathcal{M}; F)$$

is Fredholm we construct $Q$ such that $PQ = I - R$ with $R$ compact.
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic and $-\nu \notin \text{Im \ spec}_b(P)$. To prove

$$P : \mathring{\chi}^\nu H^s_b(\mathcal{M}; E) \to \mathring{\chi}^\nu H^s_b(\mathcal{M}; F)$$
is Fredholm

we construct $Q$ such that $PQ = I - R$ with $R$ compact.
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic and $0 \notin \text{Im } \text{spec}_b(P)$. To prove

$$P : \mathcal{X}^\nu H^s_b(\mathcal{M}; E) \to \mathcal{X}^\nu H^s_b(\mathcal{M}; F)$$

is Fredholm, we construct $Q$ such that $PQ = I - R$ with $R$ compact. Passing to Schwartz kernels and blow up we first find $\tilde{K}_Q$ defined on $\mathcal{M} \times \mathcal{M}$ supported near $\tilde{\Delta}$ s.t.

$$\tilde{\pi}_L^* P \tilde{K}_Q = \delta_{\tilde{\Delta}} - \tilde{K}_R$$

with $\tilde{K}_R$ smooth across $ff = \{r = 0\}$.
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic and $0 \not\in \text{Im \ spec}_b(P)$. To prove

$$P : \mathcal{X}^\nu H^s_b(\mathcal{M}; E) \to \mathcal{X}^\nu H^s_b(\mathcal{M}; F)$$

is Fredholm we construct $Q$ such that $PQ = I - R$ with $R$ compact. Passing to Schwartz kernels and blow up we first find $\tilde{K}_Q$ defined on $\tilde{\mathcal{M}} \times \tilde{\mathcal{M}}$ supported near $\tilde{\Delta}$ s.t.

$$\tilde{\pi}_L^* P \tilde{K}_Q = \delta_{\tilde{\Delta}} - \tilde{K}_R$$

with $\tilde{K}_R$ smooth across $ff = \{r = 0\}$. 

Suppose $\nu = 0$. 

$$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x') = (\frac{r(1+s)}{2}, \frac{r(1-s)}{2})$$

$$r = \frac{x + x'}{2}, s = \frac{x - x'}{x + x'}$$
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic and $0 \not\in \text{Im \ spec}_b(P)$. To prove

$$P : \pi^* H^s_b(\mathcal{M}; E) \rightarrow \pi^* H^s_b(\mathcal{M}; F)$$

is Fredholm

we construct $Q$ such that $PQ = I - R$ with $R$ compact. Passing to Schwartz kernels and blow up we first find $\tilde{K}_Q$ defined on $\mathcal{M} \times \mathcal{M}$ supported near $\tilde{\Delta}$ s.t.

$$\tilde{\pi}_L^* P \tilde{K}_Q = \delta_{\tilde{\Delta}} - \tilde{K}_R$$

with $\tilde{K}_R$ smooth across $ff = \{r = 0\}$. 

Suppose $\nu = 0$. 

[Diagram of kernel supports]

$$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x')$$

$$= \left( \frac{r(1 + s)}{2}, \frac{r(1 - s)}{2} \right)$$

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}$$
Suppose \( P \in \text{Diff}_b^m \) is \( b \)-elliptic and \( 0 \not\in \text{Im spec}_b(P) \). To prove

\[
P : \mathcal{X}^\nu H^s_b(M; E) \to \mathcal{X}^\nu H^s_b(M; F)
\]

is Fredholm we construct \( Q \) such that \( PQ = I - R \) with \( R \) compact. Passing to Schwartz kernels and blow up we first find \( \mathcal{K}_Q \) defined on \( \mathcal{M} \times \mathcal{M} \) supported near \( \mathcal{\Delta} \) s.t.

\[
\mathcal{\pi}_L^* P \mathcal{K}_Q = \delta_\mathcal{\Delta} - \mathcal{K}_R
\]

with \( \mathcal{K}_R \) smooth across \( ff = \{ r = 0 \} \).

Suppose \( \nu = 0 \).

\[
\mathcal{\pi}_L^* P = \sum_{k+|\alpha|\leq m} a_{k,\alpha}((1+s)r,\gamma)\left(\frac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^k D^\alpha
\]

\[
[0, \infty \times [-1,1] \ni (r,s) \mapsto (x, x')
\]

\[
\begin{align*}
\mathcal{\pi}_L^* P &= \left( \frac{r(1+s)}{2}, \frac{r(1-s)}{2} \right) \\
\mathbf{r} &= \frac{x + x'}{2}, \quad \mathbf{s} = \frac{x - x'}{x + x'}
\end{align*}
\]
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic and $0 \not\in \text{Im spec}_b(P)$. To prove
\[ P : \chi^\nu H^s_b(\mathcal{M}; E) \to \chi^\nu H^s_b(\mathcal{M}; F) \] is Fredholm
we construct $Q$ such that $PQ = I - R$ with $R$ compact. Passing to
Schwartz kernels and blow up we first find $\tilde{K}_Q$
defined on $\mathcal{M} \times \mathcal{M}$ supported near $\tilde{\Delta}$ s.t.
\[ \tilde{\pi}_L^* P \tilde{K}_Q = \delta_{\tilde{\Delta}} - \tilde{K}_R \]
with $\tilde{K}_R$ smooth across $\{r = 0\}$.

Suppose $\nu = 0$. 
\[ [0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x') \]
\[ r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'} \]
\[ \hat{\pi}_L^* P = \sum_{k+|\alpha| \leq m} a_{k, \alpha}((1 + s)r, y) \left( \frac{1}{2} (1 + s)r D_r + (1 - s^2)D_s \right)^k D_x^\alpha \]
\[ = \left( \frac{r(1 + s)}{2}, \frac{r(1 - s)}{2} \right) \]
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic and $0 \not\in \text{Im } \text{spec}_b(P)$. To prove

$$P : \chi^\nu H^s_b(M; E) \to \chi^\nu H^s_b(M; F)$$

is Fredholm we construct $Q$ such that $PQ = I - R$ with $R$ compact. Passing to Schwartz kernels and blow up we first find $\tilde{K}_Q$ defined on $M \tilde{\times} M$ supported near $\tilde{\Delta}$ s.t.

$$(\ast) \quad \tilde{\pi}_L^* P \tilde{K}_Q = \delta_{\tilde{\Delta}} - \tilde{K}_R$$

with $\tilde{K}_R$ smooth across $ff = \{ r = 0 \}$.

Try

$$\tilde{K}_Q = \int e^{is\sigma + i(y-y') \cdot \eta} q(r, s, \ldots, \sigma, \eta) \, d\sigma \, d\eta$$

Without much work, get $(\ast)$ with smooth $\tilde{K}_R$. 

Suppose $\nu = 0$. 

\[
\begin{align*}
\pi_L &\colon \mathcal{M} \\ \tilde{\pi}_L &\colon \mathcal{M} \tilde{\times} \mathcal{M} \\ \pi_R &\colon \mathcal{M} \\
\end{align*}
\]
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic and $0 \not\in \text{Im \ spec}_b(P)$. To prove

$$P : \mathcal{X}^\nu H^s_b(M; E) \to \mathcal{X}^\nu H^s_b(M; F)$$

is Fredholm we construct $Q$ such that $PQ = I - R$ with $R$ compact. Passing to Schwartz kernels and blow up we first find $\tilde{K}_Q$ defined on $M \times M$ supported near $\tilde{\Delta}$ s.t.

$$(*) \quad \hat{\pi}_L^* P \hat{K}_Q = \delta_{\tilde{\Delta}} - \hat{K}_R$$

with $\hat{K}_R$ smooth across $ff = \{r = 0\}$.

Try

$$\hat{\pi}_L^* P = \sum_{k,|\alpha| \leq m} a_{k, \alpha} ((1 + s) r, y) \left( \frac{1}{2} [(1 + s) r D_r + (1 - s^2) D_s] \right)^k D_y^\alpha$$

$$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x')$$

$$r = \frac{x + x'}{2}, \ s = \frac{x - x'}{x + x'}$$

Without much work, get $(*)$ with smooth $\hat{K}_R$.

Next, find $\tilde{K}_{Q_1}$ such that $\hat{\pi}_L^* P \hat{K}_{Q_1} |_{r=0} = \hat{K}_R |_{r=0}$:
Suppose \( P \in \text{Diff}_b^m \) is \( b \)-elliptic and \( 0 \not\in \text{Im \ spec}_b(P) \). To prove
\[
P : \mathfrak{X}^\nu H^s_b(\mathcal{M}; E) \to \mathfrak{X}^\nu H^s_b(\mathcal{M}; F) \text{ is Fredholm}
\]
we construct \( Q \) such that \( PQ = I - R \) with \( R \) compact. Passing to
Schwartz kernels and blow up we first find \( \check{K}_Q \)
defined on \( \check{\mathcal{M}} \times \check{\mathcal{M}} \) supported near \( \check{\Delta} \) s.t.
\[
(*) \quad \check{\pi}_L^* P \check{K}_Q = \delta_{\check{\Delta}} - \check{K}_R
\]
with \( \check{K}_R \) smooth across \( ff = \{ r = 0 \} \).

Try
\[
\check{K}_Q = \int e^{is\sigma + i(y - y') \cdot \eta} q(r, s, \ldots, \sigma, \eta) \, d\sigma \, d\eta
\]
Without much work, get \( (*) \) with smooth \( \check{K}_R \).

Next, find \( \check{K}_{Q_1} \) such that \( \check{\pi}_L^* P \check{K}_{Q_1}|_{r=0} = \check{K}_R|_{r=0} \):
\[
\check{\pi}_L^* P|_{ff} = \sum_{k + |\alpha| \leq m} a_{k, \alpha}(0, \eta) \left( \frac{1}{2} \left( 1 - s^2 \right) D_s \right)^k D_y^\alpha
\]
Suppose \( \nu = 0 \).
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic and $0 \not\in \text{Im \spec}_b(P)$. To prove

$$P : \mathcal{X}^\nu H^s_b(\mathcal{M}; E) \to \mathcal{X}^\nu H^s_b(\mathcal{M}; F)$$

we construct $Q$ such that $PQ = I - R$ with $R$ compact. Passing to Schwartz kernels and blow up we first find $\tilde{K}_Q$ defined on $\mathcal{M} \times \mathcal{M}$ supported near $\tilde{\Delta}$ s.t.

$$(*) \quad \hat{\pi}_L^* P \tilde{K}_Q = \delta_{\tilde{\Delta}} - \tilde{K}_R$$

with $\tilde{K}_R$ smooth across $\text{ff} = \{ r = 0 \}$.

Try

$$\tilde{K}_Q = \int e^{is\sigma + i(y-y')} \eta q(r, s, \ldots, \sigma, \eta) \, d\sigma \, d\eta$$

Without much work, get $(*)$ with smooth $\tilde{K}_R$.

Next, find $\tilde{K}_{Q_1}$ such that $\hat{\pi}_L^* P \tilde{K}_{Q_1}|_{r=0} = \tilde{K}_R|_{r=0}$:

$$\hat{\pi}_L^* P|_{\text{ff}} = \sum_{k + |\alpha| \leq m} a_{k, \alpha}(0, y) \left( \frac{1}{2} (1 - s^2) D_s \right)^k D_y^\alpha$$

$p = \frac{x + x'}{2}, s = \frac{x - x'}{x + x'}$
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic and $0 \not\in \text{Im} \text{ spec}_b(P)$. To prove

$$P : \mathcal{X}^\nu H_b^s(\mathcal{M}; E) \rightarrow \mathcal{X}^\nu H_b^s(\mathcal{M}; F)$$

is Fredholm we construct $Q$ such that $PQ = I - R$ with $R$ compact. Passing to Schwartz kernels and blow up we first find $\tilde{K}_Q$ defined on $\mathcal{M} \times \mathcal{M}$ supported near $\tilde{\Delta}$ s.t.

$$(\ast) \quad \tilde{\pi}_L^* P \tilde{K}_Q = \delta_{\tilde{\Delta}} - \tilde{K}_R$$

with $\tilde{K}_R$ smooth across $ff = \{r = 0\}$.

Try

$$\tilde{K}_Q = \int e^{is\sigma + i(y-y') \cdot \eta} q(r, s, \ldots, \sigma, \eta) \, d\sigma \, d\eta$$

Without much work, get $(\ast)$ with smooth $\tilde{K}_R$. Next, find $\tilde{K}_{Q_1}$ such that $\tilde{\pi}_L^* P \tilde{K}_{Q_1}|_{r=0} = \tilde{K}_R|_{r=0}$:

$$\tilde{\pi}_L^* P|_{ff} = \sum_{k+|\alpha| \leq m} a_{k,\alpha}(0, y) \left( \frac{1}{2} (1 + s^2) D_s \right)^k D_y^\alpha$$

Suppose $\nu = 0$. 

$$[0, \infty \times [-1, 1] \ni (r, s) \mapsto (x, x')$$

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}$$
Suppose \( P \in \text{Diff}_b^m \) is \( b \)-elliptic and \( 0 \not\in \text{Im spec}_b(P) \). To prove

\[
P : \mathcal{X}^\nu H_d^s(M; E) \rightarrow \mathcal{X}^\nu H_d^s(M; F)
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is Fredholm we construct \( Q \) such that \( PQ = I - R \) with \( R \) compact. Passing to Schwartz kernels and blow up we first find \( \tilde{K}_Q \) defined on \( \tilde{M} \times \tilde{M} \) supported near \( \tilde{\Delta} \) s.t.

\[
(*) \quad \tilde{\pi}_L^* P \tilde{K}_Q = \delta_{\tilde{\Delta}} - \tilde{K}_R
\]

with \( \tilde{K}_R \) smooth across \( ff = \{ r = 0 \} \).

Try

\[
\tilde{K}_Q = \int e^{is\sigma + i(y - y') \cdot \eta} q(r, s, \ldots, \sigma, \eta) \, d\sigma \, d\eta
\]

Without much work, get \( (*) \) with smooth \( \tilde{K}_R \).

Next, find \( \tilde{K}_{Q_1} \) such that \( \tilde{\pi}_L^* P \tilde{K}_{Q_1} \big|_{r=0} = \tilde{K}_R \big|_{r=0} : \)

\[
\tilde{\pi}_L^* P \big|_{ff} = \sum_{k+|\alpha| \leq m} a_{k, \alpha}(0, y) \left( \frac{1}{2} (1 + s^2) D_y + (1 - s^2) D_z \right)^k D_y^{\alpha}
\]

\( = \hat{P}(\sigma) \), invertible if \( \text{Re} \sigma = 0 \).
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic and $0 \not\in \text{Im spec}_b(P)$. To prove

$$P : \mathcal{X}^\nu H_b^s(\mathcal{M}; E) \to \mathcal{X}^\nu H_b^s(\mathcal{M}; F)$$

is Fredholm, we construct $Q$ such that $PQ = I - R$ with $R$ compact. Passing to Schwartz kernels and blow up we first find $\tilde{K}_Q$ defined on $\mathcal{M} \times \mathcal{M}$ supported near $\tilde{\Delta}$ s.t.

$$\tilde{K}_R + \tilde{\pi}_L^* P \tilde{K}_Q = 0 \text{ on } r = 0 \implies \text{op}(\tilde{K}_R + \tilde{\pi}_L^* P \tilde{K}_Q_1) \text{ compact:}$$

$$P(Q + Q_1) = I - R', \quad R' \text{ compact}$$

with $\tilde{K}_R$ smooth across $ff = \{ r = 0 \}$. Try

$$\tilde{K}_Q = \int e^{is\sigma + i(y - y')} q(r, s, \ldots, \sigma, \eta) \, d\sigma \, d\eta$$

Without much work, get $(\ast)$ with smooth $\tilde{K}_R$.

Next, find $\tilde{K}_Q_1$ such that $\tilde{\pi}_L^* P \tilde{K}_Q_1 |_{r=0} = \tilde{K}_R |_{r=0}$:

$$\tilde{\pi}_L^* P |_{ff} = \sum_{k+|\alpha| \leq m} a_k, \alpha (0, y) \left( \frac{1}{2} (1 + s^2) D_s \right) D_y^\alpha$$

$$\tilde{\pi}_L^* P(\tilde{K}_Q + \tilde{K}_Q_1) = \delta_{\tilde{\Delta}} - \tilde{\Delta}_R + \tilde{\pi}_L^* P \tilde{K}_Q_1$$

\[Suppose \nu = 0.\]
Brief explanation, by example:

The operator

$$\Lambda u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \frac{u(x')}{1 + |\xi|^2} \, dx' \, d\xi$$

maps $L^2(\mathbb{R}^n) \rightarrow H^2(\mathbb{R}^n)$ but is not compact as an operator

$$L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n).$$
Brief explanation, by example:

The operator

$$\Lambda u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \frac{u(x')}{1 + |\xi|^2} \, dx' \, d\xi$$

maps $L^2(\mathbb{R}^n) \rightarrow H^2(\mathbb{R}^n)$ but is not compact as an operator

$$L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n).$$

But the operator

$$u \mapsto (1 + |x|^2)^{-\varepsilon} \Lambda u$$

is compact as an operator

$$L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$$

for any $\varepsilon > 0$. 
Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an element of $x^{-m}\text{Diff}^m_b(\mathcal{M}; E, F)$:

$$A = x^{-m}P, \quad P \in \text{Diff}^m_b(\mathcal{M}; E, F)$$

A is $c$-elliptic if $x^mA$ is $b$-elliptic.

Assume $m > 0$. 
Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an element of $x^{-m} \text{Diff}^m_b(\mathcal{M}; E, F)$:

$$A = x^{-m}P, \quad P \in \text{Diff}^m_b(\mathcal{M}; E, F)$$

$A$ is $c$-elliptic if $x^mA$ is $b$-elliptic.

Fix $\gamma \in \mathbb{R}$, view $A$ as an unbounded operator

$$A : C^\infty_c(\mathring{\mathcal{M}}; E) \subset x^\gamma L^2_b(\mathcal{M}; E) \rightarrow x^\gamma L^2_b(\mathcal{M}; F)$$

Assume $m > 0$. 

Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an element of $x^{-m} \text{Diff}^m_b(\mathcal{M}; E, F)$:

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Fix $\gamma \in \mathbb{R}$, view $A$ as an unbounded operator

$$A : C^\infty_c(\hat{\mathcal{M}}; E) \subset x^\gamma L^2_b(\mathcal{M}; E) \to x^\gamma L^2_b(\mathcal{M}; F)$$

The domain of the closure is the “minimal domain,” $\mathcal{D}_{\text{min}}(A)$.

Assume $m > 0$. 
Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an element of $x^{-m}\text{Diff}^{m}_{b}(\mathcal{M}; E, F)$:

$$A = x^{-m}P, \quad P \in \text{Diff}^{m}_{b}(\mathcal{M}; E, F)$$

A is $c$-elliptic if $x^{m}A$ is $b$-elliptic.

Fix $\gamma \in \mathbb{R}$, view $A$ as an unbounded operator

$$A : C_{c}^{\infty}(\mathcal{M}; E) \subset x^{\gamma}L^{2}_{b}(\mathcal{M}; E) \to x^{\gamma}L^{2}_{b}(\mathcal{M}; F)$$

The domain of the closure is the “minimal domain,” $\mathcal{D}_{\text{min}}(A)$. The space

$$\mathcal{D}_{\text{max}}(A) = \{ u \in x^{\gamma}L^{2}_{b} : Au \in x^{\gamma}L^{2}_{b} \}$$

is the “maximal domain.”
Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an element of $x^{-m}\text{Diff}_b^m(\mathcal{M}; E, F)$:

$$A = x^{-m}P, \quad P \in \text{Diff}_b^m(\mathcal{M}; E, F)$$

$A$ is $c$-elliptic if $x^mA$ is $b$-elliptic.

Fix $\gamma \in \mathbb{R}$, view $A$ as an unbounded operator

$$A : C_c^\infty(\mathcal{M}; E) \subset x^\gamma L_b^2(\mathcal{M}; E) \to x^\gamma L_b^2(\mathcal{M}; F)$$

The domain of the closure is the “minimal domain,” $\mathcal{D}_{\text{min}}(A)$. The space

$$\mathcal{D}_{\text{max}}(A) = \{u \in x^\gamma L_b^2 : Au \in x^\gamma L_b^2\}$$

is the “maximal domain.” It is a Hilbert space with inner product

$$(u, v)_A = (Au, Av)_{x^\gamma L_b^2} + (u, v)_{x^\gamma L_b^2}$$
Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an element of $x^{-m}\text{Diff}^m_b(\mathcal{M}; E, F)$:

$$A = x^{-m}P, \quad P \in \text{Diff}^m_b(\mathcal{M}; E, F)$$

$A$ is $c$-elliptic if $x^mA$ is $b$-elliptic.

Fix $\gamma \in \mathbb{R}$, view $A$ as an unbounded operator

$$A : C_c^\infty(\mathcal{M}; E) \subset x^{\gamma}L^2_b(\mathcal{M}; E) \rightarrow x^{\gamma}L^2_b(\mathcal{M}; F)$$

The domain of the closure is the "minimal domain," $\mathcal{D}_{\text{min}}(A)$. The space

$$\mathcal{D}_{\text{max}}(A) = \{u \in x^{\gamma}L^2_b : Au \in x^{\gamma}L^2_b\}$$

is the "maximal domain." It is a Hilbert space with inner product

$$(u, v)_A = (Au, Av)_{x^{\gamma}L^2_b} + (u, v)_{x^{\gamma}L^2_b}$$

The closure of $C_c^\infty \subset \mathcal{D}_{\text{max}}(A)$ is $\mathcal{D}_{\text{min}}(A)$. 

Assume $m > 0$. 

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Cone operators

We continue with $\mathcal{M}$ compact.

A cone differential operator is an element of $x^{-m} \text{Diff}^m_b(\mathcal{M}; E, F)$:

$$A = x^{-m}P, \quad P \in \text{Diff}^m_b(\mathcal{M}; E, F)$$

$A$ is $c$-elliptic if $x^m A$ is $b$-elliptic.

Fix $\gamma \in \mathbb{R}$, view $A$ as an unbounded operator

$$A : C^\infty_c(\mathcal{M}; E) \subset x^\gamma L^2_b(\mathcal{M}; E) \rightarrow x^\gamma L^2_b(\mathcal{M}; F)$$

The domain of the closure is the “minimal domain,” $D_{\text{min}}(A)$. The space

$$D_{\text{max}}(A) = \{ u \in x^\gamma L^2_b : Au \in x^\gamma L^2_b \}$$

is the “maximal domain.” It is a Hilbert space with inner product

$$(u, v)_A = (Au, Av)_{x^\gamma L^2_b} + (u, v)_{x^\gamma L^2_b}$$

The closure of $C^\infty_c \subset D_{\text{max}}(A)$ is $D_{\text{min}}(A)$.

Assume $m > 0$. Both

$$A : D_{\text{min}} \subset x^\gamma L^2_b(\mathcal{M}; E) \rightarrow x^\gamma L^2_b(\mathcal{M}; F)$$

$$A : D_{\text{max}} \subset x^\gamma L^2_b(\mathcal{M}; E) \rightarrow x^\gamma L^2_b(\mathcal{M}; F)$$

are closed, densely defined.
Suppose \( A \in x^{-m} \text{Diff}_b^m \) is \( c \)-elliptic.
Suppose $A \in x^{-m} \text{Diff}_b^m$ is $c$-elliptic.

- The space $x^{\gamma+m}H^m_b(M; E)$ is a subspace of $\mathcal{D}_{\text{min}}$, and the inclusion is continuous.
Suppose $A \in x^{-m} \text{Diff}_b^m$ is $c$-elliptic.

The space $x^{\gamma+m}H^m_b(M; E)$ is a subspace of $D_{\text{min}}$, and the inclusion is continuous. Let $\omega \in C^\infty_c(\mathbb{R})$ be $1$ near $0$, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma+m}H^m_b(M; E)$ then $\omega_k u \to u$ in $x^{\gamma+m}H^m_b(M; E)$. Hence $u_k \to u$ in $x^{\gamma+m}L^2_b$ (in $x^\gamma L^2_b$) and $Au_k \to Au$ in $x^\gamma L^2_b$. 

\[ \text{\textcopyright{G. A. Mendoza}} \text{ (Temple University)} \]
Suppose $A \in x^{-m} \text{Diff}^m_b$ is c-elliptic.

- The space $x^{\gamma+m} H^m_b(\mathcal{M}; E)$ is a subspace of $\mathcal{D}_{\text{min}}$, and the inclusion is continuous. Let $\omega \in C^\infty_c(\mathbb{R})$ be $=1$ near 0, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma+m} H^m_b(\mathcal{M}; E)$ then $\omega_k u \to u$ in $x^{\gamma+m} H^m_b(\mathcal{M}; E)$.

- $x^{\gamma+m} H^m_b(\mathcal{M}; E) = \mathcal{D}_{\text{min}}$ iff $-\gamma - m \notin \text{Im spec}_b(x^m A)$.

Hence $u_k \to u$ in $x^{\gamma+m} L^2_b$ (in $x^{\gamma} L^2_b$) and $Au_k \to Au$ in $x^{\gamma} L^2_b$. 
Suppose $A \in x^{-m} \text{Diff}_b^m$ is $c$-elliptic.

- The space $x^{\gamma+m}H^m_b(M; E)$ is a subspace of $\mathcal{D}_{\text{min}}$, and the inclusion is continuous.

Let $\omega \in C^\infty_c(\mathbb{R})$ be $1$ near $0$, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma+m}H^m_b(M; E)$ then $\omega_ku \to u$ in $x^{\gamma+m}H^m_b(M; E)$. Hence $u_k \to u$ in $x^{\gamma+m}L^2_b(\mathbb{R}^n, x^{\gamma}L^2_b)$ and $Au_k \to Au$ in $x^{\gamma}L^2_b$.

If $-\gamma - m \in \text{Im spec}_b(A)$, then $\mathcal{D}_{\text{min}}$ is more complicated.
Suppose $A \in x^{-m} \text{Diff}^m_b$ is $c$-elliptic.

- The space $x^{\gamma + m} H^m_b(\mathcal{M}; E)$ is a subspace of $\mathcal{D}_{\text{min}}$, and the inclusion is continuous. Let $\omega \in C^\infty_c(\mathbb{R})$ be $1$ near $0$, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma + m} H^m_b(\mathcal{M}; E)$, then $\omega_k u \to u$ in $x^{\gamma + m} H^m_b(\mathcal{M}; E)$. Hence $u_k \to u$ in $x^{\gamma + m} L^2_b$ (in $x^{\gamma} L^2_b$).

- $x^{\gamma + m} H^m_b(\mathcal{M}; E) = \mathcal{D}_{\text{min}}$ iff $-\gamma - m \notin \text{Im spec}_b(x^m A)$.

If $-\gamma - m \in \text{Im spec}_b(A)$, then $\mathcal{D}_{\text{min}}$ is more complicated.

Whatever $\gamma$, both

$$A : \mathcal{D}_{\text{min}} \subset x^{\gamma} L^2_b \to x^{\gamma} L^2_b$$

and

$$A : \mathcal{D}_{\text{max}} \subset x^{\gamma} L^2_b \to x^{\gamma} L^2_b$$

are Fredholm operators.
Suppose $A \in x^{-m} \text{Diff}^m_b$ is $c$-elliptic.

- The space $x^{\gamma + m} H^m_b(\mathcal{M}; E)$ is a subspace of $\mathcal{D}_{\text{min}}$, and the inclusion is continuous. Let $\omega \in C^\infty_c(\mathbb{R})$ be $1$ near $0$, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma + m} H^m_b(\mathcal{M}; E)$ then $\omega_k u \to u$ in $x^{\gamma + m} H^m_b(\mathcal{M}; E)$. Hence $u_k \to u$ in $x^{\gamma + m} L^2_b$.

- $x^{\gamma + m} H^m_b(\mathcal{M}; E) = \mathcal{D}_{\text{min}}$ iff $-\gamma - m \notin \text{Im spec}_b (x^m A)$.

If $-\gamma - m \in \text{Im spec}_b(A)$, then $\mathcal{D}_{\text{min}}$ is more complicated.

Whatever $\gamma$, both

$$A : \mathcal{D}_{\text{min}} \subset x^{\gamma} L^2_b \to x^{\gamma} L^2_b$$

and

$$A : \mathcal{D}_{\text{max}} \subset x^{\gamma} L^2_b \to x^{\gamma} L^2_b$$

are Fredholm operators.

$A_D$ has finite dimensional kernel and closed range of finite codimension. $\text{Ind } A_D = \dim \ker A_D - \dim \text{coker } A_D$. 

\begin{align*}
\text{Ind } A_D & = \dim \ker A_D - \dim \text{coker } A_D. 
\end{align*}
Suppose $A \in x^{-m} \text{Diff}^m_b$ is $c$-elliptic.

- The space $x^{\gamma + m} H^m_b(\mathcal{M}; E)$ is a subspace of $\mathcal{D}_{\text{min}}$, and the inclusion is continuous. Let $\omega \in \mathcal{C}^\infty_c(\mathbb{R})$ be $=1$ near $0$, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma + m} H^m_b(\mathcal{M}; E)$ then $\omega_k u \to u$ in $x^{\gamma + m} H^m_b(\mathcal{M}; E)$. Hence $u_k \to u$ in $x^{\gamma + m} L^2_b$ and $A u_k \to A u$ in $x^{\gamma} L^2_b$.

If $-\gamma - m \in \text{Im spec}_b(A)$, then $\mathcal{D}_{\text{min}}$ is more complicated.

Whatever $\gamma$, both

$$A : \mathcal{D}_{\text{min}} \subset x^{\gamma} L^2_b \to x^{\gamma} L^2_b$$

and

$$A : \mathcal{D}_{\text{max}} \subset x^{\gamma} L^2_b \to x^{\gamma} L^2_b$$

are Fredholm operators. The inclusion $\mathcal{D}_{\text{max}} \hookrightarrow x^{\gamma} L^2_b$ is compact.

$A_D$ has finite dimensional kernel and closed range of finite codimension. Ind $A_D = \dim \ker A_D - \dim \text{coker} A_D$. 
Suppose $A \in x^{-m} \text{Diff}_b^m$ is $c$-elliptic.

- The space $x^\gamma x^m H_b^m(M; E)$ is a subspace of $D_{\min}$, and the inclusion is continuous.

  Let $\omega \in C^\infty_c(\mathbb{R})$ be $1$ near $0$, $\omega_k(x) = \omega(kx)$. If $u \in x^\gamma x^m H_b^m(M; E)$ then $\omega_k u \to u$ in $x^\gamma x^m H_b^m(M; E)$. Hence $u_k \to u$ in $x^\gamma x^m L^2_b$.

  If $-\gamma - m \in \text{Im} \, \text{spec}_b(x^m A)$, then $D_{\min}$ is more complicated.

Whatever $\gamma$, both

$$A : D_{\min} \subset x^\gamma L^2_b \to x^\gamma L^2_b$$ and $$A : D_{\max} \subset x^\gamma L^2_b \to x^\gamma L^2_b$$

are Fredholm operators. The inclusion $D_{\max} \hookrightarrow x^\gamma L^2_b$ is compact.

$A_{D}$ has finite dimensional kernel and closed range of finite codimension. $\text{Ind} \, A_{D} = \dim \ker A_{D} - \dim \text{coker} \, A_{D}$. $A_{D} - \lambda I$ is a compact perturbation of $A_{D}$ so $\text{Ind}(A_{D} - \lambda I) = \text{Ind} \, A_{D}$.
Suppose $A \in x^{-m} \text{Diff}_b^m$ is c-elliptic.

- The space $x^{\gamma + m} H^m_b(\mathcal{M}; E)$ is a subspace of $\mathcal{D}_{\text{min}}$, and the inclusion is continuous. Let $\omega \in C_c^\infty(\mathbb{R})$ be $= 1$ near 0, $\omega_k(x) = \omega(kx)$. If $u \in x^{\gamma + m} H^m_b(\mathcal{M}; E)$ then $\omega_k u \to u$ in $x^{\gamma + m} H^m_b(\mathcal{M}; E)$. Hence $u_k \to u$ in $x^{\gamma + m} L^2_b$ (in $x^{\gamma} L^2_b$) and $Au_k \to Au$ in $x^{\gamma} L^2_b$.

If $-\gamma - m \in \text{Im \ spec}_b(A)$, then $\mathcal{D}_{\text{min}}$ is more complicated.

Whatever $\gamma$, both

$$A : \mathcal{D}_{\text{min}} \subset x^{\gamma} L^2_b \to x^{\gamma} L^2_b \quad \text{and} \quad A : \mathcal{D}_{\text{max}} \subset x^{\gamma} L^2_b \to x^{\gamma} L^2_b$$

are Fredholm operators. The inclusion $\mathcal{D}_{\text{max}} \hookrightarrow x^{\gamma} L^2_b$ is compact.

$A_D$ has finite dimensional kernel and closed range of finite codimension. $\text{Ind} A_D = \dim \ker A_D - \dim \text{coker} A_D$. $A_D - \lambda I$ is a compact perturbation of $A_D$ so $\text{Ind}(A_D - \lambda I) = \text{Ind} A_D$.

This is due to Lesch. A consequence is that $\mathcal{D}_{\text{min}}$ has finite codimension in $\mathcal{D}_{\text{max}}$. Also:

Every extension of $A : \mathcal{D}_{\text{min}} \subset x^{\gamma} L^2_b \to x^{\gamma} L^2_b$ is closed.

Any extension has domain

$$\mathcal{D} = \mathcal{D}_{\text{min}} + D, \quad D \subset \mathcal{D}_{\text{max}} \text{ finite dimensional.}$$
Adjoints

Let $A^*$ denote the formal adjoint of $A$ on $x^\nu L^2_b(\mathcal{M}; E)$:

$$(Au, v)_{x^\nu L^2_b} = (u, A^* v)_{x^\nu L^2_b} \quad \forall u \in C_c^\infty(\mathcal{M}; E), \ v \in C_c^\infty(\mathcal{M}; F).$$
Adjoint

Let $A^*$ denote the formal adjoint of $A$ on $\chi^\nu L_b^2(\mathcal{M}; E)$:

$$(Au, v)_{\chi^\nu L_b^2} = (u, A^*v)_{\chi^\nu L_b^2} \quad \forall u \in C^\infty_\circ(\mathcal{M}; E), \ v \in C^\infty_\circ(\mathcal{M}; F).$$

- The Hilbert space adjoint of $A$ with domain $D_{\text{min}}(A)$ is $A^*$ with domain $D_{\text{max}}(A^*)$. 

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Adjoint

Let $A^*$ denote the formal adjoint of $A$ on $\nu L^2_b(M; E)$:

$$(Au, v)_{\nu L^2_b} = (u, A^* v)_{\nu L^2_b} \quad \forall u \in C^\infty_c(\hat{M}; E), \ v \in C^\infty_c(\hat{M}; F).$$

– The Hilbert space adjoint of $A$ with domain $D_{\min}(A)$ is $A^*$ with domain $D_{\max}(A^*)$.
– The Hilbert space adjoint of $A$ with domain $D_{\max}(A)$ is $A^*$ with domain $D_{\min}(A^*)$. 

Adjoints

Let $A^*$ denote the formal adjoint of $A$ on $\mathcal{L}^2_b(\mathcal{M}; E)$:

$$(Au, v)_{\mathcal{L}^2_b} = (u, A^* v)_{\mathcal{L}^2_b} \quad \forall u \in C^\infty(\mathcal{M}; E), \; v \in C^\infty(\mathcal{M}; F).$$

– The Hilbert space adjoint of $A$ with domain $\mathcal{D}_{\text{min}}(A)$ is $A^*$ with domain $\mathcal{D}_{\text{max}}(A^*)$.
– The Hilbert space adjoint of $A$ with domain $\mathcal{D}_{\text{max}}(A)$ is $A^*$ with domain $\mathcal{D}_{\text{min}}(A^*)$.
– The form $[u, v]_A = (Au, v) - (u, A^* v)$, $u \in \mathcal{D}_{\text{max}}(A), \; v \in \mathcal{D}_{\text{max}}(A^*)$ is n.d. on $\mathcal{E}(A) \times \mathcal{E}(A^*)$ and vanishes if $u \in \mathcal{D}_{\text{min}}(A)$ or $v \in \mathcal{D}_{\text{min}}(A^*)$. 
Adjoints

Let $A^*$ denote the *formal* adjoint of $A$ on $x^\nu L^2_b(M; E)$:

$$(Au, v)_{x^\nu L^2_b} = (u, A^* v)_{x^\nu L^2_b} \quad \forall u \in C_c^\infty(\mathcal{M}; E), \; v \in C_c^\infty(\mathcal{M}; F).$$

– The Hilbert space adjoint of $A$ with domain $\mathcal{D}_{\min}(A)$ is $A^*$ with domain $\mathcal{D}_{\max}(A^*)$.
– The Hilbert space adjoint of $A$ with domain $\mathcal{D}_{\max}(A)$ is $A^*$ with domain $\mathcal{D}_{\min}(A^*)$.
– The form $[u, v]_A = (Au, v) - (u, A^* v)$, $u \in \mathcal{D}_{\max}(A)$, $v \in \mathcal{D}_{\max}(A^*)$ is n.d. on $\mathcal{E}(A) \times \mathcal{E}(A^*)$ and vanishes if $u \in \mathcal{D}_{\min}(A)$ or $v \in \mathcal{D}_{\min}(A^*)$.
– The orthogonal complement of $\mathcal{D}_{\min}(A)$ in $\mathcal{D}_{\max}(A)$ is

$$\mathcal{E}(A) = \ker(A^* A + I) \cap \mathcal{D}_{\max}(A)$$

This is a finite-dimensional space.
Adjoint

Let $A^*$ denote the \textit{formal} adjoint of $A$ on $\chi^\nu L^2_b(M; E)$:

$$(Au, v)_{\chi^\nu L^2_b} = (u, A^* v)_{\chi^\nu L^2_b} \quad \forall u \in C_c(\hat{M}; E), \; v \in C_c(\hat{M}; F).$$

- \textit{The Hilbert space adjoint of $A$ with domain $D_{\text{min}}(A)$ is $A^*$ with domain $D_{\text{max}}(A^*)$.}
- \textit{The Hilbert space adjoint of $A$ with domain $D_{\text{max}}(A)$ is $A^*$ with domain $D_{\text{min}}(A^*)$.}
- \textit{The form $[u, v]_A = (Au, v) - (u, A^* v)$, $u \in D_{\text{max}}(A)$, $v \in D_{\text{max}}(A^*)$ is n.d. on $\mathcal{E}(A) \times \mathcal{E}(A^*)$ and vanishes if $u \in D_{\text{min}}(A)$ or $v \in D_{\text{min}}(A^*)$.}
- \textit{The orthogonal complement of $D_{\text{min}}(A)$ in $D_{\text{max}}(A)$ is}

$$\mathcal{E}(A) = \ker(A^* A + I) \cap D_{\text{max}}(A)$$

\textit{This is a finite-dimensional space.}
- \textit{if $u \in \mathcal{E}(A)$, then $Au \in \mathcal{E}(A^*)$.}
Proof of $\mathcal{E}(A) = \ker(A^*A + I) \cap D_{\text{max}}(A)$:

$u \in D_{\text{min}}^\perp \iff \forall v \in D_{\text{min}} : (u, v)_A = 0$

$\iff \forall v \in D_{\text{min}} : (Au, Av) + (u, v) = 0$

$\iff \forall v \in D_{\text{min}} : (Au, Av) = -(u, v)$
Proof of $\mathcal{E}(A) = \ker(A^*A + I) \cap D_{\max}(A)$:

\[
u \in D_{\min} \perp \iff \forall \nu \in D_{\min} : (u, \nu)_A = 0
\]

\[
\iff \forall \nu \in D_{\min} : (Au, Av) + (u, \nu) = 0
\]

\[
\iff \forall \nu \in D_{\min} : (Au, Av) = -(u, \nu)
\]

$D_{\min}(A) \ni \nu \mapsto (Au, Av) \in \mathbb{C}$ is continuous because this is just $\nu \mapsto (-u, \nu)$.
Proof of $\mathcal{E}(A) = \text{ker}(A^*A + I) \cap \mathcal{D}_{\text{max}}(A)$:

\[ u \in \mathcal{D}_{\text{min}}^{\perp} \iff \forall v \in \mathcal{D}_{\text{min}} : (u, v)_A = 0 \]

\[ \iff \forall v \in \mathcal{D}_{\text{min}} : (Au, Av) + (u, v) = 0 \]

\[ \iff \forall v \in \mathcal{D}_{\text{min}} : (Au, Av) = -(u, v) \]

\[ \iff Au \in \mathcal{D}_{\text{max}}(A^*) \text{ and } A^*Au = -u. \]
Proof of $\mathcal{E}(A) = \ker(A^*A + I) \cap D_{\text{max}}(A)$:

$$u \in D_{\min}^\perp \iff \forall \nu \in D_{\min} : (u, \nu)_A = 0$$

$$\iff \forall \nu \in D_{\min} : (Au, Av) + (u, \nu) = 0$$

$$\iff \forall \nu \in D_{\min} : (Au, Av) = -(u, \nu)$$

$$\iff Au \in D_{\text{max}}(A^*) \text{ and } A^*Au = -u.$$  

In particular, $u \in \mathcal{E}(A) \implies Au \in \mathcal{E}(A^*)$: 

$\mathcal{D}_{\min}(A) \ni \nu \mapsto (Au, Av) \in \mathbb{C}$

is continuous because this is just $\nu \mapsto (-u, \nu)$.
Proof of $\mathcal{E}(A) = \ker(A^*A + I) \cap D_{\text{max}}(A)$:

$$u \in D_{\text{min}}^\perp \iff \forall v \in D_{\text{min}} : (u, v)_A = 0$$

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In particular, $u \in \mathcal{E}(A) \implies Au \in \mathcal{E}(A^*)$:
The above says $Au \in D_{\text{max}}(A^*)$ and $A^*Au + u = 0$. 
Proof of $\mathcal{E}(A) = \ker(A^*A + I) \cap D_{\text{max}}(A)$:

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Applying $A$:

\[
A(A^*A)u + Au = 0
\]
Proof of \( \mathcal{E}(A) = \ker(A^*A + I) \cap D_{\text{max}}(A) \):

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u \in D_{\text{min}}^\perp \iff \forall v \in D_{\text{min}} : (u, v)_A = 0
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\iff \forall v \in D_{\text{min}} : (Au, Av) + (u, v) = 0
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\iff \forall v \in D_{\text{min}} : (Au, Av) = -(u, v)
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\iff Au \in D_{\text{max}}(A^*) \text{ and } A^*Au = -u.
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In particular, \( u \in \mathcal{E}(A) \implies Au \in \mathcal{E}(A^*) \):

The above says \( Au \in D_{\text{max}}(A^*) \) and \( A^*Au + u = 0 \).

Applying \( A \):

\[
A(A^*A)u + Au = 0
\]

that is

\[
(AA^*)Au + Au = 0
\]

so \( Au \in \mathcal{E}(A^*) \) by what we just proved.
The elements of $\mathcal{E}(A)$ in some sense live on the boundary of $\mathcal{M}$.

Namely, let

$$\pi_{\text{max}}, \pi_{\text{min}} : \mathcal{D}_{\text{max}} \to \mathcal{D}_{\text{max}}$$

be the orthogonal projections on, respectively, $\mathcal{E}(A)$ and $\mathcal{D}_{\text{min}}$. $\pi_{\text{max}} = I - \pi_{\text{min}}$

Since $A$ is elliptic in $\mathcal{M}$ in the usual sense, so is $A^*A + I$. If $u \in \mathcal{E}$ then $(A^*A + I)u = 0$, so $u$ is smooth in $\mathcal{M}$.

Let $\omega$ be a smooth function on $\mathcal{M}$, $= 1$ near $\partial\mathcal{M}$. If $u \in \mathcal{E}(A)$ then

$\omega u \in \mathcal{D}_{\text{max}}$ because

$$\omega u = u - (1 - \omega)u$$

and $(1 - \omega)u \in \mathcal{C}_c^\infty \subset \mathcal{D}_{\text{min}}$. And

$$\pi_{\text{max}}(\omega u) = \pi_{\text{max}}(u) - \pi_{\text{max}}((1 - \omega)u) = \pi_{\text{max}}u = u.$$  

It is in this sense that $u$ “lives” on the boundary.
Closed extensions, cont.

Any extension of \( A : D_{\text{min}} \subset x^\gamma L^2_b \to x^\gamma L^2_b \) has as domain a space

\[
D = D_{\text{min}} + D, \quad D \subset \mathcal{E}(A) \text{ finite dimensional.}
\]
Closed extensions, cont.

Any extension of \( A : D_{\min} \subset x^\gamma L^2_b \to x^\gamma L^2_b \) has as domain a space

\[ D = D_{\min} + D, \quad D \subset \mathcal{E}(A) \text{ finite dimensional}. \]

(Lesch) The index of

\[ A : D \subset x^\gamma L^2_b \to x^\gamma L^2_b \]

is

\[ \text{Ind} \ A_D = \text{Ind} \ A_{D_{\min}} + \dim D. \]
Closed extensions, cont.

Any extension of $A : D_{\text{min}} \subset x^\gamma L_b^2 \to x^\gamma L_b^2$ has as domain a space

$$D = D_{\text{min}} + D, \quad D \subset \mathcal{E}(A) \text{ finite dimensional.}$$

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$$\text{Ind } A_D = \text{Ind } A_{D_{\text{min}}} + \dim D.$$ 

So the closed extensions of $A$ of a given index are in one to one correspondence with the subspaces of $\mathcal{E}$ of a given dimension.
Closed extensions, cont.

Any extension of $A : \mathcal{D}_{\text{min}} \subset x^\gamma L^2_b \rightarrow x^\gamma L^2_b$ has as domain a space

$$\mathcal{D} = \mathcal{D}_{\text{min}} + D, \quad D \subset \mathcal{E}(A) \text{ finite dimensional.}$$

(Lesch) *The index of*

$$A : \mathcal{D} \subset x^\gamma L^2_b \rightarrow x^\gamma L^2_b$$

*is*

$$\text{Ind } A_\mathcal{D} = \text{Ind } A_{\mathcal{D}_{\text{min}}} + \dim D.$$ 

So the closed extensions of $A$ of a given index are in one to one correspondence with the subspaces of $\mathcal{E}$ of a given dimension. This is:

*The set of closed extensions of $A$ of index $\text{Ind } A_{\mathcal{D}_{\text{min}}} + k$ is parametrized by the Grassmannian $\text{Gr}_k(\mathcal{E})$ of $k$-dimensional subspaces of $\mathcal{E}$.}*
The Hilbert space adjoint of $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^*$ with domain $\mathcal{D}_{\text{min}}(A^*) + A(D)^\perp$.

Here $A(D)^\perp$ is the orthogonal of $A(D)$ in $\mathcal{D}_{\text{max}}(A^*)$. 
The Hilbert space adjoint of $A$ with domain $D = D_{\min}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^*$ with domain $D_{\min}(A^*) + A(D)\perp$.

Here $A(D)\perp$ is the orthogonal of $A(D)$ in $D_{\max}(A^*)$.

Proof: Let the domain of the adjoint of $A_D$ be $D_{\min}(A^*) + D'$, $D' \subset \mathcal{E}(A^*)$. Suppose $v \in A(D)\perp$. So $(Au, v)_{A^*} = 0$ if $u \in D$. That is,

$$0 = (Au, v)_{A^*}$$
The Hilbert space adjoint of $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^*$ with domain $\mathcal{D}_{\text{min}}(A^*) + A(D)^\perp$.

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$$0 = (Au, v)_{A^*} = (A^*Au, A^*v) + (Au, v)$$
Adjoint, cont.

The Hilbert space adjoint of $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^*$ with domain $\mathcal{D}_{\text{min}}(A^*) + A(D) \perp$.

Here $A(D) \perp$ is the orthogonal of $A(D)$ in $\mathcal{D}_{\text{max}}(A^*)$.

Proof: Let the domain of the adjoint of $A_D$ be $\mathcal{D}_{\text{min}}(A^*) + D'$, $D' \subset \mathcal{E}(A^*)$. Suppose $\nu \in A(D) \perp$. So $(Au, \nu)_{A^*} = 0$ if $u \in D$. That is,

$$0 = (Au, \nu)_{A^*} = (A^*Au, A^*\nu) + (Au, \nu) = -(u, A^*\nu) + (Au, \nu)$$
Adjoints, cont.

The Hilbert space adjoint of $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^*$ with domain $\mathcal{D}_{\text{min}}(A^*) + A(D)^\perp$.

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$$0 = (Au, \nu)_{A^*} = (A^*Au, A^*\nu) + (Au, \nu) = -(u, A^*\nu) + (Au, \nu)$$

So $D \ni u \mapsto (Au, \nu)$ is continuous ($= (A^*\nu, u)$). Since

$$(\nu, Au) = (A^*\nu, u) \text{ if } u \in \mathcal{D}_{\text{min}},$$

$\nu \in D'$. 

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The Hilbert space adjoint of $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}}(A) + D$, $D \subseteq \mathcal{E}(A)$, is $A^*$ with domain $\mathcal{D}_{\text{min}}(A^*) + A(D)^\perp$.

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Conversely, if $\nu \in D'$, then $(Au, \nu) = (u, A^*\nu)$ if $u \in D$, i.e.
Adjoints, cont.

The Hilbert space adjoint of $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}}(A) + D$, $D \subset \mathcal{E}(A)$, is $A^*$ with domain $\mathcal{D}_{\text{min}}(A^*) + A(D)^\perp$.

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Proof: Let the domain of the adjoint of $A_D$ be $\mathcal{D}_{\text{min}}(A^*) + D'$, $D' \subset \mathcal{E}(A^*)$. Suppose $v \in A(D)^\perp$. So $(Au, v)_{A^*} = 0$ if $u \in D$. That is,

$$0 = (Au, v)_{A^*} = (A^*Au, A^*v) + (Au, v) = -(u, A^*v) + (Au, v)$$

So $D \ni u \mapsto (Au, v)$ is continuous (=$(A^*v, u)$). Since

$$(v, Au) = (A^*v, u) \text{ if } u \in \mathcal{D}_{\text{min}},$$

$v \in D'$.

Conversely, if $v \in D'$, then $(Au, v) = (u, A^*v)$ if $u \in D$, i.e.

$$0 = (Au, v) - (u, A^*v) = (Au, v) + (A^*Au, A^*v) = (Au, v)_{A^*}$$
Let $d = \dim \mathcal{E}(A)$, $d^* = \dim \mathcal{E}(A^*)$. Then

$$\mathcal{J} : \text{Gr}_k(\mathcal{E}(A)) \to \text{Gr}_{d^*-k}(\mathcal{E}(A^*)), \quad \mathcal{J} D = A(D)^\perp$$

This is a continuous (holomorphic) map.
Suppose $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set: $\mathcal{D} - \lambda I$ is invertible for some $\lambda$. Then $\text{Ind}(\mathcal{D}) = 0$. Because $\mathcal{D}_{\text{max}}$ is compact, $0 = \dim \mathcal{D} + \text{Ind}(\mathcal{D})$, hence $\text{Ind}(\mathcal{D}) \leq 0$. Also $\text{Ind}(\mathcal{D}) \leq \text{Ind}(\mathcal{D}_{\text{max}})$, so $\text{Ind}(\mathcal{D}_{\text{max}}) \geq 0$. $A$ has an extension with nonempty resolvent set if $\text{Ind}(\mathcal{D}_{\text{min}}) \leq 0$ and $\text{Ind}(\mathcal{D}_{\text{max}}) \geq 0$. If $\mathcal{D} - \lambda I$ is invertible, then it is injective, so also $\mathcal{D}_{\text{min}} - \lambda I$ is injective. Also $\mathcal{D} - \lambda I$ is surjective, so $\mathcal{D}_{\text{max}} - \lambda I$ is surjective.
Suppose $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

$$A_D - \lambda_0 I$$

is invertible for some $\lambda_0$. 

So $0 = \dim \mathcal{D} + \text{Ind}(A_{\mathcal{D}_{\text{min}}})$, hence $\text{Ind}(A_{\mathcal{D}_{\text{min}}}) \leq 0$. Also $\text{Ind}(A_D) \leq \text{Ind}(A_{\mathcal{D}_{\text{max}}})$, so $\text{Ind}(A_{\mathcal{D}_{\text{max}}}) \geq 0$. 

$A$ has an extension with nonempty resolvent set if $\text{Ind}(A_{\mathcal{D}_{\text{min}}}) \leq 0$ and $\text{Ind}(A_{\mathcal{D}_{\text{max}}}) \geq 0$. 

If $A_D - \lambda_0 I$ is invertible, then it is injective, so also $A_{\mathcal{D}_{\text{min}}} - \lambda_0 I$ is injective. Also $A_D - \lambda_0 I$ is surjective, so $A_{\mathcal{D}_{\text{max}}} - \lambda_0 I$ is surjective.
Suppose $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

$A_D - \lambda_0 I$ is invertible for some $\lambda_0$

Then $\text{Ind}(A_D) = 0$. 
Spectra

Suppose $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

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is invertible for some $\lambda_0$

Then $\text{Ind}(A_D) = 0$. So $0 = \dim D + \text{Ind} A_{\mathcal{D}_{\text{min}}}$, hence $\text{Ind} A_{\mathcal{D}_{\text{min}}} \leq 0$.

Ind$(A_D - \lambda_0 I) = \text{Ind}(A_D)$ because $\mathcal{D}_{\text{max}} \leftrightarrow x^\gamma L_B^2$ is compact.
Spectra

Suppose $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

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is invertible for some $\lambda_0$.

Then $\text{Ind}(A_{\mathcal{D}}) = 0$. So $0 = \dim D + \text{Ind} A_{\mathcal{D}_{\text{min}}}$, hence $\text{Ind} A_{\mathcal{D}_{\text{min}}} \leq 0$. Also $\text{Ind} A_{\mathcal{D}} \leq \text{Ind} A_{\mathcal{D}_{\text{max}}}$, so $\text{Ind} A_{\mathcal{D}_{\text{max}}} \geq 0$.
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Suppose $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

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A has an extension with nonempty resolvent set iff

$$\text{Ind } A_{\mathcal{D}_{\text{min}}} \leq 0 \quad \text{and} \quad \text{Ind } A_{\mathcal{D}_{\text{max}}} \geq 0.$$
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If $A_D - \lambda_0 I$ is invertible, then it is injective,
Spectra

Suppose $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

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is invertible for some $\lambda_0$.

Then $\text{Ind}(A_{\mathcal{D}}) = 0$. So $0 = \dim D + \text{Ind} A_{\mathcal{D}_{\text{min}}}$, hence $\text{Ind} A_{\mathcal{D}_{\text{min}}} \leq 0$. Also $\text{Ind} A_{\mathcal{D}} \leq \text{Ind} A_{\mathcal{D}_{\text{max}}}$, so $\text{Ind} A_{\mathcal{D}_{\text{max}}} \geq 0$.

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If $A_{\mathcal{D}} - \lambda_0 I$ is invertible, then it is injective, so also

$$A_{\mathcal{D}_{\text{min}}} - \lambda_0 I$$

is injective.
Suppose $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

$$A_D - \lambda_0 I \text{ is invertible for some } \lambda_0$$

Then $\text{Ind}(A_D) = 0$. So $0 = \dim D + \text{Ind } A_{D_{\text{min}}}$, hence $\text{Ind } A_{D_{\text{min}}} \leq 0$. Also $\text{Ind } A_D \leq \text{Ind } A_{D_{\text{max}}}$, so $\text{Ind } A_{D_{\text{max}}} \geq 0$.

$A$ has an extension with nonempty resolvent set iff

$$\text{Ind } A_{D_{\text{min}}} \leq 0 \quad \text{and} \quad \text{Ind } A_{D_{\text{max}}} \geq 0.$$

If $A_D - \lambda_0 I$ is invertible, then it is injective, so also

$$A_{D_{\text{min}}} - \lambda_0 I \text{ is injective.}$$

Also $A_D - \lambda_0 I$ is surjective,
Suppose $A$ with domain $\mathcal{D} = \mathcal{D}_{\text{min}} + D$ has nonempty resolvent set:

\[ A_D - \lambda_0 I \text{ is invertible for some } \lambda_0 \]

Then $\text{Ind}(A_D) = 0$. So $0 = \dim D + \text{Ind } A_{\mathcal{D}_{\text{min}}}$, hence $\text{Ind } A_{\mathcal{D}_{\text{min}}} \leq 0$. Also $\text{Ind } A_D \leq \text{Ind } A_{\mathcal{D}_{\text{max}}}$, so $\text{Ind } A_{\mathcal{D}_{\text{max}}} \geq 0$.

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If $A_D - \lambda_0 I$ is invertible, then it is injective, so also

\[ A_{\mathcal{D}_{\text{min}}} - \lambda_0 I \text{ is injective.} \]

Also $A_D - \lambda_0 I$ is surjective, so

\[ A_{\mathcal{D}_{\text{max}}} - \lambda_0 I \text{ is surjective.} \]
The background resolvent of $A$ is

$$\text{bg-res}(A) = \{ \lambda \in \mathbb{C} : A_{D_{\text{min}}} - \lambda_0 I \text{ is injective, } A_{D_{\text{max}}} - \lambda_0 I \text{ is surjective} \}$$
The background resolvent of $A$ is

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The background spectrum is the complement of this:

$$bg\text{-}spec(A) = \mathbb{C} \setminus bg\text{-}res(A).$$
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Relevancy:

1. The background spectrum of $A$ is contained in the spectrum of any extension of $A$. 
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Relevancy:

1. The background spectrum of $A$ is contained in the spectrum of any extension of $A$.

2. For $\lambda \in \text{bg-res}(A)$ let

$$\mathcal{K}_\lambda = \ker(A_{D_{\text{max}}} - \lambda I).$$
The background resolvent of $A$ is

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Then

$$\text{spec } A_D = \text{bg-spec}(A) \cup \{ \lambda \in \text{bg-res}(A) : \mathcal{K}_\lambda \cap D \neq 0 \}$$

This is a disjoint union.
The background resolvent of $A$ is

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Relevancy:

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This is a disjoint union.
The operator \( A_{D_{\text{max}}} - \lambda I \) has a right inverse if \( \lambda \in \text{bg-res}(A) \) because it is surjective. Choose it so that it maps onto \( K_{\lambda}^\perp \), call it \( B_{\text{max}}(\lambda) \):

\[
B_{\text{max}}(\lambda) \text{ is the inverse of } (A_{D_{\text{max}}} - \lambda I) : K_{\lambda}^\perp \to x^\gamma L_b^2;
\]

it depends smoothly (real-analytically) on \( \lambda \).

\[
B_{\text{max}}(\lambda)(A_{D_{\text{max}}} - \lambda I) = \pi_{K_{\lambda}^\perp}
\]
The operator $A_{D_{\text{max}}} - \lambda I$ has a right inverse if $\lambda \in \text{bg-res}(A)$ because it is surjective. Choose it so that it maps onto $\mathcal{K}_{\lambda}^\perp$, call it $B_{\text{max}}(\lambda)$:

$$B_{\text{max}}(\lambda) \text{ is the inverse of } (A_{D_{\text{max}}} - \lambda I) : \mathcal{K}_{\lambda}^\perp \rightarrow x^\gamma L_b^2;$$

it depends smoothly (real-analytically) on $\lambda$.

Let $\mathcal{D} = \mathcal{D}_{\text{min}} + \mathcal{D}$ be any domain of index 0. If $\mathcal{K}_{\lambda} \cap \mathcal{D} = 0$, then $\mathcal{D} + \mathcal{K}_{\lambda} = D_{\text{max}}$.

$$\pi_{\mathcal{K}_{\lambda} \mathcal{D}} = \text{projection on } \mathcal{K}_{\lambda} \text{ along } \mathcal{D}.$$
The operator $A_{\mathcal{D}_{\text{max}}} - \lambda I$ has a right inverse if $\lambda \in \text{bg-res}(A)$ because it is surjective. Choose it so that it maps onto $\mathcal{K}_{\lambda}^\perp$, call it $B_{\text{max}}(\lambda)$:

$$B_{\text{max}}(\lambda) \text{ is the inverse of } (A_{\mathcal{D}_{\text{max}}} - \lambda I) : \mathcal{K}_{\lambda}^\perp \to x^\gamma L^2_b;$$

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$$\pi_{\mathcal{K}_{\lambda}, \mathcal{D}} = \text{ projection on } \mathcal{K}_{\lambda} \text{ along } \mathcal{D}.$$  

Then

$$(A - \lambda I)[B_{\text{max}}(\lambda) - \pi_{\mathcal{K}_{\lambda}, \mathcal{D}}B_{\text{max}}(\lambda)] = I$$

and

$$[B_{\text{max}}(\lambda) - \pi_{\mathcal{K}_{\lambda}, \mathcal{D}}B_{\text{max}}(\lambda)](A - \lambda I) = (I - \pi_{\mathcal{K}_{\lambda}, \mathcal{D}})B_{\text{max}}(\lambda)(A - \lambda I)$$

$$= \pi_{\mathcal{D}, \mathcal{K}_{\lambda}} \pi_{\mathcal{K}_{\lambda}^\perp} = \pi_{\mathcal{D}, \mathcal{K}_{\lambda}} (\pi_{\mathcal{K}_{\lambda}^\perp} + \pi_{\mathcal{K}_{\lambda}}) = \pi_{\mathcal{D}, \mathcal{K}_{\lambda}}$$

So

$$B_{\mathcal{D}}(\lambda) = B_{\text{max}}(\lambda) - \pi_{\mathcal{K}_{\lambda}, \mathcal{D}}B_{\text{max}}(\lambda)$$

is the resolvent of $A_{\mathcal{D}} - \lambda I$.  

$B_{\text{max}}(\lambda)(A_{\mathcal{D}_{\text{max}}} - \lambda I) = \pi_{\mathcal{K}_{\lambda}^\perp}$
Suppose \(-d' = \text{Ind } A_{\mathcal{D}_{\text{min}}} < 0\) and \(\text{Ind } A_{\mathcal{D}_{\text{max}}} > 0\). For any \(\lambda \in \mathbb{C}\) there is \(\mathcal{D} = \mathcal{D}_{\text{min}} + D\) with \(D \in \text{Gr}_{d'}\) such that \(\lambda \in \text{spec}(A_{\mathcal{D}})\).
Suppose \(-d' = \text{Ind } A_{\mathcal{D}_{\text{min}}} < 0 \) and \(\text{Ind } A_{\mathcal{D}_{\text{max}}} > 0\). For any \(\lambda \in \mathbb{C}\) there is \(\mathcal{D} = \mathcal{D}_{\text{min}} + D\) with \(D \in \text{Gr}_{d'}\) such that \(\lambda \in \text{spec}(A_{\mathcal{D}})\).

Proof. If \(\lambda \in \text{bg-spec}(A)\), then \(\lambda \in \text{spec}(A_{\mathcal{D}})\) for any \(\mathcal{D}\).
Suppose $-d' = \text{Ind } A_{D_{\text{min}}} < 0$ and $\text{Ind } A_{D_{\text{max}}} > 0$. For any $\lambda \in \mathbb{C}$ there is $D = D_{\text{min}} + D$ with $D \in \text{Gr}_{d'}$ such that $\lambda \in \text{spec}(A_D)$.

Proof. If $\lambda \in \text{bg-spec}(A)$, then $\lambda \in \text{spec}(A_D)$ for any $D$. So suppose $\lambda \in \text{bg-spec}(A)$. We have $K_\lambda \cap D_{\text{min}} = 0$ (otherwise $A_{D_{\text{min}}} - \lambda I$ is not injective). Also,

$$d'' = \text{Ind}(A_{D_{\text{max}}} - \lambda I) = \dim \ker(A_{D_{\text{max}}} - \lambda I) - \dim \text{coker}(A_{D_{\text{max}}} - \lambda I)$$

so $\dim K_\lambda = d'' > 0$. Pick $\phi \in K_\lambda$, $\phi \neq 0$. Pick a subspace $D_0$ of $E$ of dimension $d' - 1$ such that $\phi \notin D' + D_{\text{min}}$ and let $D = D_{\text{min}} + \text{span } \phi + D' = D$. Then $D = D_{\text{min}} + D$ with $D \in \text{Gr}_{d'}$. 


Elliptic operators on manifolds with conical singularities, III: asymptotics of resolvent

Gerardo Mendoza
Temple University

joint work with
Juan Gil and Thomas Krainer

Temple University

Paderborn, June 9, 2011

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Research partially supported by NSF Grants DMS-0901173 and DMS-0901202
This talk is based on extensive joint work with T. Krainer and J. Gil of Penn State, in particular our last paper.\textsuperscript{2}

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- Briefly review the elliptic case on closed manifolds;
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- discuss the structure of the resolvent of such operators in the presence of a ray of minimal growth.

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Classical case

The classical theorem\(^3,^4\) goes as follows:

Let \( M \) be a compact \( n \)-manifold without boundary, \( A : C^\infty(M; E) \to C^\infty(M; E) \) a positive elliptic differential operator of order \( m \). The \( \zeta \) function of \( A \),

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\zeta_A(s) = \sum_k \frac{\dim \mathcal{E}_k}{\lambda_k^s}, \quad \Re s > n/m
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Letting $\pi_k : L^2 \to \mathcal{E}_k$ be the orthogonal projection, the resolvent of $A$ is

$$B(\lambda) = \sum_k \frac{1}{\lambda_k - \lambda} \pi_k, \quad \lambda \notin \overline{\mathbb{R}}_+.$$
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$B(\lambda)$ is pseudodifferential of order $-m$. If $-m \geq -n$, then $B(\lambda)$ is not trace-class. If so, note that $\partial_\lambda^\ell B(\lambda)$ is of order $-m\ell$ which is trace-class if $\ell$ is large enough.
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Seeley first defines complex powers $A^s = \frac{1}{2\pi i} \int_{\gamma} \lambda^s B(\lambda) d\lambda$, $\Re s << 0$, then shows that $s \mapsto A^s$ extends meromorphically, finally $\zeta(s) = \text{Tr} A^{-s}$. 

To do this one does not need $A$ to be symmetric semibounded, just that it has a ray of minimal growth: $\exists \lambda_0 \in \mathbb{C} \setminus \{0\}$, s.t $B(\lambda_0) \text{ exists if } s>>0$, and $B(\lambda) \leq C/s$. The existence of such ray of minimal growth is equivalent to $\exists \lambda_0 \in \mathbb{C} \setminus \{0\}$, s.t $(\sigma(A) - s\lambda_0)^{-1}$ exists if $s>>0$. 

(Temple University) Zeta functions Paderborn, June 9, 2011 5 / 30
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Anomalies in the location of the poles of the zeta function of an ordinary differential operator with regular singular points were described first by H. Falomir, P. A. G. Pisani, and A. Wipf\(^5\), additional examples were analyzed by H. Falomir, M. A. Muschietti, P. A. G. Pisani and R. T. Seeley\(^6\)


K. Kirsten, P. Loya, and J.-S. Park\textsuperscript{7} showed that the $\zeta$ function may have no meromorphic extension at all.

\textsuperscript{7} The very unusual properties of the resolvent, heat kernel, and zeta function for the operator $-d^2/dr^2 - 1/(4r^2)$. J. Math. Phys. 47 (2006), no. 4, 043506, 27 pp.

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In all these case the operators are “cone operators.”

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A = \frac{1}{\chi^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(x, y) D_y^\alpha (xD_x)^k
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\Delta = \frac{1}{r^2} ((rD_r)^2 - i(n-2)rD_r + \Delta_{S^{n-1}}) = \frac{1}{r^2}(r^2 \partial_r^2 + r \partial_r)
\]

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In general \(M\) will be a compact manifold with boundary, \(E \to M\) a smooth vector bundle, and \(A : C^\infty(\tilde{M}; E) \to C^\infty(\tilde{M}; E)\) a linear differential operator of order \(m\) such that

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Cone Operators  \(^9\)  

Cone differential operators are differential operators modeled on what one gets when introducing polar (spherical) coordinates about a point. For example, the (positive) Laplacian in polar coordinates is 

\[
\Delta = \frac{1}{r^2} \left( (rD_r)^2 - i(n-2)rD_r + \Delta_{S^{n-1}} \right)
\]

viewed as an operator on \([0, \infty) \times S^{n-1}\), a manifold with boundary. 

In general \(M\) will be a compact manifold with boundary, \(E \rightarrow M\) a smooth vector bundle, and \(A : C^\infty(\dot{M}; E) \rightarrow C^\infty(\dot{M}; E)\) a linear differential operator of order \(m\) such that

\[A = \frac{1}{x^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(x, y) D_y^\alpha (xD_x)^k\]

loosely near \(\partial M\) with \(a_{k\alpha} \in C^\infty\) up to \(\partial M\). 

We say
\[ A \in x^{-m} \text{Diff}^m_b(\mathcal{M}; E) \]
if \( A \in \text{Diff}^m(\mathcal{M}; E) \) and
\[
A = \frac{1}{x^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(x, y) D_y^\alpha (xD_x)^k \quad \text{locally near } \partial \mathcal{M}, \quad a_{k\alpha} \in C^\infty
\]

\[\text{---}\]
\[
^{10}\text{This and much more in J. B. Gil, T. Krainer, and —, Geometry and spectra of closed extensions of elliptic cone operators, Canad. J. Math. 59 (2007) 742–794.}
\]
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\[ A \in x^{-m} \text{Diff}^m_b(\mathcal{M}; E) \]

if \( A \in \text{Diff}^m(\mathring{\mathcal{M}}; E) \) and

\[ A = \frac{1}{x^m} \sum_{|\alpha+k|\leq m} a_{k\alpha}(x, y) D_y^\alpha (xD_x)^k \quad \text{locally near } \partial \mathcal{M}, \ a_{k\alpha} \in C^\infty \]

The principal symbol of \( A \) is a combination of its usual principal symbol in the interior and a symbol along \( \partial \mathcal{M} \) that lives invariantly on \( cT^*\mathcal{M} \).\(^{10}\)

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For the local expression of our operator

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The principal symbol of \( A \) is a combination of its usual principal symbol in the interior and a symbol along \( \partial \mathcal{M} \) that lives invariantly on \( cT^*\mathcal{M} \).

For the local expression of our operator the symbol is

\[
c\omega(A) = \sum_{|\alpha+k|=m} a_{k\alpha}(x, y) \eta^\alpha \xi^k.
\]

---

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We say

$$A \in x^{-m} \text{Diff}^{m}_b(\mathcal{M}; E)$$

if $A \in \text{Diff}^{m}(\overset{\circ}{\mathcal{M}}; E)$ and

$$A = \frac{1}{x^m} \sum_{|\alpha+k|\leq m} a_{k\alpha}(x, y) D_y^\alpha (xD_x)^k$$

locally near $\partial \mathcal{M}$, $a_{k\alpha} \in C^\infty$

The principal symbol of $A$ is a combination of its usual principal symbol in the interior and a symbol along $\partial \mathcal{M}$ that lives invariantly on $cT^*\mathcal{M}$.\(^{10}\)

For the local expression of our operator the symbol is

$$c\sigma(A) = \sum_{|\alpha+k|=m} a_{k\alpha}(x, y) \eta^\alpha \xi^k.$$  

Ellipticity:

Invertibility of $c\sigma(A)$.

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Closed extensions, domains

View $A$ initially as the unbounded operator

$$A : C^\infty_c(\mathcal{M}; E) \subset x^{-m/2} L^2_b(\mathcal{M}; E) \rightarrow x^{-m/2} L^2_b(\mathcal{M}; E)$$
Closed extensions, domains

View $A$ initially as the unbounded operator

$$A : C_c^\infty (\mathcal{M}; E) \subset x^{-m/2} L^2_b (\mathcal{M}; E) \to x^{-m/2} L^2_b (\mathcal{M}; E)$$
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$$A : C_c^\infty(\mathcal{M}; E) \subset x^{-m/2}L^2_b(\mathcal{M}; E) \rightarrow x^{-m/2}L^2_b(\mathcal{M}; E)$$

where $u = x^{-m/2}w$, $w \in L^2_b$

$$\|u\| \overset{\text{def}}{=} \|w\|_{L^2_b}$$
Closed extensions, domains

View $A$ initially as the unbounded operator

$$A : C_c^\infty(\mathcal{M}; E) \subset x^{-m/2}L^2_b(\mathcal{M}; E) \rightarrow x^{-m/2}L^2_b(\mathcal{M}; E)$$

$L^2(\mathcal{M}, \frac{1}{x}m)$, $m$ smooth positive measure on $\mathcal{M}$

$u = x^{-m/2}w$, $w \in L^2_b$

$\|u\| \overset{\text{def}}{=} \|w\|_{L^2_b}$

same weights—but $-m/2$ is arbitrary
Closed extensions, domains

View \( A \) initially as the unbounded operator

\[
A : C_c^\infty(\mathcal{M}; E) \subset \times x^{-m/2} L^2_b(\mathcal{M}; E) \rightarrow \times x^{-m/2} L^2_b(\mathcal{M}; E)
\]

Its closed extensions are determined by their domains.
Closed extensions, domains

View $A$ initially as the unbounded operator

$$A : C_0^\infty(\mathcal{M}; E) \subset x^{-m/2} L^2_b(\mathcal{M}; E) \to x^{-m/2} L^2_b(\mathcal{M}; E)$$

Its closed extensions are determined by their domains. If $\mathcal{D}$ is one such domain then

$$\mathcal{D}_{\text{min}} \subset \mathcal{D} \subset \mathcal{D}_{\text{max}}$$
Closed extensions, domains

View $A$ initially as the unbounded operator

$$A : C_c^\infty(\mathcal{M}; E) \subset x^{-m/2} L^2_b(\mathcal{M}; E) \to x^{-m/2} L^2_b(\mathcal{M}; E)$$

Its closed extensions are determined by their domains. If $\mathcal{D}$ is one such domain then

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- $\mathcal{D}_{\min}$ = domain of the closure of
Closed extensions, domains

View $A$ initially as the unbounded operator

$$A : C_c^\infty (\mathcal{M}; E) \subset x^{\frac{-m}{2}} L_b^2 (\mathcal{M}; E) \to x^{\frac{-m}{2}} L_b^2 (\mathcal{M}; E)$$

Its closed extensions are determined by their domains. If $\mathcal{D}$ is one such domain then

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Closed extensions, domains

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- $\mathcal{D}_{\text{min}} = \text{domain of the closure of}$
- $\mathcal{D}_{\text{max}} = \{ u \in x^{-m/2}L_b^2(\mathcal{M}; E) : Au \in x^{-m/2}L_b^2(\mathcal{M}; E) \}$

$L^2(\mathcal{M}, \frac{1}{x}m)$, $m$ smooth positive measure on $\mathcal{M}$

same weights—-but $-m/2$ is arbitrary

minimal domain
Closed extensions, domains

View $A$ initially as the unbounded operator

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Its closed extensions are determined by their domains. If $\mathcal{D}$ is one such domain then

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Closed extensions, domains

View $A$ initially as the unbounded operator

$$A : C_c^\infty(\hat{M}; E) \subset x^{-m/2} L^2_b(M; E) \to x^{-m/2} L^2_b(M; E)$$

Its closed extensions are determined by their domains. If $\mathcal{D}$ is one such domain then

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Closed extensions, domains

View $A$ initially as the unbounded operator

$$A : C_c^\infty(\mathcal{M}^0; E) \subset x^{-m/2} L_b^2(\mathcal{M}; E) \rightarrow x^{-m/2} L_b^2(\mathcal{M}; E)$$

Its closed extensions are determined by their domains. If $\mathcal{D}$ is one such domain then

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where

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$\mathcal{D}_{\text{max}}$ is a Hilbert space with inner product

$$(u, v)_A = (Au, Av) + (u, v), \quad u, v \in \mathcal{D}_{\text{max}}.$$
Closed extensions, domains

View \( A \) initially as the unbounded operator
\[
A : C_c^\infty(\mathcal{M}; E) \subset x^{-m/2} L^2_b(\mathcal{M}; E) \rightarrow x^{-m/2} L^2_b(\mathcal{M}; E)
\]

Its closed extensions are determined by their domains. If \( D \) is one such domain then
\[
\mathcal{D}_{\text{min}} \subset D \subset \mathcal{D}_{\text{max}}
\]
where
- \( \mathcal{D}_{\text{min}} = \) domain of the closure of
- \( \mathcal{D}_{\text{max}} = \{ u \in x^{-m/2} L^2_b(\mathcal{M}; E) : Au \in x^{-m/2} L^2_b(\mathcal{M}; E) \} \)

\( \mathcal{D}_{\text{max}} \) is a Hilbert space with inner product
\[
(u, \nu)_A = (Au, Av) + (u, \nu), \quad u, \nu \in \mathcal{D}_{\text{max}}.
\]
Closed extensions, domains

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Its closed extensions are determined by their domains. If $\mathcal{D}$ is one such domain then

$$\mathcal{D}_{\text{min}} \subset \mathcal{D} \subset \mathcal{D}_{\text{max}}$$

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Its closed extensions are determined by their domains. If $\mathcal{D}$ is one such domain then

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$$\mathcal{D} = D \oplus \mathcal{D}_{\text{min}} \text{ with } D \subset \mathcal{D}_{\text{max}} \text{ closed, orthogonal to } \mathcal{D}_{\text{min}}.$$
The domains of closed extensions of

\[ A : C_c^\infty (\mathcal{M}; E) \subset x^{-m/2} L^2_b (\mathcal{M}; E) \rightarrow x^{-m/2} L^2_b (\mathcal{M}; E) \]

are closed subspaces \( \mathcal{D} \subset \mathcal{D}_{\text{max}} \) with \( \mathcal{D}_{\text{min}} \subset \mathcal{D} \).

Let \( \mathcal{D}_{\text{min}} \) be the orthogonal complement of \( \mathcal{D}_{\text{min}} \) in \( \mathcal{D}_{\text{max}} \).

So \( \mathcal{D} \subset \mathcal{E} \).

Fact: ellipticity of \( A \) implies \( \text{Lesch} \mathcal{D}_{\text{min}} \) has finite codimension in \( \mathcal{D}_{\text{max}} \) (so \( \dim \mathcal{E} < \infty \)).

Choosing a domain corresponds classically to choosing a homogeneous boundary condition.

Every closed extension of \( A \) is Fredholm. The set of domains of closed extensions with a given index is in one-to-one correspondence with the Grassmannian of \( k \)-dimensional subspaces of \( \mathcal{E} \), for some \( k \).

End of review
The domains of closed extensions of

\[ A : C_c^\infty(\mathcal{M}; E) \subset x^{-m/2}L_b^2(\mathcal{M}; E) \to x^{-m/2}L_b^2(\mathcal{M}; E) \]

are closed subspaces \( \mathcal{D} \subset \mathcal{D}_{\text{max}} \) with \( \mathcal{D}_{\text{min}} \subset \mathcal{D} \),

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Let \( \mathcal{E} = \text{orthogonal complement of } \mathcal{D}_{\text{min}} \text{ in } \mathcal{D}_{\text{max}}. \)

So \( D \subset \mathcal{E} \).
The domains of closed extensions of

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\[ \mathcal{D} = \mathcal{D} \oplus \mathcal{D}_{\text{min}} \text{ with } \mathcal{D} \subset \mathcal{D}_{\text{max}} \text{ orthogonal to } \mathcal{D}_{\text{min}}. \]

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Fact: ellipticity of \( A \) implies

\[ \mathcal{D}_{\text{min}} \text{ has finite codimension in } \mathcal{D}_{\text{max}} \text{ (so } \dim \mathcal{E} < \infty \text{).} \]

Choosing a domain corresponds classically to choosing a homogeneous boundary condition.
The domains of closed extensions of

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Fact: ellipticity of \( A \) implies

\( \mathcal{D}_{\text{min}} \) has finite codimension in \( \mathcal{D}_{\text{max}} \) (so \( \dim \mathcal{E} < \infty \)).

Choosing a domain corresponds classically to choosing a homogeneous boundary condition.

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End of review
If $A$ is symmetric and semibounded on $C_c^\infty(\mathcal{M}; E)$, and $\mathcal{D}_{\text{min}} \neq \mathcal{D}_{\text{max}}$, then it admits many selfadjoint extensions.

The set of domains of selfadjoint extensions, $\mathcal{S}A$, is a smooth real-analytic variety of $\text{Gr}(E)$ of dimension $d$. For example, if $\dim E = 2$, then $\text{Gr}_1(E) \approx \mathbb{C}P^1 = S^2$, and $\mathcal{S}A$ is a circle in $S^2$. 

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*The set of domains of selfadjoint extensions, $\mathcal{SA}$, is a smooth real-analytic variety of $\text{Gr}_d(\mathcal{E})$ of dimension $d^2$.***
If $A$ is symmetric and semibounded on $C_c^{\infty}(\mathcal{M}; E)$, and $\mathcal{D}_{\text{min}} \neq \mathcal{D}_{\text{max}}$, then it admits many selfadjoint extensions.

The set of domains of selfadjoint extensions, $\mathcal{SA}$, is a smooth real-analytic variety of $\text{Gr}_d(\mathcal{E})$ of dimension $d^2$.

For example, if $\dim \mathcal{E} = 2$, then $\text{Gr}_1(\mathcal{E}) \approx \mathbb{C}P^1 = S^2$, and $\mathcal{SA}$ is a circle in $S^2$. 

$E$ = orthogonal complement of $\mathcal{D}_{\text{min}}$ in $\mathcal{D}_{\text{max}}$, $\dim \mathcal{E} = 2d$
If $A$ is symmetric and semibounded on $C_c^\infty(\mathcal{M}; E)$, and $\mathcal{D}_{\text{min}} \neq \mathcal{D}_{\text{max}}$, then it admits many selfadjoint extensions.

The set of domains of selfadjoint extensions, $\mathcal{S}A$, is a smooth real-analytic variety of $\text{Gr}_d(\mathcal{E})$ of dimension $d^2$.

For example, if $\dim \mathcal{E} = 2$, then $\text{Gr}_1(\mathcal{E}) \approx \mathbb{C}P^1 = S^2$, and $\mathcal{S}A$ is a circle in $S^2$. 

$\mathcal{E} = \text{orthogonal complement of } \mathcal{D}_{\text{min}} \text{ in } \mathcal{D}_{\text{max}}, \dim \mathcal{E} = 2d$
Associated with $A$ there is another operator, $A^\wedge$, on the boundary.
Associated with $A$ there is another operator, $A_\wedge$, on the boundary. If

$$A = \frac{1}{x^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(x, y) D_y^\alpha (xD_x)^k,$$

then

$$A_\wedge = \frac{1}{x^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(0, y) D_y^\alpha (xD_x)^k$$

on $[0, \infty) \times \partial \mathcal{M}$. 
Associated with $A$ there is another operator, $A_\wedge$, on the boundary. If

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on $[0, \infty) \times \partial M$. freeze coefficients at the boundary
Associated with \( A \) there is another operator, \( A_\wedge \), on the boundary. If

\[
A = \frac{1}{x^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(x, y) D_y^\alpha (x D_x)^k,
\]

then

\[
A_\wedge = \frac{1}{x^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(0, y) D_y^\alpha (x D_x)^k
\]
on \([0, \infty) \times \partial M\).

\( L^2 \) space is \( x^{-m/2} L^2_b(\partial M \times [0, \infty)) \), measure \( \frac{1}{x} (dx \otimes m_{\text{bdy}}) \).
Associated with $A$ there is another operator, $A_{\wedge}$, on the boundary. If

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then

$$A_{\wedge} = \frac{1}{x^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(0, y) D_y^\alpha(xD_x)^k$$

on $[0, \infty) \times \partial \mathcal{M}$. The operator

$$A_{\wedge} : C_c^\infty(\mathcal{Y}^\wedge; E) \subset x^{-m/2} L^2_b(\mathcal{Y}; E) \rightarrow x^{-m/2} L^2_b(\mathcal{Y}; E)$$

has its own minimal and maximal domains $\mathcal{D}_{\wedge,\text{min}}$, $\mathcal{D}_{\wedge,\text{max}}$. 

freeze coefficients at the boundary
Associated with $A$ there is another operator, $A_\wedge$, on the boundary. If

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has its own minimal and maximal domains $\mathcal{D}_{\wedge, \text{min}}, \mathcal{D}_{\wedge, \text{max}}$.

There is a natural isomorphism $\theta : \mathcal{E} \rightarrow \mathcal{E}_\wedge$. $\mathcal{E}_\wedge = \text{orthogonal of } \mathcal{D}_{\wedge, \text{min}} \text{ in } \mathcal{D}_{\wedge, \text{max}}$.

$L^2$ space is $x^{-m/2}L^2_b(\partial \mathcal{M} \times [0, \infty))$, measure $\frac{1}{x}(dx \otimes m_{\text{bdy}})$. Freeze coefficients at the boundary.
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then

$$A^\wedge = \frac{1}{x^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(0, y) D_y^\alpha (xD_x)^k$$

on $[0, \infty) \times \partial M$. The operator

$$A^\wedge : C^\infty_c (\mathcal{V}^\wedge; E) \subset x^{-m/2} L^2_b (\partial M \times [0, \infty)), \text{ measure } \frac{1}{x} (dx \otimes m_bdy).$$

has its own minimal and maximal domains $\mathcal{D}^\wedge_{\text{min}}, \mathcal{D}^\wedge_{\text{max}}$.

There is a natural isomorphism $\theta : \mathcal{E} \rightarrow \mathcal{E}^\wedge$.

This isomorphism gives a correspondence between domains of $A$ and $A^\wedge$:

$$\mathcal{D} = \mathcal{D} + \mathcal{D}_{\text{min}} \longleftrightarrow \mathcal{D}^\wedge = \theta(D) + \mathcal{D}^\wedge_{\text{min}}.$$
Associated with $A$ there is another operator, $A_\wedge$, on the boundary. If

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then

$$A_\wedge = \frac{1}{x^m} \sum_{|\alpha+k| \leq m} a_{k\alpha}(0, y) D_y^\alpha(x D_x)^k$$

on $[0, \infty) \times \partial \mathcal{M}$. The operator

$$A_\wedge : C_c^\infty(\mathcal{Y}_\wedge; E) \subset x^{-m/2} L^2_b(\partial \mathcal{M} \times [0, \infty)), \text{ measure } \frac{1}{x}(dx \otimes m_{\text{bdy}}).$$

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There is a natural isomorphism $\theta : \mathcal{E} \to \mathcal{E}_\wedge$.

This isomorphism gives a correspondence between domains of $A$ and $A_\wedge$:

$$\mathcal{D} = \mathcal{D} + \mathcal{D}_{\text{min}} \longleftrightarrow \mathcal{D}_\wedge = \theta(\mathcal{D}) + \mathcal{D}_{\wedge,\text{min}}.$$

Call this $\mathcal{D}_\wedge$. 

$L^2$ space is $x^{-m/2} L^2_b(\partial \mathcal{M} \times [0, \infty))$, measure $\frac{1}{x}(dx \otimes m_{\text{bdy}})$. 

freeze coefficients at the boundary

$\mathcal{E}_\wedge = \text{orthogonal of } \mathcal{D}_{\wedge,\text{min}} \text{ in } \mathcal{D}_{\wedge,\text{max}}$
Sectors of minimal growth

We showed:\textsuperscript{11}

\textit{Let }$\Lambda \subset \mathbb{C}$\textit{ be a closed sector.}

Sectors of minimal growth


\emph{Let} $\Lambda \subset \mathbb{C}$ \emph{be a closed sector. Suppose} 

- $\mathcal{C}(A) - \lambda$ \emph{is invertible for} $\lambda \in \Lambda,$

Sectors of minimal growth

We showed:\(^{11}\)

\[
\text{Let } \Lambda \subset \mathbb{C} \text{ be a closed sector. Suppose}
\]

- \(\mathfrak{c}(A) - \lambda \text{ is invertible for } \lambda \in \Lambda, \text{ and}
- \Lambda \text{ is a sector of minimal growth for } A_{\Lambda} \text{ with domain}

\[
D_{\Lambda} = \theta(D) + D_{\Lambda, \text{min}}.
\]

Sectors of minimal growth

We showed:11

Let $\Lambda \subset \mathbb{C}$ be a closed sector. Suppose

- $\mathcal{C}(A) - \lambda$ is invertible for $\lambda \in \Lambda$, and
- $\Lambda$ is a sector of minimal growth for $A_{\Lambda}$ with domain $\mathcal{D}_{\Lambda} = \theta(D) + \mathcal{D}_{\Lambda,\min}$.

---

Sectors of minimal growth

We showed:\(^{11}\)

Let \( \Lambda \subset \mathbb{C} \) be a closed sector. Suppose

- \( \mathfrak{c}_0(A) - \lambda \) is invertible for \( \lambda \in \Lambda \), and
- \( \Lambda \) is a sector of \textbf{minimal growth} for \( A_\Lambda \) with domain
  \[ D_\Lambda = \theta(D) + D_{\Lambda, \min}. \]

Then \( \Lambda \) is a sector of minimal growth for \( A \) with domain
\[ D = D + D_{\min}. \]

---

\(^{11}\) J. B. Gil, T. Krainer, and —, Resolvents of elliptic cone operators, J. Funct. Anal.

\[ A_\Lambda - \lambda : D_\Lambda \subset x^{-m/2} L^2_b \to x^{-m/2} L^2_b \]
has inverse \( B_{\Lambda, D_\Lambda}(\lambda), \lambda \in \Lambda, |\lambda| \text{ large} \)
and \( \| B_{\Lambda, D_\Lambda}(\lambda) \| \leq C/|\lambda|, \text{ some } C > 0. \)
Sectors of minimal growth

We showed:¹¹

Let \( \Lambda \subset \mathbb{C} \) be a closed sector. Suppose

- \( \mathcal{C}(A) - \lambda \) is invertible for \( \lambda \in \Lambda \), and
- \( \Lambda \) is a sector of minimal growth for \( A_{\Lambda} \) with domain
  \( \mathcal{D}_{\Lambda} = \theta(D) + \mathcal{D}_{\Lambda,\text{min}}. \)

Then \( \Lambda \) is a sector of minimal growth for \( A \) with domain

\( \mathcal{D} = D + \mathcal{D}_{\text{min}}. \)

The proof consists of building \( B_D(\lambda) \) starting with \( B_{\Lambda,D_{\Lambda}}(\lambda) \) and a classical pseudodifferential parametrix in the interior.

Sectors of minimal growth

We showed:\textsuperscript{11}

\textit{Let }\Lambda \subset \mathbb{C} \textit{ be a closed sector. Suppose}

- \(c_0(A) - \lambda\) is invertible for \(\lambda \in \Lambda\), and
- \(\Lambda\) is a sector of \underline{minimal growth} for \(A_\Lambda\) with domain
  \(\mathcal{D}_\Lambda = \theta(D) + \mathcal{D}_{\Lambda, \min}\).

\textit{Then }\Lambda \textit{ is a sector of minimal growth for }A\textit{ with domain}

\(\mathcal{D} = D + \mathcal{D}_{\min}\).

The proof consists of building \(B_D(\lambda)\) starting with \(B_{\Lambda, \mathcal{D}_{\Lambda}}(\lambda)\) and a classical pseudodifferential parametrix in the interior.

- \(\mathcal{D}_\Lambda\) is a kind of principal symbol for \(\mathcal{D}\);

Sectors of minimal growth

We showed:  

Let \( \Lambda \subset \mathbb{C} \) be a closed sector. Suppose

- \( \mathcal{C}(A) - \lambda \) is invertible for \( \lambda \in \Lambda \), and
- \( \Lambda \) is a sector of \textit{minimal growth} for \( A_{\Lambda} \) with domain \( \mathcal{D}_{\Lambda} = \theta(D) + \mathcal{D}_{\Lambda, \text{min}} \).

Then \( \Lambda \) is a sector of minimal growth for \( A \) with domain \( \mathcal{D} = D + \mathcal{D}_{\text{min}} \).

The proof consists of building \( B_{\mathcal{D}}(\lambda) \) starting with \( B_{\Lambda, \mathcal{D}_{\Lambda}}(\lambda) \) and a classical pseudodifferential parametrix in the interior.

- \( \mathcal{D}_{\Lambda} \) is a kind of principal symbol for \( \mathcal{D} \);
- the pair \( \mathcal{C}(A), A_{\Lambda, \mathcal{D}_{\Lambda}} \) is the principal symbol of \( A_{\mathcal{D}} \).

---

Sectors of minimal growth

We showed:¹¹

Let \( \Lambda \subset \mathbb{C} \) be a closed sector. Suppose

- \( \varphi(A) - \lambda \) is invertible for \( \lambda \in \Lambda \), and
- \( \Lambda \) is a sector of **minimal growth** for \( A^{\wedge} \) with domain \( D^{\wedge} = \theta(D) + D^{\wedge, \min} \).

Then \( \Lambda \) is a sector of minimal growth for \( A \) with domain \( D = D + D_{\min} \).

The proof consists of building \( B_D(\lambda) \) starting with \( B^{\wedge,D^{\wedge}}(\lambda) \) and a classical pseudodifferential parametrix in the interior.

- \( D^{\wedge} \) is a kind of principal symbol for \( D \);
- the pair \( \varphi(A), A^{\wedge,D^{\wedge}} \) is the principal symbol of \( A_D \).

Sectors of minimal growth

We showed:\textsuperscript{11}

\begin{itemize}
  \item \(\mathcal{s}(A) \rightarrow \lambda\) is invertible for \(\lambda \in \Lambda\), and
  \item \(\Lambda\) is a sector of \textit{minimal growth} for \(A_\Lambda\) with domain \(\mathcal{D}_\Lambda = \theta(D) + \mathcal{D}_{\Lambda, \text{min}}\).
\end{itemize}

Then \(\Lambda\) is a sector of minimal growth for \(A\) with domain \(\mathcal{D} = \mathcal{D} + \mathcal{D}_{\text{min}}\).

The proof consists of building \(B_\mathcal{D}(\lambda)\) starting with \(B_{\Lambda, \mathcal{D}_{\Lambda}}(\lambda)\) and a classical pseudodifferential parametrix in the interior.

- \(\mathcal{D}_\Lambda\) is a kind of principal symbol for \(\mathcal{D}\);
- the pair \(\mathcal{s}(A), A_{\Lambda, \mathcal{D}_\Lambda}\) is the principal symbol of \(A_{\mathcal{D}}\).

\textbf{Issue:} can we determine when is a given sector a sector of minimal growth for \(A_\Lambda\) with domain \(\mathcal{D}_\Lambda\) ?

Spectrum of $A_{\wedge,D_{\wedge}}$

Let $\lambda \in \mathbb{C}$. The condition $\lambda \notin \text{spec } A_{\wedge,D_{\wedge}}$ is equivalent to
- $(A_{\wedge,\text{min}} - \lambda)$ is injective, $(A_{\wedge,\text{max}} - \lambda)$ is surjective,
- $D_{\wedge} \cap \mathcal{K}_{\lambda} = 0$, $\mathcal{K}_{\lambda} = \{u \in D_{\wedge,\text{max}} : (A_{\wedge} - \lambda)u = 0\}$.

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Let $\lambda \in \mathbb{C}$. The condition $\lambda \notin \text{spec } A_{\wedge, D_{\wedge}}$ is equivalent to

- $(A_{\wedge, \text{min}} - \lambda)$ is injective, $(A_{\wedge, \text{max}} - \lambda)$ is surjective,
- $D_{\wedge} \cap K_{\lambda} = 0$, $K_{\lambda} = \{ u \in D_{\wedge, \text{max}} : (A_{\wedge} - \lambda)u = 0 \}$.

We say $\lambda \in \text{bg-res}(A_{\wedge})$.

\[\text{(Temple University) Zeta functions \hspace{1cm} Paderborn, June 9, 2011 15 / 30}\]

---

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We say $\lambda \in \text{bg-res}(A_{\wedge})$.

$D_{\wedge, \text{min}} \subset D \subset D_{\wedge, \text{max}}$.
Spectrum of $A_{\wedge}, D_{\wedge}$

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Given $\lambda \in \text{bg-res}(A_{\wedge})$, the condition is equivalent to

$$D_{\wedge} \cap \pi_{\wedge, \text{max}} K_{\lambda} = 0.$$
Spectrum of $A_{\wedge}, \mathcal{D}_{\wedge}$

Let $\lambda \in \mathbb{C}$. The condition $\lambda \notin \text{spec } A_{\wedge}, \mathcal{D}_{\wedge}$ is equivalent to

1. $(A_{\wedge, \min} - \lambda)$ is injective, $(A_{\wedge, \max} - \lambda)$ is surjective,
2. $\mathcal{D}_{\wedge} \cap \mathcal{K}_{\lambda} = 0$, $\mathcal{K}_{\lambda} = \{ u \in \mathcal{D}_{\wedge, \max} : (A_{\wedge} - \lambda)u = 0 \}$.

Given $\lambda \in \text{bg-res}(A_{\wedge})$, the condition is equivalent to

$$D_{\wedge} \cap \pi_{\wedge, \max} \mathcal{K}_{\lambda} = 0.$$
Spectrum of $A_\wedge, D_\wedge$

Let $\lambda \in \mathbb{C}$. The condition $\lambda \not\in \text{spec } A_\wedge, D_\wedge$ is equivalent to

- $(A_\wedge, \text{min} - \lambda)$ is injective, $(A_\wedge, \text{max} - \lambda)$ is surjective,

- $D_\wedge \cap \mathcal{K}_\lambda = 0$, $\mathcal{K}_\lambda = \{u \in D_\wedge, \text{max} : (A_\wedge - \lambda)u = 0\}$.

Given $\lambda \in \text{bg-res}(A_\wedge)$, the condition is equivalent to

$$D_\wedge \cap \pi_\wedge, \text{max} \mathcal{K}_\lambda = 0.$$

equivalently $D_\wedge \ominus \pi_\wedge, \text{max} \mathcal{K}_\lambda = \mathcal{E}_\wedge$

We say $\lambda \in \text{bg-res}(A_\wedge)$

$D_\wedge, \text{min} \subset D \subset D_\wedge, \text{max}$

$D_\wedge = D_\wedge + D_\wedge, \text{min}$

$\pi_\wedge, \text{max} : D_\wedge, \text{max} \rightarrow D_\wedge, \text{max}$

is orthogonal projection on

$\mathcal{E}_\wedge = \text{orthogonal of } D_\text{min} \text{ in } D_\text{max}$

---

Spectrum of $A_\wedge, D_\wedge$

Let $\lambda \in \mathbb{C}$. The condition $\lambda \notin \text{spec } A_\wedge, D_\wedge$ is equivalent to

- $(A_\wedge, \min - \lambda)$ is injective, $(A_\wedge, \max - \lambda)$ is surjective,
- $D_\wedge \cap K_\lambda = 0$, $K_\lambda = \{ u \in D_\wedge, \max : (A_\wedge - \lambda)u = 0 \}$.

Given $\lambda \in \text{bg-res}(A_\wedge)$, the condition is equivalent to

$$D_\wedge \cap \pi_\wedge, \max K_\lambda = 0,$$
equivalently $D_\wedge \oplus \pi_\wedge, \max K_\lambda = \mathcal{E}_\wedge$

Let $k = \dim D_\wedge$. For $K \in \text{Gr}_{d-k}(\mathcal{E})$ define

$$\mathcal{V}_K = \{D_\wedge \in \text{Gr}_k(\mathcal{E}_\wedge) : K \cap D_\wedge \neq 0 \}.$$
Spectrum of $A_\wedge, \mathcal{D}_\wedge$

Let $\lambda \in \mathbb{C}$. The condition $\lambda \notin \text{spec}\, A_\wedge, \mathcal{D}_\wedge$ is equivalent to

- $(A_{\wedge, \text{min}} - \lambda)$ is injective, $(A_{\wedge, \text{max}} - \lambda)$ is surjective,
- $\mathcal{D}_\wedge \cap \mathcal{K}_\lambda = 0$, $\mathcal{K}_\lambda = \{ u \in \mathcal{D}_{\wedge, \text{max}} : (A_{\wedge} - \lambda)u = 0 \}$.

Given $\lambda \in \text{bg-res}(A_\wedge)$, the condition (*) is equivalent to

$$D_\wedge \cap \pi_{\wedge, \text{max}} \mathcal{K}_\lambda = 0. \tag{*}$$

equivalently $D_\wedge \oplus \pi_{\wedge, \text{max}} \mathcal{K}_\lambda = \mathcal{E}_\wedge$

Let $k = \dim D_\wedge$. For $K \in \text{Gr}_{d-k}(\mathcal{E})$ define

$$\mathcal{V}_K = \{ D_\wedge \in \text{Gr}_k(\mathcal{E}_\wedge) : K \cap D_\wedge \neq 0 \}. \quad \text{We say } \lambda \in \text{bg-res}(A_\wedge)$$

$$\mathcal{D}_{\wedge, \text{min}} \subset \mathcal{D} \subset \mathcal{D}_{\wedge, \text{max}} \quad \mathcal{D}_\wedge = \mathcal{D}_\wedge + \mathcal{D}_{\wedge, \text{min}} \quad \pi_{\wedge, \text{max}} : \mathcal{D}_{\wedge, \text{max}} \to \mathcal{D}_{\wedge, \text{max}}$$

$$\dim \mathcal{E}_\wedge \quad \mathcal{E}_\wedge = \text{orthogonal of } \mathcal{D}_{\text{min}} \text{ in } \mathcal{D}_{\text{max}} \quad \text{is orthogonal projection on}$$

Setting $K_\lambda = \pi_{\wedge, \text{max}} \mathcal{K}_\lambda$, (*) says

$$D_\wedge \notin \mathcal{V}_{K_\lambda}$$

---

Spectrum of $A_{\lambda,D_{\lambda}}$

Let $\lambda \in \mathbb{C}$. The condition $\lambda \notin \text{spec } A_{\lambda,D_{\lambda}}$ is equivalent to

- $(A_{\lambda, \text{min}} - \lambda)$ is injective, $(A_{\lambda, \text{max}} - \lambda)$ is surjective,
- $D_{\lambda} \cap K_{\lambda} = 0$, $K_{\lambda} = \{ u \in D_{\lambda, \text{max}} : (A_{\lambda} - \lambda)u = 0 \}$.

Given $\lambda \in \text{bg-res}(A_{\lambda})$, the condition is equivalent to

(*)

$$D_{\lambda} \cap \pi_{\lambda, \text{max}} K_{\lambda} = 0.$$

equivalently $D_{\lambda} \oplus \pi_{\lambda, \text{max}} K_{\lambda} = E_{\lambda}^\dim$

Let $k = \dim D_{\lambda}$. For $K \in \text{Gr}_{d-k}(E)$ define

$$\mathcal{V}_K = \{ D_{\lambda} \in \text{Gr}_k(E_{\lambda}) : K \cap D_{\lambda} \neq 0 \}.$$

Setting $K_{\lambda} = \pi_{\lambda, \text{max}} K_{\lambda}$, (*) says

$$D_{\lambda} \notin \mathcal{V}_{K_{\lambda}}$$

This gives$^{12}$

$$\text{spec } A_{\lambda,D_{\lambda}} = \{ \lambda \in \text{bg-res } A_{\lambda} : D_{\lambda} \in \mathcal{V}_{K_{\lambda}} \} \cup \text{bg-spec}(A_{\lambda}).$$

---

For functions $u$ on $[0, \infty) \times \partial M$ define

$$(\kappa_\varrho u)(x, y) = \varrho^{m/2} u(\varrho x, y), \quad \varrho > 0.$$ 

This is a one-parameter group of unitary maps

$$\mathbb{R}_+ \overset{\varrho \mapsto \kappa_\varrho}{\longrightarrow} \text{U}(x^{-m/2} L^2_b([0, \infty) \times \partial M; E)).$$
For functions $u$ on $[0, \infty) \times \partial M$ define

$$(\kappa_\varphi u)(x, y) = \varphi^{m/2} u(\varphi x, y), \quad \varphi > 0.$$ 

This is a one-parameter group of unitary maps

$$\mathbb{R}_+ \xrightarrow{\varphi \mapsto \kappa_\varphi} U(x^{-m/2} L^2_b([0, \infty) \times \partial M; E)).$$

multiplicative group; $x \partial_x$ is translation invariant,

$\kappa_\varphi$ is translation
For functions $u$ on $[0, \infty) \times \partial M$ define

$$(\kappa_\rho u)(x, y) = \rho^{m/2} u(\rho x, y), \quad \rho > 0.$$ 

This is a one-parameter group of unitary maps

$$\mathbb{R}_+ \xrightarrow{\rho \mapsto \kappa_\rho} U(x^{-m/2} L_b^2([0, \infty) \times \partial M; E)).$$

The formula

$$\kappa_\rho^{-1} A \wedge \kappa_\rho = \rho^m A$$

holds.
For functions $u$ on $[0, \infty) \times \partial M$ define

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holds.
For functions $u$ on $[0, \infty) \times \partial M$ define

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Multiplicative group; $x\partial_x$ is translation invariant, $\kappa_\varrho$ is translation

The formula

$$\kappa^{-1}_\varrho A \land \kappa_\varrho = \varrho^m A \land$$

holds.
For functions \( u \) on \([0, \infty) \times \partial \mathcal{M}\) define

\[
(\kappa_{\varrho} u)(x, y) = \varrho^{m/2} u(\varrho x, y), \quad \varrho > 0.
\]

This is a one-parameter group of unitary maps

\[
\mathbb{R}_+ \xrightarrow{\varrho \mapsto \kappa_{\varrho}} \text{U}(x^{-m/2} L^2_b([0, \infty) \times \partial \mathcal{M}; E)).
\]

The formula

\[
\kappa_{\varrho}^{-1} A \wedge \kappa_{\varrho} = \varrho^m A \wedge
\]

holds.

\[
\begin{align*}
\kappa_{\varrho}^{-1} &\left[ \frac{1}{x^m} a_{k\alpha}(0, y) D_y^\alpha (xD_x)^k \kappa_{\varrho} \right] \\
&= \frac{1}{(x/\varrho)^m} \kappa_{\varrho}^{-1} \left[ a_{k\alpha}(0, y) D_y^\alpha (xD_x)^k \kappa_{\varrho} \right] \\
&= \frac{\varrho^m}{x^m} a_{k\alpha}(0, y) D_y^\alpha (xD_x)^k
\end{align*}
\]
For functions \( u \) on \([0, \infty) \times \partial M\) define

\[
(\kappa_{\varrho} u)(x, y) = \varrho^{m/2} u(\varrho x, y), \quad \varrho > 0.
\]

This is a one-parameter group of unitary maps

\[
\mathbb{R}_+ \xrightarrow{\varrho \mapsto \kappa_{\varrho}} U(x^{-m/2}L^2_b([0, \infty) \times \partial M; E)).
\]

The formula

\[
\kappa_{\varrho}^{-1} A \wedge \kappa_{\varrho} = \varrho^m A \wedge
\]

holds. It gives that \( \kappa_{\varrho} \) maps \( D_{\wedge,\max} \) to itself.
For functions $u$ on $[0, \infty) \times \partial M$ define

$$(\kappa_\rho u)(x, y) = \rho^{m/2} u(\rho x, y), \quad \rho > 0.$$ 

This is a one-parameter group of unitary maps

$$\mathbb{R}_+ \xrightarrow{\rho \mapsto \kappa_\rho} \text{U}(x^{-m/2}L^2_b([0, \infty) \times \partial M; E)).$$

The formula

$$\kappa_\rho^{-1} A \wedge \kappa_\rho = \rho^m A \wedge$$

holds. It gives that $\kappa_\rho$ maps $\mathcal{D}_{\wedge, \text{max}}$ to itself. Get action on $\mathcal{E}_\wedge$:

$$\kappa_\rho |_{\mathcal{E}_\wedge} : \mathcal{E}_\wedge \to \mathcal{E}_\wedge.$$
For functions \( u \) on \([0, \infty) \times \partial M\) define

\[
(k_\varrho u)(x, y) = \varrho^{m/2} u(\varrho x, y), \quad \varrho > 0.
\]

This is a one-parameter group of unitary maps

\[
\mathbb{R}_+ \xrightarrow{\varrho \mapsto k_\varrho} U(x^{-m/2} L^2_b([0, \infty) \times \partial M; E)).
\]

multiplicative group; \( x\partial_x \) is translation invariant, \( k_\varrho \) is translation

The formula

\[
k_\varrho^{-1} A \wedge k_\varrho = \varrho^m A \wedge
\]

holds. It gives that \( k_\varrho \) maps \( D_{\wedge, \max} \) to itself. Get action on \( E_\wedge \):

\[
\varrho \mapsto k_\varrho = \pi_{\wedge, \max} k_\varrho |_{E_\wedge} : E_\wedge \to E_\wedge.
\]

The \( k_\varrho \) give diffeomorphisms

\[
k_\varrho : \text{Gr}_k(E_\wedge) \to \text{Gr}_k(E_\wedge), \quad D_\wedge \mapsto k_\varrho D_\wedge.
\]
Let $D_{\Lambda} \in \text{Gr}_k(\mathcal{E}_{\Lambda})$. Define

$$\Omega^{-}(D_{\Lambda}) = \{ D'_{\Lambda} \in \text{Gr}_k(\mathcal{E}_{\Lambda}) : \exists \{\varrho_{\nu}\} \text{ s.t. } \varrho_{\nu} \to \infty, \kappa_{\varrho_{\nu}}^{-1}D_{\Lambda} \to D'_{\Lambda}\}$$

When is a sector a sector of minima growth?

---


Let $D^\wedge \in \text{Gr}_k(\mathcal{E}^\wedge)$. Define

$$\Omega^-(D^\wedge) = \{ D'_\wedge \in \text{Gr}_k(\mathcal{E}^\wedge) : \exists \{ \varrho_\nu \} \text{ s.t. } \varrho_\nu \to \infty, \ \kappa^{\frac{-1}{\varrho_\nu}}D^\wedge \to D'_\wedge \}$$

Let $\Lambda$ be a closed sector and $a$ an arc subtending $\Lambda$. Suppose $a \subset \text{bg-res } A^\wedge$.

When is a sector a sector of minima growth?

---


When is a sector a sector of minima growth?

Let $D^\wedge \in \text{Gr}_k(\mathcal{E}^\wedge)$. Define

$$\Omega^-(D^\wedge) = \{ D'_\wedge \in \text{Gr}_k(\mathcal{E}^\wedge) : \exists \{ \varrho_\nu \} \text{ s.t. } \varrho_\nu \to \infty, \kappa^{-1}_{\varrho_\nu} D^\wedge \to D'_\wedge \}$$

Let $\Lambda$ be a closed sector and $\alpha$ an arc subtending $\Lambda$. Suppose $\alpha \subset \text{bg-res } A^\wedge$.

Let $\mathcal{V} = \bigcup_{\lambda \in \alpha} \mathcal{V}_{K^\wedge}$

D^\wedge \in \mathcal{V}_{K^\wedge} \iff D^\wedge \cap K^\wedge \neq 0$

---


Let $D^\wedge \in \text{Gr}_k(\mathcal{E}^\wedge)$. Define

$$\Omega^-(D^\wedge) = \{D'_\wedge \in \text{Gr}_k(\mathcal{E}^\wedge) : \exists \{\varrho_\nu\} \text{ s.t. } \varrho_\nu \to \infty, \kappa^{-1}_{\varrho_\nu} D^\wedge \to D'_\wedge\}$$

Let $\Lambda$ be a closed sector and $\alpha$ an arc subtending $\Lambda$. Suppose $\alpha \subset \text{bg-res } A^\wedge$.

Let

$$\mathcal{V} = \bigcup_{\lambda \in \alpha} \mathcal{V}_{K^\wedge}$$

$\Lambda$ is a sector of minimal growth for $A^\wedge, D^\wedge$ if and only if $\Omega^-(D^\wedge) \cap \mathcal{V} = \emptyset$.\textsuperscript{13,14}

This is an ellipticity condition on $D^\wedge$.


Let $D_\Lambda \in \text{Gr}_k(\mathcal{E}_\Lambda)$. Define

$$
\Omega^-(D_\Lambda) = \{ D'_\Lambda \in \text{Gr}_k(\mathcal{E}_\Lambda) : \exists \{ \varrho_\nu \} \text{ s.t. } \varrho_\nu \to \infty, \kappa_{\varrho_\nu}^{-1} D_\Lambda \to D'_\Lambda \} 
$$

Let $\Lambda$ be a closed sector and $\alpha$ an arc subtending $\Lambda$. Suppose

$$
\alpha \subset \text{bg-res } A_\Lambda.
$$

Let

$$
\mathcal{Y} = \bigcup_{\lambda \in \alpha} \mathcal{Y}_{K_\lambda}
$$

When is a sector a sector of minima growth?

$\Lambda$ is a sector of minimal growth for $A_\Lambda, D_\Lambda$ if and only if

$$
\Omega^-(D_\Lambda) \cap \mathcal{Y} = \emptyset. \text{ \footnote{13, 14}}
$$

This is an ellipticity condition on $D_\Lambda$.


Let $D_\wedge \in \text{Gr}_k(E_\wedge)$. Define

$$\Omega^-(D_\wedge) = \{ D'_\wedge \in \text{Gr}_k(E_\wedge) : \exists \{ q_\nu \} \text{ s.t. } q_\nu \to \infty, \kappa_{q_\nu}^{-1} D_\wedge \to D'_\wedge \}$$

Let $\Lambda$ be a closed sector and $a$ an arc subtending $\Lambda$. Suppose $a \subset \text{bg-res } A_\wedge$.

Let

$$\mathcal{V} = \bigcup_{\lambda \in a} \mathcal{V}_{K_\lambda}$$

Then

$$\Lambda \text{ is a sector of minimal growth for } A_\wedge, D_\wedge \text{ if and only if } \Omega^-(D_\wedge) \cap \mathcal{V} = \emptyset.$$ 

This is an ellipticity condition on $D_\wedge$.

What is $\Omega^-(D_\wedge)$?

---


\( \Omega^-(D) \subset \text{Gr}_k(\mathcal{E}) \) is an embedded torus \( G \) (a point for generic \( A \)), and there are \( D_\infty \in \Omega^-(D) \), \( v \in \mathfrak{g} \) such that

\[
\text{dist}\left( \kappa^{-1}_q D, D_\infty \exp(-\log q \, v) \right) \to 0
\]
as \( q \to \infty \).
\( \Omega^-(D^\wedge) \subset \text{Gr}_k(\mathcal{E}^\wedge) \) is an embedded torus \( G \) (a point for generic \( A \)), and there are \( D_\infty \in \Omega^-(D^\wedge) \), \( \nu \in \mathfrak{g} \) such that 
\[
\text{dist} \left( \kappa_q^{-1} D^\wedge, D_\infty \exp(-\log \rho \nu) \right) \to 0
\]
as \( \rho \to \infty \).
\(\Omega^{-}(D_{\wedge}) \subset \text{Gr}_{k}(E_{\wedge})\) is an embedded torus \(G\) (a point for generic \(A\)), and there are \(D_{\infty} \in \Omega^{-}(D_{\wedge})\), \(v \in \mathfrak{g}\) such that \(\text{dist} \left( \kappa_{q}^{-1}D_{\wedge}, D_{\infty} \exp(-\log q \; v) \right) \to 0\) as \(q \to \infty\).

The map \(\kappa_{q} : E_{\wedge} \to E_{\wedge}\) is of the form \(e^{\log q \; \alpha}\) for some linear \(\alpha : E_{\wedge} \to E_{\wedge}\). Given \(D_{\wedge}\), there are \(v_{\ell}(q)\) forming a basis of \(\kappa_{q}^{-1}D_{\wedge}\) \((q \gg 0)\),

\[
v_{\ell}(q) = e^{-\log q \; \alpha'} g_{\ell}(\log q) + \sum_{\lambda \in \text{spec} \; \alpha} e^{-\left(\lambda - \mu_{\ell}\right) \log q} p_{\ell, \lambda}(\log q).
\]
\( \Omega^-(D^\wedge) \subset \text{Gr}_k(\mathcal{E}^\wedge) \) is an embedded torus \( G \) (a point for generic \( A \)), and there are \( D^\infty \in \Omega^-(D^\wedge) \), \( \nu \in \mathfrak{g} \) such that
\[
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\]

The map \( \kappa_q : \mathcal{E}^\wedge \to \mathcal{E}^\wedge \) is of the form \( e^{\log \varrho \alpha} \) for some linear \( \alpha : \mathcal{E}^\wedge \to \mathcal{E}^\wedge \). Given \( D^\wedge \), there are \( \nu_\ell(\varrho) \) forming a basis of \( \kappa_q^{-1}D^\wedge \) \( (\varrho \gg 0) \),
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\nu_\ell(\varrho) = e^{-\log \varrho \alpha'} g_\ell(\log \varrho) + \sum_{\lambda \in \text{spec } \alpha} e^{-(\lambda - \mu_\ell) \log \varrho} p_{\ell, \lambda}(\log \varrho).
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\( g_\ell(t) = \text{polynomials in } 1/t \text{ with values in } \bigoplus_{\Re \lambda = \mu_\ell} \mathcal{E}_\lambda; \mathcal{E}_\lambda = \text{generalized eigenspaces of } \alpha \)
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\(\alpha'\) is diagonal with eigenvalues \(i \Im \lambda, \lambda \in \text{spec } \alpha\)
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The map $\kappa_q : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$ is of the form $e^{\log q \alpha}$ for some linear $\alpha : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$. Given $D_\wedge$, there are $\nu_\ell(q)$ forming a basis of $\kappa_q^{-1}D_\wedge$ ($q \gg 0$),

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The collection of vectors

\[
g_{\infty,\ell} = \lim_{t \to \infty} g_\ell(t)
\]
is a basis of \( D_\infty \).
Example

Let $\mathcal{M}$ be a 2-manifold with boundary $\mathcal{Y} = S^1$ and cone metric

$$g = dx^2 + x^2 g_\mathcal{Y}(x) \text{ near } \mathcal{Y}, \quad g_\mathcal{Y}(0) = d\theta^2$$
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$$\Delta = x^{-2} \left( (xD_x)^2 + p(x, y)(xD_x) + \Delta g_\mathcal{Y}(x) \right)$$
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$\Delta_\wedge$ is the Laplacian on $\mathcal{Y}^{\wedge} = \mathbb{R}^2 \setminus \{(0, 0)\}$ in polar coordinates, considered as an unbounded operator in $L^2(\mathcal{Y}^{\wedge})$. 
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Domains for $\Delta_\wedge$:

$$\mathcal{D}_\wedge,_{\text{min}} = \{ u \in H^2(\mathbb{R}^2) : u(0,0) = 0 \}$$

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Grassmannian of extensions: one dimensional subspaces of

$$\mathcal{E}_\wedge = \mathcal{D}_{\wedge, \text{max}}/\mathcal{D}_{\wedge, \text{min}} \approx \{1, \log(x)\}$$
Example

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By the way, the domain of the Friedrichs extension of $\Delta_\wedge$ is $H^2(\mathbb{R}^2) = \mathcal{D}_{\min} + \text{span}_\mathbb{C}\{\omega\}$. 

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Example, cont.

Grassmannian of extensions of index 0: one dimensional subspaces of

\[ E^\wedge = D^\wedge,_{\text{max}}/D^\wedge,_{\text{min}} \approx \{1, \log(x)\} \]
Example, cont.

Grassmannian of extensions of index 0: one dimensional subspaces of

\[ \mathcal{E}^\wedge = \mathcal{D}^\wedge,_{\text{max}}/\mathcal{D}^\wedge,_{\text{min}} \approx \{1, \log(x)\} \]

parametrized by

\[ \mathbb{C}P^1 \ni [a : b] \leftrightarrow D_{[a:b]} = \text{span}_\mathbb{C}\{a + b \log(x)\} \quad (a, b) \neq 0 \]
Example, cont.

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Example, cont.

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Now

\[ D_F \leftrightarrow [1 : 0] \]

\[ \kappa_\varphi D_{[a:b]} = \text{span}_\mathbb{C}\{a + b \log(\varphi x)\} = \text{span}_\mathbb{C}\{(a + \log \varphi) + b \log(\varphi x)\} = D_{[a + \log \varphi : b]} \]
Example, cont.

Grassmannian of extensions of index 0: one dimensional subspaces of

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For \( |\log \varrho| \) large,

\[ D_{[a+\log \varrho:b]} = D_{[1:b/(a+\log \varrho)]} \]
Example, cont.

Grassmannian of extensions of index 0: one dimensional subspaces of

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so \(\kappa_\varrho D_{[a:b]} \to D_F\).
Asymptotics

\[ \ker \hat{A}(\sigma) \neq 0, \quad \hat{A} = \sum_{|\alpha| + k \leq m} a_k \alpha(0, y) \sigma^k D^\alpha \]

Associated with \( A \) there is a set \( \text{spec}_b A \subset \mathbb{C} \).
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Suppose \( V \cap \Omega = (D \wedge ) = \emptyset \).

\[ = \Rightarrow E \wedge = \lambda \oplus \kappa \]

Then \( \pi K \lambda, \kappa \kappa \kappa \kappa \kappa \sim \nu \in M \nu(\lambda, \nu_1, \nu_N), \log \pi \nu \rightarrow \infty \).

\[ r \nu = p \nu / q \nu, |q \nu(\lambda, \nu_{1, \ldots, \nu_N}, \nu_{N+1})| \geq C > 0, \]

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Asymptotics

Associated with $A$ there is a set $\text{spec}_b A \subset \mathbb{C}$. Let

$$\Sigma = \{ \sigma \in \text{spec}_b A : -m/2 < \Im \sigma < m/2 \},$$

let $M$ be the additive semigroup generated by

$$\{ \Im \sigma_1 - \Im \sigma_2 : \sigma_1, \sigma_2 \in \Sigma, \Im \sigma_1 \leq \Im \sigma_2 \} \subset (-\infty, 0]$$
Asymptotics

Associated with $A$ there is a set $\text{spec}_b A \subset \mathbb{C}$. Let

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and let

$$\mathcal{P} = \{ \Re \sigma : \sigma \in \Sigma \} = \{ \mu_1, \mu_2, \ldots, \mu_N \}$$

Suppose $\mathcal{V} \cap \Omega^-(D_\wedge) = \emptyset$. Then $\pi_{K_\lambda, \kappa_q^{-1}D_\wedge}$ has an asymptotic expansion

$$\pi_{K_\lambda, \kappa_q^{-1}D_\wedge} \sim \sum_{\nu \in \mathcal{M}} r_\nu(\lambda, q^{i\mu_1}, \ldots, q^{i\mu_N}, \log q) q^{-\nu}$$

as $q \to \infty$. The $r_\nu(\lambda, z_1, \ldots, z_N, z_{N+1})$ are rational functions of $z$ with values in $\text{End}(\mathcal{E}_\wedge)$. 
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“Proof”: The map $\kappa_\varrho : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$ is of the form $e^{\log \varrho \alpha}$ for some linear $\alpha : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$. Given $D_\wedge$, there are $v_\ell(\varrho)$ forming a basis of $\kappa_\varrho^{-1}D_\wedge \ (\varrho \gg 0)$,

$$v_\ell(\varrho) = e^{-\log \varrho \alpha'} g_\ell(\log \varrho) + \sum_{\lambda \in \text{spec } \alpha} e^{-\mu_\ell} p_{\ell, \lambda}(\log \varrho).$$

$g_\ell(t)$ = polynomials in $1/t$ with values in $\bigoplus_{\mu_\lambda = \mu_\ell} \mathcal{E}_\lambda$; $\mathcal{E}_\lambda$ = generalized eigenspaces of $\alpha$ $\alpha'$ is diagonal with eigenvalues $i \Im \lambda$, $\lambda \in \text{spec } \alpha$

The collection of vectors $g_{\infty, \ell} = \lim_{t \to \infty} g_\ell(t)$ is a basis of $D_\infty$. Let $\phi$ be a basis of $K_\lambda$, $u$ a complementary basis. Then

$$[\phi, v] = [\phi, u] \cdot \begin{bmatrix} I & P \\ 0 & Q \end{bmatrix}, [\phi, u] = [\phi, v] \cdot \begin{bmatrix} I & -PQ^{-1} \\ 0 & Q^{-1} \end{bmatrix}.$$
“Proof”: The map $\kappa_{\varrho} : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$ is of the form $e^{\log\varrho \alpha}$ for some linear $\alpha : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$. Given $D_\wedge$, there are $v_\ell(\varrho)$ forming a basis of $\kappa_{\varrho}^{-1}D_\wedge$ ($\varrho \gg 0$),

$$v_\ell(\varrho) = e^{-\log\varrho \alpha'}g_\ell(\log \varrho) + \sum_{\lambda \in \text{spec} \alpha, R\lambda > \mu_\ell} e^{-(\lambda - \mu_\ell) \log\varrho} p_{\ell,\lambda}(\log \varrho).$$

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and

$$w = [\phi, u] \cdot \begin{bmatrix} x \\ y \end{bmatrix} = [\phi, v] \begin{bmatrix} I & -PQ^{-1} \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \phi \cdot (x - PQ^{-1}y) + v \cdot Q^{-1}y.$$
"Proof": The map $\kappa_\varrho : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$ is of the form $e^{\log \varrho \alpha}$ for some linear $\alpha : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$. Given $D_\wedge$, there are $v_\ell(\varrho)$ forming a basis of $\kappa_\varrho^{-1} D_\wedge (\varrho \gg 0)$,

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$g_\ell(t) = \text{polynomials in } 1/t$ with values in $\bigoplus_{\mathbb{R} \lambda = \mu_\ell} \mathcal{E}_\lambda$; $\mathcal{E}_\lambda$ is the space of generalized eigenspaces of $\alpha$ and $\alpha'$ is diagonal with eigenvalues $i \Im \lambda$, $\lambda \in \text{spec} \alpha$.

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so

$$\pi_{K_\lambda, \kappa_\varrho^{-1} D_\wedge}(w) = \phi \cdot (x - PQ^{-1} y)$$
Example

Let $A_\wedge = \Delta$ on $\mathbb{R}^2 \setminus 0$. Then

$$u_1 = \frac{1}{(2\pi)^2} \int e^{ix\cdot \xi} \frac{1}{1 + |\xi|^4} \, d\xi, \quad u_2 = \frac{1}{(2\pi)^2} \int e^{ix\cdot \xi} \frac{|\xi|^2}{1 + |\xi|^4} \, d\xi$$

These span the orthogonal of $D_{\text{min}}$ in $D_{\text{max}}$ with respect to $(u, v)_A = (Au, Av) + (u, v)$, $u, v \in D_{\text{max}}$ and one has

$$\pi_{\wedge, \text{max}}(\kappa_\varrho u_1) = u_1, \quad \pi_{\wedge, \text{max}}(\kappa_\varrho u_2) = -\frac{4}{\pi} \log \varrho \, u_1 + u_2.$$  

Say

$$K = \text{span} \, \phi, \phi = pu_1 + qu_2, \quad D_\wedge = \text{span} \, u, u = au_1 + bu_2.$$  

$$[\phi \, \kappa_\varrho u] = [u_1 \, u_2] \begin{bmatrix} p & a - (4b/\pi) \log \varrho \\ q & b \end{bmatrix}$$  

$$[\phi \, \kappa_\varrho u] \begin{bmatrix} b \\ -q \end{bmatrix} = \begin{bmatrix} -a + (4b/\pi) \log \varrho & p \end{bmatrix} C_\varrho [u_1 \, u_2]$$
Let $A_\wedge = \Delta$ on $\mathbb{R}^2 \setminus 0$. Then

$$u_1 = \frac{1}{(2\pi)^2} \int e^{ix \cdot \xi} \frac{1}{1 + |\xi|^4} \, d\xi, \quad u_2 = \frac{1}{(2\pi)^2} \int e^{ix \cdot \xi} \frac{|\xi|^2}{1 + |\xi|^4} \, d\xi$$

These span the orthogonal of $\mathcal{D}_{\text{min}}$ in $\mathcal{D}_{\text{max}}$ with respect to

$$(u, v)_A = (Au, Av) + (u, v), \quad u, v \in \mathcal{D}_{\text{max}}$$

and one has

$$\pi_\wedge, \text{max}(k_q u_1) = u_1, \quad \pi_\wedge, \text{max}(k_q u_2) = -\frac{4}{\pi} \log \varrho \ u_1 + u_2.$$
The link with the asymptotics of $B_{\wedge, \mathcal{D}_{\wedge}}(\lambda)$ is the formula

$$B_{\wedge, \mathcal{D}_{\wedge}}(\lambda) = B_{\wedge, \text{max}}(\lambda) - \left[ I - B_{\wedge, \text{min}}(\lambda)(A_{\wedge} - \lambda) \right] \pi_{\mathcal{K}_{\wedge}, \lambda, \mathcal{D}} B_{\wedge, \text{max}}(\lambda)$$

(for $\lambda \notin \text{spec } A_{\wedge, \mathcal{D}_{\wedge}}$)

in which

$$(A_{\wedge} - \lambda)B_{\wedge, \text{max}}(\lambda) = I, \quad B_{\wedge, \text{min}}(\lambda)(A_{\wedge} - \lambda) = \Pi(\lambda).$$
The link with the asymptotics of $B_{\land,D}(\lambda)$ is the formula

$$B_{\land,D}(\lambda) = B_{\land,\max}(\lambda) - \left[ I - B_{\land,\min}(\lambda)(A_{\land} - \lambda) \right] \pi_{K_{\land,\lambda},D} B_{\land,\max}(\lambda)$$

in which

$$(A_{\land} - \lambda)B_{\land,\max}(\lambda) = I, \quad B_{\land,\min}(\lambda)(A_{\land} - \lambda) = \Pi(\lambda).$$
The link with the asymptotics of $B_{\wedge, D_{\wedge}}(\lambda)$ is the formula

$$B_{\wedge, D_{\wedge}}(\lambda) = B_{\wedge, \text{max}}(\lambda) - \left[ I - B_{\wedge, \text{min}}(\lambda) (A_{\wedge} - \lambda) \right] \pi_{K_{\wedge}, \lambda, D} B_{\wedge, \text{max}}(\lambda)$$

in which

$$(A_{\wedge} - \lambda) B_{\wedge, \text{max}}(\lambda) = I, \quad B_{\wedge, \text{min}}(\lambda) (A_{\wedge} - \lambda) = \Pi(\lambda).$$

The property $\kappa_\varrho^{-1} A_{\wedge} \kappa_\varrho = \varrho^m A_{\wedge}$ implies

$$\kappa_\varrho^{-1} (A_{\wedge} - \varrho^m \lambda) \kappa_\varrho = \varrho^m (A_{\wedge} - \lambda).$$
The link with the asymptotics of $B_{\wedge,\mathcal{D}_{\wedge}}(\lambda)$ is the formula

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in which

$$(A_{\wedge} - \lambda)B_{\wedge,\text{max}}(\lambda) = I, \quad B_{\wedge,\text{min}}(\lambda)(A_{\wedge} - \lambda) = \Pi(\lambda).$$

The property $\kappa_q^{-1}A_{\wedge}\kappa_q = q^m A_{\wedge}$ implies

$$\kappa_q^{-1}(A_{\wedge} - q^m \lambda)\kappa_q = q^m (A_{\wedge} - \lambda).$$

$\Rightarrow \mathcal{K}_q^{m\lambda} = \kappa_q \mathcal{K}_\lambda$
The link with the asymptotics of $B_{\wedge,D_{\wedge}}(\lambda)$ is the formula

$$B_{\wedge,D_{\wedge}}(\lambda) = B_{\wedge,\text{max}}(\lambda) - \left[ I - B_{\wedge,\text{min}}(\lambda)(A_{\wedge} - \lambda) \right] \pi_{K_{\wedge,\lambda},D} B_{\wedge,\text{max}}(\lambda)$$

in which

$$(A_{\wedge} - \lambda)B_{\wedge,\text{max}}(\lambda) = I, \quad B_{\wedge,\text{min}}(\lambda)(A_{\wedge} - \lambda) = \Pi(\lambda).$$

The property $\kappa_\varrho^{-1} A_{\wedge,\lambda} \kappa_\varrho = \varrho^m A_{\wedge}$ implies

$$\kappa_\varrho^{-1} (A_{\wedge} - \varrho^m \lambda) \kappa_\varrho = \varrho^m (A_{\wedge} - \lambda).$$

This permits choosing $B_{\wedge,\text{min}}(\lambda)$ and $B_{\wedge,\text{max}}(\lambda)$ so that

$$\kappa_\varrho^{-1} B_{\wedge,\text{min}}(\varrho^m \lambda) \kappa_\varrho = \varrho^{-m} B_{\wedge,\text{min}}(\lambda), \quad \kappa_\varrho^{-1} B_{\wedge,\text{max}}(\varrho^m \lambda) \kappa_\varrho = \varrho^{-m} B_{\wedge,\text{max}}(\lambda)$$

(for $\lambda \notin \text{spec } A_{\wedge,D_{\wedge}}$)

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(Temple University) | Zeta functions | Paderborn, June 9, 2011
The link with the asymptotics of $B_{\land,\mathcal{D}_\land}(\lambda)$ is the formula

$$B_{\land,\mathcal{D}_\land}(\lambda) = B_{\land,\max}(\lambda) - \left[I - B_{\land,\min}(\lambda)(A_{\land} - \lambda)\right] \pi_{K_{\land,\lambda},\mathcal{D}} B_{\land,\max}(\lambda)$$

in which

$$(A_{\land} - \lambda)B_{\land,\max}(\lambda) = I, \quad B_{\land,\min}(\lambda)(A_{\land} - \lambda) = \Pi(\lambda).$$

The property $\kappa_q^{-1}A_{\land}\kappa_q = \varrho^m A_{\land}$ implies

$$\kappa_q^{-1}(A_{\land} - \varrho^m \lambda)\kappa_q = \varrho^m (A_{\land} - \lambda).$$

This permits choosing $B_{\land,\min}(\lambda)$ and $B_{\land,\max}(\lambda)$ so that

$$\kappa_q^{-1}B_{\land,\min}(\varrho^m \lambda)\kappa_q = \varrho^{-m} B_{\land,\min}(\lambda), \quad \kappa_q^{-1}B_{\land,\max}(\varrho^m \lambda)\kappa_q = \varrho^{-m} B_{\land,\max}(\lambda)$$

The point is that we can analyze what happens as $|\lambda| \to \infty$ by looking at $B_{\land,\mathcal{D}_\land}(\varrho^m \lambda)$ with $\lambda$ fixed (or in a) and $\varrho \to \infty$. 
The link with the asymptotics of $B_{\land, D_{\land}}(\lambda)$ is the formula

$$B_{\land, D_{\land}}(\lambda) = B_{\land, \text{max}}(\lambda) - \left[ I - B_{\land, \text{min}}(\lambda)(A_{\land} - \lambda) \right] \pi_{K_{\land, \lambda}, D} B_{\land, \text{max}}(\lambda)$$

in which

$$(A_{\land} - \lambda)B_{\land, \text{max}}(\lambda) = I, \quad B_{\land, \text{min}}(\lambda)(A_{\land} - \lambda) = \Pi(\lambda).$$

The property $\kappa_q^{-1}A_{\land}\kappa_q = \varrho^m A_{\land}$ implies

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This permits choosing $B_{\land, \text{min}}(\lambda)$ and $B_{\land, \text{max}}(\lambda)$ so that

$$\kappa_q^{-1}B_{\land, \text{min}}(\varrho^m \lambda)\kappa_q = \varrho^{-m}B_{\land, \text{min}}(\lambda), \quad \kappa_q^{-1}B_{\land, \text{max}}(\varrho^m \lambda)\kappa_q = \varrho^{-m}B_{\land, \text{max}}(\lambda)$$

The point is that we can analyze what happens as $|\lambda| \to \infty$ by looking at $B_{\land, D_{\land}}(\varrho^m \lambda)$ with $\lambda$ fixed (or in $\alpha$) and $\varrho \to \infty$.

All depends on $\pi_{K_{\land, \lambda}, D}$. This is the projection on $K_{\lambda}$ according to $D_{\text{max}} = K_{\lambda} \oplus D_{\land}$.
The link with the asymptotics of $B_{\wedge, D_{\wedge}}(\lambda)$ is the formula

$$B_{\wedge, D_{\wedge}}(\lambda) = B_{\wedge, \text{max}}(\lambda) - \left[ I - B_{\wedge, \text{min}}(\lambda)(A_{\wedge} - \lambda) \right] \pi_{\mathcal{K}_{\wedge, \lambda}, D} B_{\wedge, \text{max}}(\lambda)$$

in which

$$(A_{\wedge} - \lambda)B_{\wedge, \text{max}}(\lambda) = I, \quad B_{\wedge, \text{min}}(\lambda)(A_{\wedge} - \lambda) = \Pi(\lambda).$$

The property $\kappa_{\rho}^{-1} A_{\wedge} \kappa_{\rho} = \rho^m A_{\wedge}$ implies

$$\kappa_{\rho}^{-1} (A_{\wedge} - \rho^m \lambda) \kappa_{\rho} = \rho^m (A_{\wedge} - \lambda).$$

This permits choosing $B_{\wedge, \text{min}}(\lambda)$ and $B_{\wedge, \text{max}}(\lambda)$ so that

$$\kappa_{\rho}^{-1} B_{\wedge, \text{min}}(\rho^m \lambda) \kappa_{\rho} = \rho^{-m} B_{\wedge, \text{min}}(\lambda), \quad \kappa_{\rho}^{-1} B_{\wedge, \text{max}}(\rho^m \lambda) \kappa_{\rho} = \rho^{-m} B_{\wedge, \text{max}}(\lambda).$$

The point is that we can analyze what happens as $|\lambda| \to \infty$ by looking at $B_{\wedge, D_{\wedge}}(\rho^m \lambda)$ with $\lambda$ fixed (or in $a$) and $\rho \to \infty$.

All depends on $\pi_{\mathcal{K}_{\wedge, \lambda}, D}$. This is the projection on $\mathcal{K}_\lambda$ according to $D_{\text{max}} = \mathcal{K}_\lambda \oplus D_{\wedge}$.
The link with the asymptotics of $B_{\wedge,D}(\lambda)$ is the formula

$$B_{\wedge,D}(\lambda) = B_{\wedge,max}(\lambda) - \left[ I - B_{\wedge,min}(\lambda)(A_{\wedge} - \lambda) \right] \pi_{\mathcal{K}_{\wedge,\lambda,D}} B_{\wedge,max}(\lambda)$$

in which

$$(A_{\wedge} - \lambda)B_{\wedge,max}(\lambda) = I, \quad B_{\wedge,min}(\lambda)(A_{\wedge} - \lambda) = \Pi(\lambda).$$

The property $\kappa_q^{-1}A_{\wedge}\kappa_q = \varrho^m A_{\wedge}$ implies

$$\kappa_q^{-1}(A_{\wedge} - \varrho^m \lambda)\kappa_q = \varrho^m(A_{\wedge} - \lambda).$$

This permits choosing $B_{\wedge,min}(\lambda)$ and $B_{\wedge,max}(\lambda)$ so that

$$\kappa_q^{-1}B_{\wedge,min}(\varrho^m \lambda)\kappa_q = \varrho^{-m} B_{\wedge,min}(\lambda), \quad \kappa_q^{-1}B_{\wedge,max}(\varrho^m \lambda)\kappa_q = \varrho^{-m} B_{\wedge,max}(\lambda).$$

The point is that we can analyze what happens as $|\lambda| \to \infty$ by looking at $B_{\wedge,D}(\varrho^m \lambda)$ with $\lambda$ fixed (or in $a$) and $\varrho \to \infty$.

All depends on $\pi_{\mathcal{K}_{\wedge,\lambda,D}}$. This is the projection on $\mathcal{K}_{\lambda}$ according to $D_{\text{max}} = \mathcal{K}_{\lambda} \oplus D_{\wedge}$. 
\[ B_{\Lambda, \text{max}}(q^m \lambda) - \left[ I - B_{\Lambda, \text{min}}(q^m \lambda) (A_{\Lambda} - q^m \lambda) \right] \pi_{\mathcal{K}, q^m \lambda, D_{\Lambda}} B_{\Lambda, \text{max}}(q^m \lambda) \]

The asymptotics and bounds of \( B_{\Lambda, D_{\Lambda}}(q^m \lambda) \) as \( q \to \infty \) are determined by those of \( \pi_{\mathcal{K}, q^m \lambda, D_{\Lambda}} \):

\[ \]
The asymptotics and bounds of $B_{\wedge,\max}(\rho^m \lambda)$ as $\rho \to \infty$ are determined by those of $\pi_{\mathcal{K}_\wedge,\rho^m \lambda,\mathcal{D}_\wedge}$: Using $\kappa$-homogeneity,

$$B_{\wedge,\mathcal{D}_\wedge}(\rho^m \lambda) = \rho^{-m} \kappa_\rho B_{\wedge,\max}(\lambda) \kappa_\rho^{-1}$$

$$+ [I - \rho^{-m} \kappa_\rho B_{\wedge,\min}(\lambda) \kappa_\rho^{-1} \rho^m \kappa_\rho (A_\wedge - \lambda) \kappa_\rho^{-1}]$$

$$\pi_{\mathcal{K}_\wedge,\rho^m \lambda,\mathcal{D}_\wedge} \rho^{-m} \kappa_\rho B_{\wedge,\max}(\lambda) \kappa_\rho^{-1}$$
The asymptotics and bounds of $B_{\Lambda, D_{\Lambda}}(\varrho^m \lambda)$ as $\varrho \to \infty$ are determined by those of $\pi_{\mathcal{K}_{\Lambda, \varrho^m \lambda, D_{\Lambda}}}$: Using $\kappa$-homogeneity,

\[
B_{\Lambda, D_{\Lambda}}(\varrho^m \lambda) = \varrho^{-m} \kappa_{\varrho} B_{\Lambda, \max}(\lambda) \kappa_{\varrho}^{-1} \\
- \left[ I - \varrho^{-m} \kappa_{\varrho} B_{\Lambda, \min}(\lambda) \kappa_{\varrho}^{-1} \varrho^m \kappa_{\varrho} (A_{\Lambda} - \lambda) \kappa_{\varrho}^{-1} \right]
\]

\[
\pi_{\mathcal{K}_{\Lambda, \varrho^m \lambda, D_{\Lambda}}} \varrho^{-m} \kappa_{\varrho} B_{\Lambda, \max}(\lambda) \kappa_{\varrho}^{-1}
\]
The asymptotics and bounds of $B_{\wedge,D}(q^m \lambda)$ as $q \to \infty$ are determined by those of $\pi_{K,\wedge,q^m \lambda,D}$: Using $\kappa$-homogeneity,

$$B_{\wedge,D}(q^m \lambda) = q^{-m} \kappa_q B_{\wedge,max}(\lambda) \kappa_q^{-1}$$

$$- [I - q^{-m} \kappa_q B_{\wedge,min}(\lambda) \kappa_q^{-1} q^m \kappa_q (A_{\wedge} - \lambda) \kappa_q^{-1}]$$

$$\pi_{K,\wedge,q^m \lambda,D} q^{-m} \kappa_q B_{\wedge,max}(\lambda) \kappa_q^{-1}$$
\[ B_{\wedge, \text{max}}(q^m \lambda) = [I - B_{\wedge, \text{min}}(q^m \lambda)(A_{\wedge} - q^m \lambda)] \pi_{\kappa_{\wedge}, q^m \lambda, D_{\wedge}} B_{\wedge, \text{max}}(q^m \lambda) \]

The asymptotics and bounds of \( B_{\wedge, D_{\wedge}}(q^m \lambda) \) as \( q \rightarrow \infty \) are determined by those of \( \pi_{\kappa_{\wedge}, q^m \lambda, D_{\wedge}} \): Using \( \kappa \)-homogeneity,

\[
\begin{align*}
B_{\wedge, D_{\wedge}}(q^m \lambda) &= q^{-m} \kappa_q B_{\wedge, \text{max}}(\lambda) \kappa_q^{-1} \\
&- [I - q^{-m} \kappa_q B_{\wedge, \text{min}}(\lambda) \kappa_q^{-1} q^m \kappa_q (A_{\wedge} - \lambda) \kappa_q^{-1}]
\end{align*}
\]

\[
\pi_{\kappa_{\wedge}, q^m \lambda, D_{\wedge}} q^{-m} \kappa_q B_{\wedge, \text{max}}(\lambda) \kappa_q^{-1}
\]
\[ B_{\lambda, \max}(q^{m} \lambda) - [I - B_{\lambda, \min}(q^{m} \lambda)(A_{\lambda} - q^{m} \lambda)] \pi_{K_{\lambda}, q^{m} \lambda, D_{\lambda}} B_{\lambda, \max}(q^{m} \lambda) \]

The asymptotics and bounds of \( B_{\lambda, D_{\lambda}}(q^{m} \lambda) \) as \( q \to \infty \) are determined by those of \( \pi_{K_{\lambda}, q^{m} \lambda, D_{\lambda}} \): Using \( \kappa \)-homogeneity,

\[
B_{\lambda, D_{\lambda}}(q^{m} \lambda) = q^{-m} \kappa_{q} B_{\lambda, \max}(\lambda) \kappa_{q}^{-1} \\
- [I - q^{-m} \kappa_{q} B_{\lambda, \min}(\lambda) \kappa_{q}^{-1} (A_{\lambda} - \lambda) \kappa_{q}^{-1}] \\
\pi_{K_{\lambda}, q^{m} \lambda, D_{\lambda}} q^{-m} \kappa_{q} B_{\lambda, \max}(\lambda) \kappa_{q}^{-1} \\
= q^{-m} \kappa_{q} \{ \]

\( \kappa_{q}^{-1}(A_{\lambda} - q^{m} \lambda) \kappa_{q} = q^{m}(A_{\lambda} - \lambda) \) \\
\( \kappa_{q}^{-1} B_{\lambda, \cdot}(q^{m} \lambda) \kappa_{q} = q^{-m} B_{\lambda, \cdot}(\lambda) \)
The asymptotics and bounds of $B_{\hat{\omega}, \mathcal{D}}(q^m \lambda)$ as $q \to \infty$ are determined by those of $\pi_{K_{\hat{\omega}}, q^m \lambda, \mathcal{D}}$: Using $\kappa$-homogeneity,

\[
B_{\hat{\omega}, \mathcal{D}}(q^m \lambda) = q^{-m} \kappa q B_{\hat{\omega}, \max(\lambda)} \kappa q^{-1} \\
- \left[ I - q^{-m} \kappa q B_{\hat{\omega}, \min(\lambda)} \kappa q^{-1} q^m \kappa q(A_{\hat{\omega}} - \lambda) \kappa q^{-1} \right] \\
\pi_{K_{\hat{\omega}}, q^m \lambda, \mathcal{D}} q^{-m} \kappa q B_{\hat{\omega}, \max(\lambda)} \kappa q^{-1} \\
= q^{-m} \kappa q \{ \} 
\]
The asymptotics and bounds of $B_{\lambda, \max}(\varrho^m \lambda)$ as $\varrho \to \infty$ are determined by those of $\pi_{K, \varrho^m \lambda, D}$: Using $\kappa$-homogeneity,

$$B_{\lambda, D}(\varrho^m \lambda) = \varrho^{-m} \kappa_\varrho B_{\lambda, \max}(\lambda) \kappa_\varrho^{-1}$$

$$- \left[ I - \varrho^{-m} \kappa_\varrho B_{\lambda, \min}(\lambda) \kappa_\varrho^{-1} \right] \varrho^m \kappa_\varrho (A_{\lambda} - \lambda) \kappa_\varrho^{-1}$$

$$= \varrho^{-m} \kappa_\varrho \{ \}$$
\[ B_{\wedge, \max}(q^m \lambda) - \left[ I - B_{\wedge, \min}(q^m \lambda)(A_{\wedge} - q^m \lambda) \right] \pi_{K_{\wedge}, q^m \lambda, D_{\wedge}} B_{\wedge, \max}(q^m \lambda) \]

The asymptotics and bounds of \( B_{\wedge, D}(q^m \lambda) \) as \( q \to \infty \) are determined by those of \( \pi_{K_{\wedge}, q^m \lambda, D_{\wedge}} \). Using \( \kappa \)-homogeneity,

\[ B_{\wedge, D}(q^m \lambda) = q^{-m} \kappa_q B_{\wedge, \max}(\lambda) \kappa_q^{-1} \]

\[ - \left[ I - q^{-m} \kappa_q B_{\wedge, \min}(\lambda) \kappa_q^{-1} q^m \kappa_q(A_{\wedge} - \lambda) \kappa_q^{-1} \right] \]

\[ \pi_{K_{\wedge}, q^m \lambda, D_{\wedge}} q^{-m} \kappa_q B_{\wedge, \max}(\lambda) \kappa_q^{-1} \]

\[ = q^{-m} \kappa_q \{ \]

\[ \} \kappa_q^{-1} \]
\[
B_{\wedge, \text{max}}(q^m \lambda) - [I - B_{\wedge, \text{min}}(q^m \lambda)(A_{\wedge} - q^m \lambda)] \pi_{\mathcal{K}, q^m \lambda, D} B_{\wedge, \text{max}}(q^m \lambda)
\]

The asymptotics and bounds of \( B_{\wedge, D}(q^m \lambda) \) as \( q \to \infty \) are determined by those of \( \pi_{\mathcal{K}, q^m \lambda, D} \): Using \( \kappa \)-homogeneity,

\[
B_{\wedge, D}(q^m \lambda) = q^{-m} \kappa_q B_{\wedge, \text{max}}(\lambda) \kappa_q^{-1}
\]

\[
- [I - q^{-m} \kappa_q B_{\wedge, \text{min}}(\lambda \kappa_q^{-1} q^m \kappa_q^{-1} \kappa_q^{-1} A_{\wedge} - \lambda \kappa_q^{-1} q^m \kappa_q^{-1} \kappa_q^{-1} I] \pi_{\mathcal{K}, q^m \lambda, D} q^{-m} \kappa_q B_{\wedge, \text{max}}(\lambda) \kappa_q^{-1}
\]

\[
= q^{-m} \kappa_q \{ B_{\wedge, \text{max}}(\lambda) 
\]

\[
- [I - B_{\wedge, \text{min}}(\lambda)(A_{\wedge} - \lambda)] \kappa_q^{-1} \pi_{\mathcal{K}, q^m \lambda, D} \kappa_q B_{\wedge, \text{max}}(\lambda) \kappa_q^{-1}
\]
The asymptotics and bounds of $B_{\Lambda, D}(q^m \lambda)$ as $q \to \infty$ are determined by those of $\pi_{K, \epsilon m, D}$: 

Using $\kappa$-homogeneity, 

$$B_{\Lambda, D}(q^m \lambda) = q^{-m} \kappa_q B_{\Lambda, \text{max}}(\lambda) \kappa_q^{-1}$$ 

$$- [I - q^{-m} \kappa_q B_{\Lambda, \text{min}}(\lambda) \kappa_q^{-1} q^m \kappa_q (A_{\Lambda} - \lambda) \kappa_q^{-1}]$$ 

$$\pi_{K, \epsilon m, D} q^{-m} \kappa_q B_{\Lambda, \text{max}}(\lambda) \kappa_q^{-1}$$ 

$$= q^{-m} \kappa_q \{B_{\Lambda, \text{max}}(\lambda)$$ 

$$- [I - B_{\Lambda, \text{min}}(\lambda) (A_{\Lambda} - \lambda)] \kappa_q^{-1} \pi_{K, \epsilon m, D} \kappa_q B_{\Lambda, \text{max}}(\lambda) \} \kappa_q^{-1}$$ 

Further analysis shows that this can be replaced by 

$$\pi_{K, \epsilon q^{-1}, D} \pi_{\Lambda, \text{max}}$$ 

$$\kappa_q = \pi_{\Lambda, \text{max}} K_{\epsilon}$$ 

$$D = D \oplus D_{\Lambda, \text{min}}$$
The asymptotics and bounds of $B_{\Lambda,m}(\rho m \lambda)$ as $\rho \to \infty$ are determined by those of $\pi_{\mathcal{K}_{\Lambda,m},\mathcal{D}_{\Lambda}}$: Using $\kappa$-homogeneity,

$$B_{\Lambda,m}(\rho m \lambda) = \rho^{-m} \kappa_{\rho} B_{\Lambda,\max}(\lambda) \kappa_{\rho}^{-1}$$

$$- \left[ I - \rho^{-m} \kappa_{\rho} B_{\Lambda,\min}(\lambda) \kappa_{\rho}^{-1} \right] \rho^{-m} \kappa_{\rho} (A_{\Lambda} - \lambda) \kappa_{\rho}^{-1}$$

$$= \rho^{-m} \kappa_{\rho} \left\{ B_{\Lambda,\max}(\lambda) - \left[ I - B_{\Lambda,\min}(\lambda) (A_{\Lambda} - \lambda) \right] \kappa_{\rho}^{-1} \pi_{\mathcal{K}_{\Lambda,m},\mathcal{D}_{\Lambda}} \kappa_{\rho} B_{\Lambda,\max}(\lambda) \right\} \kappa_{\rho}^{-1}$$

Further analysis shows that this can be replaced by using $\ker \left[ I - B_{\Lambda,\min}(\lambda) (A_{\Lambda} - \lambda) \right] \subset \mathcal{D}_{\Lambda,\min}$

$$K_{\lambda} = \pi_{\Lambda,\max} \mathcal{K}_{\lambda}$$

$$\mathcal{D}_{\Lambda} = \mathcal{D}_{\Lambda,\max} \oplus \mathcal{D}_{\Lambda,\min}$$
The asymptotics and bounds of $B_{\lambda,D}(\varrho^{m}\lambda)$ as $\varrho \to \infty$ are determined by those of $\pi_{K_{\lambda},\varrho^{m}\lambda,D}$: Using $\kappa$-homogeneity,

\[
B_{\lambda,D}(\varrho^{m}\lambda) = \varrho^{-m}\kappa_{\varrho}B_{\lambda,\max}(\lambda)\kappa_{\varrho}^{-1} - [I - \varrho^{-m}\kappa_{\varrho}B_{\lambda,\min}(\lambda)] \kappa_{\varrho}^{-1}(A_{\lambda} - \varrho^{m}\lambda)\kappa_{\varrho}^{-1}]
\]

\[
\pi_{K_{\lambda},\varrho^{m}\lambda,D}\varrho^{-m}\kappa_{\varrho}B_{\lambda,\max}(\lambda)\kappa_{\varrho}^{-1}
\]

Further analysis shows that this can be replaced by

using $\ker [I - B_{\lambda,\min}(\lambda)(A_{\lambda} - \lambda)] \subset D_{\lambda,\min}$

\[
\pi_{K_{\lambda},\kappa_{\varrho}^{-1}D_{\lambda}}\pi_{\lambda,\max}.
\]

So

\[
B_{\lambda,D}(\varrho^{m}\lambda) = \varrho^{-m}\kappa_{\varrho}\{B_{\lambda,\max}(\lambda) - [I - B_{\lambda,\min}(\lambda)(A_{\lambda} - \lambda)] \pi_{K_{\lambda},\kappa_{\varrho}^{-1}D_{\lambda}}\pi_{\lambda,\max}B_{\lambda,\max}(\lambda)\} \kappa_{\varrho}^{-1}.
\]
The asymptotics and bounds of $B_{\land,\mathcal{D}}(q^m \lambda)$ as $q \to \infty$ are determined by those of $\pi_{\mathcal{K}_{\land},q^m\lambda,\mathcal{D}_{\land}}$: Using $\kappa$-homogeneity,

\[
B_{\land,\mathcal{D}}(q^m \lambda) = q^{-m} \kappa^{-1}_q B_{\land,\max}(\lambda) \kappa^{-1}_q \\
- \left[ I - q^{-m} \kappa^{-1}_q B_{\land,\min}(\lambda) \right] \kappa^{-1}_q (A_{\land} - \lambda) \kappa^{-1}_q \\
= q^{-m} \kappa^{-1}_q \left\{ B_{\land,\max}(\lambda) - \left[ I - B_{\land,\min}(\lambda) (A_{\land} - \lambda) \right] \kappa^{-1}_q \pi_{\mathcal{K}_{\land},q^m\lambda,\mathcal{D}_{\land}} \kappa^{-1}_q B_{\land,\max}(\lambda) \right\} \kappa^{-1}_q
\]

Further analysis shows that this can be replaced by

\[
\pi_{\mathcal{K}_{\land},q^m\lambda,\mathcal{D}_{\land}} q^{-m} \kappa^{-1}_q B_{\land,\max}(\lambda) \kappa^{-1}_q
\]

using ker $[I - B_{\land,\min}(\lambda)(A_{\land} - \lambda)] \subset \mathcal{D}_{\land,\min}$

\[
\pi_{\mathcal{K}_{\land},q^m\lambda,\mathcal{D}_{\land}} q^{-m} \kappa^{-1}_q \pi_{\land,\max} \cdot \kappa^{-1}_q
\]

So

\[
B_{\land,\mathcal{D}}(q^m \lambda) = q^{-m} \kappa^{-1}_q \left\{ B_{\land,\max}(\lambda) - \left[ I - B_{\land,\min}(\lambda) (A_{\land} - \lambda) \right] \pi_{\mathcal{K}_{\land},q^m\lambda,\mathcal{D}_{\land}} \pi_{\land,\max} B_{\land,\max}(\lambda) \right\} \kappa^{-1}_q \cdot \kappa^{-1}_q
\]

After taking $\| \cdot \|$ or trace, the conjugation with $\kappa^{-1}_q$ disappears.
With \( \lambda \) in the arc \( \alpha \subset \Lambda \setminus 0 \),

\[
B_{\Lambda, D}(\varrho^m \lambda) = \varrho^{-m} \kappa_{\varrho} \left\{ B_{\Lambda, \text{max}}(\lambda) - \left[ I - B_{\Lambda, \text{min}}(\lambda)(A_{\Lambda} - \lambda) \right] \pi_{K_{\lambda}, \kappa_{\varrho}^{-1} D_{\Lambda} \pi_{\Lambda, \text{max}} B_{\Lambda, \text{max}}(\lambda) } \right\} \kappa_{\varrho}^{-1}
\]

gives \( 1/|\varrho^m| \) decay of \( \| B_{\Lambda, D}(\varrho^m \lambda) \| \) as \( \varrho \to \infty \).
With $\lambda$ in the arc $\alpha \subset \Lambda \setminus 0$,

$$B_{\Lambda, D_{\Lambda}}(\varrho^m \lambda) = \varrho^{-m} \kappa_{\varrho} \left\{ B_{\Lambda, \max}(\lambda) - \left[ I - B_{\Lambda, \min}(\lambda)(A_{\Lambda} - \lambda) \right] \pi K_{\lambda, \kappa_{\varrho}^{-1}} D_{\Lambda} \pi_{\Lambda, \max} B_{\Lambda, \max}(\lambda) \right\} \kappa_{\varrho}^{-1}$$

gives $1/|\varrho^m|$ decay of $\|B_{\Lambda, D_{\Lambda}}(\varrho^m \lambda)\|$ as $\varrho \to \infty$

(keep $K_{\lambda} \cap \kappa_{\varrho}^{-1} D_{\Lambda} = 0$ in a uniform way)
With $\lambda$ in the arc $\alpha \subset \Lambda \setminus 0$,

$$B_{\Lambda,D}(q^m \lambda) = \frac{q^{-m} \kappa_q \{ B_{\Lambda,\max}(\lambda) - [I - B_{\Lambda,\min}(\lambda)(A_{\Lambda} - \lambda)] \pi_{K_{\lambda},\kappa_q^{-1} D_{\Lambda}} \pi_{\Lambda,\max} B_{\Lambda,\max}(\lambda) \} \kappa_q^{-1}}{\lambda}$$

gives $1/|q^m|$ decay of $\|B_{\Lambda,D}(q^m \lambda)\|$ as $q \to \infty$ if

$$\Omega^-(D_{\Lambda}) = \{ D_{\Lambda}' \in \text{Gr}_k(\mathcal{E}) : \exists \{ \theta_\nu \} \text{ s.t. } \theta_\nu \to \infty, \ k_{\nu}^{-1} D_{\Lambda} \to D_{\Lambda}' \}$$

is disjoint from

$$\mathcal{V} = \bigcup_{\lambda \in \alpha} \mathcal{V}_{K_{\lambda}}.$$
With $\lambda$ in the arc $\alpha \subset \Lambda \setminus 0$,

$$B_{\Lambda, D_{\Lambda}}(q^m \lambda) = q^{-m} \kappa_{q} \{ B_{\Lambda, \max}(\lambda) - [I - B_{\Lambda, \min}(\lambda)(A_{\Lambda} - \lambda)] \pi_{K_{\lambda}, \kappa_{q}^{-1} D_{\Lambda}} \pi_{\Lambda, \max} B_{\Lambda, \max}(\lambda) \} \kappa_{q}^{-1}$$

gives $1/|q^m|$ decay of $\|B_{\Lambda, D_{\Lambda}}(q^m \lambda)\|$ as $q \to \infty$ if

$$\Omega^{-}(D_{\Lambda}) = \{ D'_{\Lambda} \in \text{Gr}_k(\mathcal{E}) : \exists \{ q_{\nu} \} \text{ s.t. } q_{\nu} \to \infty, \ k_{q_{\nu}}^{-1} D_{\Lambda} \to D'_{\Lambda} \}$$

is disjoint from

$$\mathcal{V} = \bigcup_{\lambda \in \alpha} \mathcal{V}_{K_{\lambda}}.$$
With $\lambda$ in the arc $a \subset \Lambda \setminus 0$,

\[
B_{\Lambda, D_{\Lambda}}(\varrho^m \lambda) = \varrho^{-m} \kappa_{\varrho} \left\{ B_{\Lambda, \max}(\lambda) - \left[ I - B_{\Lambda, \min}(\lambda)(A_{\Lambda} - \lambda) \right] \pi_{K_{\lambda}, \kappa_{\varrho}^{-1}} D_{\Lambda} \pi_{\Lambda, \max} B_{\Lambda, \max}(\lambda) \right\}^{\kappa_{\varrho}^{-1}}
\]

gives $1/|\varrho^m|$ decay of $\|B_{\Lambda, D_{\Lambda}}(\varrho^m \lambda)\|$ as $\varrho \to \infty$ if

\[
\Omega^{-}(D_{\Lambda}) = \{ D'_{\Lambda} \in \text{Gr}_{k}(\mathcal{E}) : \exists \{ \varrho_{\nu} \} \text{ s.t. } \varrho_{\nu} \to \infty, \ k_{\varrho_{\nu}} D_{\Lambda} \to D'_{\Lambda} \}
\]

is disjoint from

\[
\mathcal{V} = \bigcup_{\lambda \in a} \mathcal{V}_{K_{\lambda}}.
\]

Conversely, decay implies $\Omega^{-}(D_{\Lambda}) \cap \mathcal{V} = \emptyset$.

This requires some work!
**Theorem**

For any $\ell \in \mathbb{N}$ with $m\ell > n$,

$$
\text{Tr}((A_D - \lambda)^{-\ell}) = \sum_{j=0}^{n-1} \alpha_j \lambda^{\frac{n-\ell m-j}{m}} + \alpha_n \log(\lambda)\lambda^{-\ell} + s_D(\lambda)
$$
Theorem

For any $\ell \in \mathbb{N}$ with $m\ell > n$,

$$\text{Tr}((A_D - \lambda)^{-\ell}) = \sum_{j=0}^{n-1} \alpha_j \lambda^{\frac{n-\ell m-j}{m}} + \alpha_n \log(\lambda) \lambda^{-\ell} + s_D(\lambda)$$

with

$$s_D(\lambda) \sim \sum_{\nu \in \mathcal{E}} r_{\nu}(\lambda^{i\mu_1/m}, \ldots, \lambda^{i\mu_N/m}, \log \lambda) \lambda^{\nu/m} \text{ as } |\lambda| \to \infty.$$
Theorem

For any \( \ell \in \mathbb{N} \) with \( ml > n \),

\[
\text{Tr}((A_D - \lambda)^{-\ell}) = \sum_{j=0}^{n-1} \alpha_j \lambda^{\frac{n-\ell m-j}{m}} + \alpha_n \log(\lambda) \lambda^{-\ell} + s_D(\lambda)
\]

with

\[
s_D(\lambda) \sim \sum_{\nu \in \mathcal{E}} r_\nu(\lambda^{i\mu_1/m}, \ldots, \lambda^{i\mu_N/m}, \log \lambda) \lambda^{\nu/m} \quad \text{as } |\lambda| \to \infty.
\]

where

\[\nu \leq -\ell m\]

\[\mathcal{E} = \{ \mu : \mu \in \text{spec}_b A, -m/2 < \mu < m/2 \}\]
Theorem

For any \( \ell \in \mathbb{N} \) with \( m\ell > n \),

\[
\text{Tr}((A_D - \lambda)^{-\ell}) = \sum_{j=0}^{n-1} \alpha_j \lambda^{\frac{n-\ell m-j}{m}} + \alpha_n \log(\lambda) \lambda^{-\ell} + s_D(\lambda)
\]

with

\[
s_D(\lambda) \sim \sum_{\nu \in \mathcal{E}} r_{\nu}(\lambda^{i\mu_1/m}, \ldots, \lambda^{i\mu_N/m}, \log \lambda) \lambda^{\nu/m} \text{ as } |\lambda| \to \infty.
\]

\( \mathcal{E} = \text{additive semigroup generated by } (-\mathbb{N}_0) \text{ and } \{ \Im(\sigma - \sigma') : \sigma, \sigma' \in \text{spec}_b(A), -m/2 < \Im \sigma \leq \Im \sigma' < m/2 \} \)
Theorem

For any \( \ell \in \mathbb{N} \) with \( m\ell > n \),

\[
\text{Tr}((A_D - \lambda)^{-\ell}) = \sum_{j=0}^{n-1} \alpha_j \lambda^{n-\ell m-j} + \alpha_n \log(\lambda)\lambda^{-\ell} + s_D(\lambda)
\]

with

\[
s_D(\lambda) \sim \sum_{\nu \in \mathcal{E}} r_{\nu}(\lambda^{i\mu_1/m}, \ldots, \lambda^{i\mu_N/m}, \log \lambda)\lambda^{\nu/m} \quad \text{as } |\lambda| \to \infty.
\]

\( \mathcal{E} = \text{additive semigroup generated by } (-N_0) \) and
\( \nu \leq -\ell m \)

Each \( r_{\nu} \) is a rational function in \( N + 1 \) variables. We have
\( r_{\nu} = p_{\nu}/q_{\nu} \) with \( p_{\nu}, q_{\nu} \in \mathbb{C}[z_1, \ldots, z_{N+1}] \) such that
\( q_{\nu}(\lambda^{i\mu_1}, \ldots, \lambda^{i\mu_N}, \log \lambda) \) is uniformly bounded away from zero for large \( \lambda \in \Lambda \).
Assume $\text{spec}_b(A) \cap \{-m/2 < \Re(\sigma) < m/2\}$ is vertically aligned. Then

$$s_D(\lambda) \sim \sum_{\nu \in \mathcal{E}} r_\nu(\log(\lambda))\lambda^{\nu/m},$$

i.e., the $r_\nu$ are rational functions of $\log(\lambda)$ only.
Special Cases

\[ \text{Tr} \varphi(A_D - \lambda)^{-\ell} \sim \sum_{j=0}^{n-1} c_j \lambda^{n-j-\ell} + c_n \lambda^{-\ell} \log(\lambda) + s_D(\lambda) \]

Assume \( \text{spec}_b(A) \cap \{-m/2 < \Re(\sigma) < m/2\} \) is vertically aligned. Then

\[ s_D(\lambda) \sim \sum_{\nu \in \mathfrak{c}} r_\nu(\log(\lambda)) \lambda^{\nu/m}, \]

i.e., the \( r_\nu \) are rational functions of \( \log(\lambda) \) only.

For example, let \( A \) be of Laplace type, near \( \mathcal{Y} \) of the form\(^{15,16}\)

\[ A = D_x^2 + \frac{B(x)}{x^2} = x^{-2}((xD_x)^2 + i(xD_x) + B(x)), \]

\( B(x) \in C^\infty([0, \varepsilon), \text{Diff}^2(\mathcal{Y})) \) semibounded, of Laplace type, \( B(0) \geq -\frac{1}{4} \).

---


Special Cases

\[ \text{Tr} \varphi(A_D - \lambda)^{-\ell} \sim \sum_{j=0}^{n-1} c_j \lambda^{\frac{n-j}{m} - \ell} + c_n \lambda^{-\ell} \log(\lambda) + s_D(\lambda) \]

Assume \( \text{spec}_b(A) \cap \{-m/2 < \Re(\sigma) < m/2\} \) is vertically aligned. Then

\[ s_D(\lambda) \sim \sum_{\nu \in \mathcal{E}} r_\nu(\log(\lambda)) \lambda^{\nu/m}, \]

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For example, let \( A \) be of Laplace type, near \( \mathcal{Y} \) of the form\(^{15,16}\)

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\( B(x) \in C^\infty([0, \varepsilon), \text{Diff}^2(\mathcal{Y})) \) semibounded, of Laplace type, \( B(0) \geq -\frac{1}{4} \).

Then \( \hat{A}(\sigma) = \sigma^2 + i\sigma + B(0) \), and

\[ \text{spec}_b(A) = \{ \sigma : \sigma^2 + i\sigma + \lambda = 0, \lambda \in \text{spec}(B(0)) \} \]


\(^{16}\) —, —, *The expansion of the resolvent near a singular stratum of conical type*, J. Funct. Anal. 95 (1991), no. 2, 255–290
Special Cases

\[ \text{Tr } \varphi (A_D - \lambda)^{-\ell} \sim \sum_{j=0}^{n-1} c_j \lambda^{n-j} - \ell + c_n \lambda^{-\ell} \log(\lambda) + s_D(\lambda) \]

Assume \( \text{spec}_b(A) \cap \{-m/2 < \Im(\sigma) < m/2\} \) is vertically aligned. Then

\[ s_D(\lambda) \sim \sum_{\nu \in \mathcal{E}, \nu \leq -m\ell} r_{\nu}(\log(\lambda)) \lambda^{\nu/m}, \]

i.e., the \( r_{\nu} \) are rational functions of \( \log(\lambda) \) only.

For example, let \( A \) be of Laplace type, near \( \mathcal{Y} \) of the form\(^{15,16}\)

\[ A = D_x^2 + \frac{B(x)}{x^2} = x^{-2} ((xD_x)^2 + i(xD_x) + B(x)), \]

\( B(x) \in C^\infty([0, \varepsilon), \text{Diff}^2(\mathcal{Y})) \) semibounded, of Laplace type, \( B(0) \geq -\frac{1}{4} \).

Then \( \hat{A}(\sigma) = \sigma^2 + i\sigma + B(0) \), and

\[ \text{spec}_b(A) = \{ \sigma : \sigma^2 + i\sigma + \lambda = 0, \lambda \in \text{spec}(B(0)) \} \subset i\mathbb{R}. \]


\(^{16}\) ——, ——, *The expansion of the resolvent near a singular stratum of conical type*, J. Funct. Anal. 95 (1991), no. 2, 255–290
Special Cases

\[
\text{Tr} \varphi (A_D - \lambda)^{-\ell} \sim \sum_{j=0}^{n-1} c_j \lambda^{n-j-\ell} + c_n \lambda^{-\ell} \log(\lambda) + s_D(\lambda)
\]

Assume \( \text{spec}_b(A) \cap \{ -m/2 < \Im(\sigma) < m/2 \} \) is vertically aligned. Then

\[
s_D(\lambda) \sim \sum_{\nu \in \mathcal{E}} \sum_{\nu \leq -m \ell} r_\nu (\log(\lambda)) \lambda^{\nu/m},
\]

i.e., the \( r_\nu \) are rational functions of \( \log(\lambda) \) only.

For example, let \( A \) be of Laplace type, near \( \mathcal{Y} \) of the form\(^{15,16}\)

\[
A = D_x^2 + \frac{B(x)}{x^2} = x^{-2}((xD_x)^2 + i(xD_x) + B(x)),
\]

\( B(x) \in C^\infty([0, \varepsilon), \text{Diff}^2(\mathcal{Y})) \) semibounded, of Laplace type, \( B(0) \geq -\frac{1}{4} \).

Then \( \hat{A}(\sigma) = \sigma^2 + i\sigma + B(0) \), and

\[
\text{spec}_b(A) = \{ \sigma : \sigma^2 + i\sigma + \lambda = 0, \lambda \in \text{spec}(B(0)) \} \subset i\mathbb{R}.
\]


\(^{16}\) —, —, The expansion of the resolvent near a singular stratum of conical type, J. Funct. Anal. 95 (1991), no. 2, 255–290
Special Cases, cont.

If $\mathcal{D}$ is stationary, then

$$\text{Tr} \varphi(A_{\mathcal{D}} - \lambda)^{-\ell} \sim \sum_{j=0}^{n-1} c_j \lambda^{\frac{n-j}{m} - \ell} + c_n \lambda^{-\ell} \log(\lambda) + s_{\mathcal{D}}(\lambda)$$

$$s_{\mathcal{D}}(\lambda) \sim \sum_{\nu \in \mathcal{E} \cap (-\mathbb{N}_0)} p_{\nu}(\log(\lambda)) \lambda^{\nu/m}$$

with polynomials $p_{\nu}$.


If $\mathcal{D}$ is stationary, then

$$s_{\mathcal{D}}(\lambda) \sim \sum_{\nu \in \mathcal{E} \cap (-\mathbb{N}_0) \atop \nu \leq -m \ell} p_{\nu}(\log(\lambda)) \lambda^{\nu/m}$$

with polynomials $p_{\nu}$. 

$$\mathcal{D} = \mathcal{D} + \mathcal{D}_{\text{min}} \leftarrow \theta \rightarrow \mathcal{D}_\wedge = \mathcal{D}_\wedge + \mathcal{D}_{\wedge,\text{min}}$$

---


Special Cases, cont.

If $\mathcal{D}$ is stationary, then

$$s_\mathcal{D}(\lambda) \sim \sum_{\nu \in \mathcal{E} \cap (-\mathbb{N}_0)} p_\nu(\log(\lambda))\lambda^{\nu/m}$$

with polynomials $p_\nu$.

$$\text{Tr} \varphi(A \mathcal{D} - \lambda)^{-\ell} \sim \sum_{j=0}^{n-1} c_j \lambda^{\frac{n-j}{m} - \ell} + c_n \lambda^{-\ell} \log(\lambda) + s_\mathcal{D}(\lambda)$$

$$\mathcal{D} = \mathcal{D} + \mathcal{D}_{\text{min}} \iff \mathcal{D}_\wedge = \mathcal{D}_\wedge + \mathcal{D}_\wedge,\text{min}$$

$\forall \varrho : \kappa_\varrho \mathcal{D}_\wedge = \mathcal{D}_\wedge$

---


Special Cases, cont.

If $\mathcal{D}$ is stationary, then

\[
\text{Tr} \varphi(A_{\mathcal{D}} - \lambda)^{-\ell} \sim \sum_{j=0}^{n-1} c_j \lambda^{n-j-\ell} + c_n \lambda^{-\ell} \log(\lambda) + s_{\mathcal{D}}(\lambda)
\]

\[
s_{\mathcal{D}}(\lambda) \sim \sum_{\nu \in \mathcal{E} \cap \mathbb{N}_0, \nu \leq -m\ell} p_\nu(\log(\lambda)) \lambda^{\nu/m}
\]

with polynomials $p_\nu$.

Notably, the domain of the Friedrichs extension of any elliptic semibounded cone operator $A$ is stationary\textsuperscript{17}.

\textsuperscript{17} J. B. Gil, \textit{—, Adjoins of elliptic cone operators}, Amer. J. Math. 125 (2003), 357–408.

\textsuperscript{18} P. Loya, \textit{The structure of the resolvent of elliptic pseudodifferential operators}, J. Funct. Anal. 184 (2001), no. 1, 77–135

Special Cases, cont.

If $\mathcal{D}$ is stationary, then

$$s_\mathcal{D}(\lambda) \sim \sum_{\nu \in \mathcal{E} \cap (-N_0)} \sum_{\nu \leq -m \ell} p_\nu (\log(\lambda)) \lambda^{\nu/m}$$

with polynomials $p_\nu$.

Notably, the domain of the Friedrichs extension of any elliptic semibounded cone operator $A$ is stationary\(^{17}\).

If $\mathcal{D}$ is stationary and $A$ has constant coefficients near the boundary, then all $p_\nu$ are constants.


\(^{19}\) J. Gil, Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators, Math. Nachr. 250 (2003), 25–57
Special Cases, cont.

If $\mathcal{D}$ is stationary, then

$$s_\mathcal{D}(\lambda) \sim \sum_{\nu \in \mathfrak{C} \cap (-N_0)} \sum_{\nu \leq -m\ell} p_\nu(\log(\lambda)) \lambda^{\nu/m}$$

with polynomials $p_\nu$.

If $\mathcal{D}$ is stationary and $A$ has constant coefficients near the boundary, then all $p_\nu$ are constants.

If $\mathcal{D} = \mathcal{D}_{\text{min}}$, then $\deg p_\nu \leq 1$ for all $\nu$.\textsuperscript{18, 19}

\textsuperscript{17} J. B. Gil, —, Adjoinths of elliptic cone operators, Amer. J. Math. 125 (2003), 357–408.


\textsuperscript{19} J. Gil, Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators, Math. Nachr. 250 (2003), 25–57
End