Elliptic operators on manifolds with conical singularities, I

Gerardo A. Mendoza

Temple University

Paderborn, May 19, 2011
Cylindrical ends

For the purposes of this talk, a manifold with cylindrical ends is a smooth manifold $\mathcal{M}$ in which there is a compact submanifold $\mathcal{K}$ with smooth boundary such that $\mathcal{M} \setminus \mathcal{K}$ is diffeomorphic to a disjoint union of cylinders

$$\mathcal{M} \setminus \mathcal{K} \approx \bigsqcup_j (-\infty, 0) \times \mathcal{N}_j$$

where the $\mathcal{N}_j$ are the components of $\partial \mathcal{K}$. 
Cylindrical ends

For the purposes of this talk, a manifold with cylindrical ends is a smooth manifold $\mathcal{M}$ in which there is a compact submanifold $\mathcal{K}$ with smooth boundary such that $\mathcal{M}\setminus\mathcal{K}$ is diffeomorphic to a disjoint union of cylinders

$$\mathcal{M}\setminus\mathcal{K} \cong \bigsqcup_j (-\infty, 0) \times \mathcal{N}_j$$

where the $\mathcal{N}_j$ are the components of $\partial\mathcal{K}$.

The cylindrical compactification of $\mathcal{M}$ is the manifold with boundary obtained by

1. attaching each one of the $\mathcal{N}_j$ at $-\infty$ to the corresponding component of $\mathcal{M}\setminus\mathcal{K}$ in the natural way,
Cylindrical ends

For the purposes of this talk, a manifold with cylindrical ends is a smooth manifold $\mathcal{M}$ in which there is a compact submanifold $\mathcal{K}$ with smooth boundary such that $\mathcal{M} \setminus \mathcal{K}$ is diffeomorphic to a disjoint union of cylinders

$$\mathcal{M} \setminus \mathcal{K} \approx \bigsqcup_j (−∞, 0) \times \mathcal{N}_j$$

where the $\mathcal{N}_j$ are the components of $\partial \mathcal{K}$.

The cylindrical compactification of $\mathcal{M}$ is the manifold with boundary obtained by

1. attaching each one of the $\mathcal{N}_j$ at $−∞$ to the corresponding component of $\mathcal{M} \setminus \mathcal{K}$ in the natural way,

   This is a topological construction.
Cylindrical ends

For the purposes of this talk, a manifold with cylindrical ends is a smooth manifold $\mathcal{M}$ in which there is a compact submanifold $\mathcal{K}$ with smooth boundary such that $\mathcal{M} \setminus \mathcal{K}$ is diffeomorphic to a disjoint union of cylinders

$$\mathcal{M} \setminus \mathcal{K} \approx \bigsqcup_j (\mathbb{R} \cap (0, \infty), 0) \times \mathcal{N}_j$$

where the $\mathcal{N}_j$ are the components of $\partial \mathcal{K}$.

The cylindrical compactification of $\mathcal{M}$ is the manifold with boundary obtained by

1. attaching each one of the $\mathcal{N}_j$ at $-\infty$ to the corresponding component of $\mathcal{M} \setminus \mathcal{K}$ in the natural way,

2. while at the same time regarding the functions

$$(\mathbb{R} \cap (0, \infty), 0) \times \mathcal{N}_j \ni (t, p) \mapsto x = e^t, \quad j = 1, 2, \ldots$$

as smooth defining functions of the new boundary component.
Cylindrical ends

For the purposes of this talk, a manifold with cylindrical ends is a smooth manifold $\mathcal{M}$ in which there is a compact submanifold $\mathcal{K}$ with smooth boundary such that $\mathcal{M} \setminus \mathcal{K}$ is diffeomorphic to a disjoint union of cylinders

$$\mathcal{M} \setminus \mathcal{K} \approx \bigsqcup_{j} (\mathbb{R}, 0) \times \mathcal{N}_j$$

where the $\mathcal{N}_j$ are the components of $\partial \mathcal{K}$.

The cylindrical compactification of $\mathcal{M}$ is the manifold with boundary obtained by

1. attaching each one of the $\mathcal{N}_j$ at $-\infty$ to the corresponding component of $\mathcal{M} \setminus \mathcal{K}$ in the natural way,
2. while at the same time regarding the functions

$$(-\infty, 0) \times \mathcal{N}_j \ni (t, p) \mapsto x = e^t, \quad j = 1, 2, \ldots$$

as smooth defining functions of the new boundary component.

This is a topological construction.

This gives the $C^\infty$ structure.
Cylindrical ends

For the purposes of this talk, a manifold with cylindrical ends is a smooth manifold $\mathcal{M}$ in which there is a compact submanifold $\mathcal{K}$ with smooth boundary such that $\mathcal{M}\setminus\mathcal{K}$ is diffeomorphic to a disjoint union of cylinders

$$\mathcal{M}\setminus\mathcal{K} \approx \bigsqcup_j (\mathcal{N}_j)$$

where the $\mathcal{N}_j$ are the components of $\partial\mathcal{K}$.

The cylindrical compactification of $\mathcal{M}$ is the manifold with boundary obtained by

1. attaching each one of the $\mathcal{N}_j$ at $-\infty$ to the corresponding component of $\mathcal{M}\setminus\mathcal{K}$ in the natural way,
2. while at the same time regarding the functions

$$(\mathcal{N}_j, t) \ni (t, p) \mapsto x = e^t, \quad j = 1, 2, \ldots$$

as smooth defining functions of the new boundary component.

*The compactification is diffeomorphic to $\mathcal{K}$.*
**b-metrics**

The cylindrical metric $dt^2 + h_j$ on $(-\infty, 0) \times \mathcal{N}_j$, where $h_j$ is a metric on $\mathcal{N}_j$, is transformed to

$$\frac{dx^2}{x^2} + h_j$$

under the change $t = \log x$. 

**b-metrics**

The cylindrical metric $dt^2 + h_j$ on $(-\infty, 0) \times N_j$, where $h_j$ is a metric on $N_j$, is transformed to

$$dx^2 + \frac{h_j}{x^2}$$

under the change $t = \log x$.

Let $\mathcal{M}$ be a compact manifold with boundary, let $x$ be a defining function for its boundary, positive in $\mathcal{M}$.

**A b-metric on a compact manifold $\mathcal{M}$ with boundary together is a smooth Riemannian metric on $\mathcal{M}$ which is of the form**

$$g = a \frac{dx^2}{x^2} + h$$

**near $\partial \mathcal{M}$, where $h$ is such smooth and whose restriction to $\{x = \varepsilon\}$ is a Riemannian metric for each $\varepsilon \geq 0$.**
A $b$-Laplacian is, naturally, the Laplace operator of a $b$-metric. Near $\partial M$ (on functions) it has the general form

\[
a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j} xD_x D_{y_j} + \sum_{i,j=1}^{n} a_{ij} D_{y_i} D_{y_j} + a_0 xD_x + \sum_{j=1}^{n} a_j D_{y_j} \quad (*)
\]
A $b$-Laplacian is, naturally, the Laplace operator of a $b$-metric. Near $\partial M$ (on functions) it has the general form

\[(x, y) \text{ are coordinates near } \partial M\]

\[a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j} xD_x D_y j + \sum_{i,j=1}^{n} a_{ij} D_y i D_y j + a_0 xD_x + \sum_{j=1}^{n} a_j D_y j \quad (*)\]
A \(b\)-Laplacian is, naturally, the Laplace operator of a \(b\)-metric. Near \(\partial M\) (on functions) it has the general form

\[
a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j} x D_x D_y + \sum_{i,j=1}^{n} a_{ij} D_y D_{y_j} + a_0 x D_x + \sum_{j=1}^{n} a_j D_{y_j}\quad(\ast)
\]

\((x, y)\) are coordinates near \(\partial M\)

For example, if \(g = \frac{dx^2}{x^2}\) on \([0, \infty)\), then \(|g| = \det[g_{ij}] = \frac{1}{x^2}\) and

\([g^{ij}] = x^2\),
**A b-Laplacian**

A b-Laplacian is, naturally, the Laplace operator of a b-metric. Near $\partial M$ (on functions) it has the general form

$$(x, y) \text{ are coordinates near } \partial M$$

$$a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j} xD_x D_y j + \sum_{i,j=1}^{n} a_{ij} D_{yi} D_{yj} + a_0 xD_x + \sum_{j=1}^{n} a_j D_{yj} \quad (*)$$

For example, if $g = \frac{dx^2}{x^2}$ on $[0, \infty)$, then $|g| = \det[g_{ij}] = \frac{1}{x^2}$ and $[g_{ij}] = x^2$, so

$$\Delta = -\frac{1}{|g|^{1/2}} \sum \frac{\partial}{\partial x^i} |g|^{1/2} g^{ij} \frac{\partial}{\partial x^j}$$
**b-Laplacian**

A *b*-Laplacian is, naturally, the Laplace operator of a *b*-metric. Near $\partial M$ (on functions) it has the general form

$$a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j} xD_x D_y j + \sum_{i,j=1}^{n} a_{ij} D_y i D_y j + a_0 xD_x + \sum_{j=1}^{n} a_j D_y j \quad (*)$$

*(x, y) are coordinates near $\partial M$

For example, if $g = \frac{dx^2}{x^2}$ on $[0, \infty)$, then $|g| = \text{det}[g_{ij}] = \frac{1}{x^2}$ and $[g^{ij}] = x^2$, so

$$\Delta = -\frac{1}{|g|^{1/2}} \sum \frac{\partial}{\partial x^i} |g|^{1/2} g^{ij} \frac{\partial}{\partial x^j} = -\frac{1}{1/x} \frac{\partial x}{x} x^2 \frac{\partial x}{x}$$
**b-Laplacian**

A *b*-Laplacian is, naturally, the Laplace operator of a *b*-metric. Near \( \partial M \) (on functions) it has the general form

\[
(a_0 \partial_x)^2 + \sum_{j=1}^n a_{0j} x \partial_x \partial_{y_j} + \sum_{i,j=1}^n a_{ij} \partial_{y_i} \partial_{y_j} + a_0 x \partial_x + \sum_{j=1}^n a_j \partial_{y_j}
\]

\( (x, y) \) are coordinates near \( \partial M \)

For example, if \( g = \frac{dx^2}{x^2} \) on \( [0, \infty) \), then \( |g| = \det[g_{ij}] = \frac{1}{x^2} \) and \( [g^{ij}] = x^2 \), so

\[
\Delta = -\frac{1}{|g|^{1/2}} \sum \frac{\partial}{\partial x_i} |g|^{1/2} g^{ij} \frac{\partial}{\partial x_j} = -\frac{1}{1/x} \frac{1}{x} x^2 \partial_x = -(x \partial_x)^2
\]
A $b$-Laplacian is, naturally, the Laplace operator of a $b$-metric. Near $\partial M$ (on functions) it has the general form

\[(x, y) \text{ are coordinates near } \partial M\]

\[a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j}xD_xD_yj + \sum_{i,j=1}^{n} a_{ij}D_yiD_yj + a_0xD_x + \sum_{j=1}^{n} a_jD_yj \quad (*)\]

For example, if $g = \frac{dx^2}{x^2}$ on $[0, \infty)$, then $|g| = \det[g_{ij}] = \frac{1}{x^2}$ and $[g^{ij}] = x^2$, so

\[\Delta = -\frac{1}{|g|^{1/2}} \sum \frac{\partial}{\partial x^i}|g|^{1/2} g^{ij} \frac{\partial}{\partial x^j} = -\frac{1}{1/x} \frac{1}{x} \frac{1}{x^2} \partial_x = -(x\partial_x)^2\]

The operator $(*)$ is constructed using vector the fields $xD_x$ and $\partial_y$:
A \( b \)-Laplacian is, naturally, the Laplace operator of a \( b \)-metric. Near \( \partial M \) (on functions) it has the general form

\[
a_{00}(xD_x)^2 + \sum_{j=1}^{n} a_{0j} xD_x D_y_j + \sum_{i,j=1}^{n} a_{ij} D_{y_i} D_y_j + a_0 xD_x + \sum_{j=1}^{n} a_j D_{y_j} \quad (*)
\]

\((x, y)\) are coordinates near \( \partial M \)

For example, if \( g = \frac{dx^2}{x^2} \) on \([0, \infty)\), then \( |g| = \det[g_{ij}] = \frac{1}{x^2} \) and \( [g^{ij}] = x^2 \), so

\[
\Delta = -\frac{1}{|g|^{1/2}} \sum \frac{\partial}{\partial x_i} |g|^{1/2} g_{ij} \frac{\partial}{\partial x_j} = -\frac{1}{1/x} \cdot \frac{1}{x} x^2 \partial_x = -(x \partial_x)^2
\]

The operator \((*)\) is constructed using vector the fields \( x \partial_x \) and \( \partial_y_j \): these are tangential vector vector fields.
$b$-Manifolds

A $b$-manifold is a manifold in which the role tangent bundle is replaced by that of the $b$-tangent bundle:
A \( b \)-manifold is a manifold in which the role tangent bundle is replaced by that of the \( b \)-tangent bundle:

The \( b \)-tangent bundle is the vector bundle \( bT\mathcal{M} \to \mathcal{M} \) whose smooth sections are the vector fields of \( \mathcal{M} \) which along \( \partial \mathcal{M} \) are tangential.
A $b$-manifold is a manifold in which the role tangent bundle is replaced by that of the $b$-tangent bundle:

The $b$-tangent bundle is the vector bundle $bT\mathcal{M} \to \mathcal{M}$ whose smooth sections are the vector fields of $\mathcal{M}$ which along $\partial \mathcal{M}$ are tangential.

A \textit{b-manifold} is a manifold in which the role tangent bundle is replaced by that of the \textit{b-tangent bundle}:

\textbf{The b-tangent bundle is the vector bundle} $bT\mathcal{M} \to \mathcal{M}$ \textit{whose smooth sections are the vector fields of} $\mathcal{M}$ \textit{which along} $\partial\mathcal{M}$ \textit{are tangential.}


Let

$$C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M}) = \{X \in C^\infty(\mathcal{M}; T\mathcal{M}) : X|_{\partial\mathcal{M}} \text{ is tangent to } \partial\mathcal{M}\}$$

There is

$$\text{ev} : bT\mathcal{M} \to T\mathcal{M}$$

\textit{(a bundle homomorphism)}

such that

$$\text{ev}_* : C^\infty(\mathcal{M}; bT\mathcal{M}) \xrightarrow{\sim} C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M})$$
**b-Manifolds**

A *b*-manifold is a manifold in which the role tangent bundle is replaced by that of the *b*-tangent bundle:

The *b*-tangent bundle is the vector bundle $bT\mathcal{M} \to \mathcal{M}$ whose smooth sections are the vector fields of $\mathcal{M}$ which along $\partial\mathcal{M}$ are tangential.


Let

$$C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M}) = \{ X \in C^\infty(\mathcal{M}; T\mathcal{M}) : X|_{\partial\mathcal{M}} \text{ is tangent to } \partial\mathcal{M}\}$$

There is

$$\text{ev} : bT\mathcal{M} \to T\mathcal{M}$$

(a bundle homomorphism)

such that

$$\text{ev}_* : C^\infty(\mathcal{M}; bT\mathcal{M}) \congto C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M})$$

$C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M})$ is a finitely generated projective module over $C(\mathcal{M})$. By a theorem of Swan there is a vector bundle $bT\mathcal{M} \to \mathcal{M}$ whose space of continuous sections is $C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M})$. 

G. A. Mendoza (Temple University)
A $b$-manifold is a manifold in which the role tangent bundle is replaced by that of the $b$-tangent bundle:

The $b$-tangent bundle is the vector bundle $bT\mathcal{M} \to \mathcal{M}$ whose smooth sections are the vector fields of $\mathcal{M}$ which along $\partial\mathcal{M}$ are tangential.


Let

$$C_{\text{tan}}^\infty(\mathcal{M}; T\mathcal{M}) = \{ X \in C^\infty(\mathcal{M}; T\mathcal{M}) : X|_{\partial\mathcal{M}} \text{ is tangent to } \partial\mathcal{M} \}$$

There is

$$\text{ev} : bT\mathcal{M} \to T\mathcal{M}$$

(a bundle homomorphism)

such that

$$\text{ev}_* : C^\infty(\mathcal{M}; bT\mathcal{M}) \overset{\sim}{\longrightarrow} C_{\text{tan}}^\infty(\mathcal{M}; T\mathcal{M})$$

If $x, y_1, \ldots, y_n$ is a local chart of $\mathcal{M}$ near $p \in \partial\mathcal{M}$, then

$$x\partial_x, \partial_{y_1}, \ldots, \partial_{y_n}$$

are the image elements of a frame of $bT\mathcal{M}$.
**b-Manifolds**

A *b-manifold* is a manifold in which the role tangent bundle is replaced by that of the *b-tangent bundle:*

*The b-tangent bundle is the vector bundle* $bT\mathcal{M} \to \mathcal{M}$ *whose smooth sections are the vector fields of* $\mathcal{M}$ *which along* $\partial \mathcal{M}$ *are tangential.*


Let

$$C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M}) = \{X \in C^\infty(\mathcal{M}; T\mathcal{M}) : X|_{\partial\mathcal{M}} \text{ is tangent to } \partial\mathcal{M}\}$$

There is

$$\text{ev} : bT\mathcal{M} \to T\mathcal{M}$$

(a bundle homomorphism)

such that

$$\text{ev}_* : C^\infty(\mathcal{M}; bT\mathcal{M}) \xrightarrow{\cong} C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M})$$

If $x, y_1, \ldots, y_n$ is a local chart of $\mathcal{M}$ near $p \in \partial\mathcal{M}$, then

$$x\partial_x, \partial_{y_1}, \ldots, \partial_{y_n}$$

are the image elements of a frame of $bT\mathcal{M}$.

$\text{ev}$ is a bundle isomorphism over $\mathcal{M}$, has 1-dimensional kernel over $\partial\mathcal{M}$. $x\partial_x$ is a canonical section of $bT\mathcal{M}$ along $\partial\mathcal{M}$.
The $b$-cotangent bundle; $b$-metrics

The $b$-cotangent bundle is the dual of the $b$-tangent bundle:

\[ bT^*M = (bT M)^* \]

If $x \partial_x, \partial_{y_1}, \ldots, \partial_{y_n}$ is a frame of $bT M$ near $p_0 \in \partial M$, then

\[ \frac{dx}{x}, dy_1, \ldots, dy_n \]

is a frame of $bT^*M$ near $p_0$. The bundle map $\text{ev} : bT M \to T M$ gives

\[ \text{ev}^* : T^*M \to bT^*M \]
The $b$-cotangent bundle; $b$-metrics

The $b$-cotangent bundle is the dual of the $b$-tangent bundle:

$$bT^*M = (bTM)^*$$

If $x\partial_x, \partial_{y_1}, \ldots, \partial_{y_n}$ is a frame of $bTM$ near $p_0 \in \partial M$, then

$$\frac{dx}{x}, dy_1, \ldots, dy_n$$

is a frame of $bT^*M$ near $p_0$. The bundle map $ev : bTM \to TM$ gives

$$ev^* : T^*M \to bT^*M$$

ev* is a bundle isomorphism over $\hat{M}$, has 1-dimensional kernel over $\partial M$.

A $b$-metric is a Riemannian metric on $bTM$. Locally such metrics have the form

$$g_{00} \frac{dx}{x} \otimes \frac{dx}{x} + \sum_j g_{0j} \left( \frac{dx}{x} \otimes dy_j + dy_j \otimes \frac{dx}{x} \right) + \sum_{i,j} a_{ij} dy_i \otimes dy_j.$$
**b-Differential operators**

A linear $b$-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial\mathcal{M}$ has the form

\[
\sum_{\alpha+k\leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial\mathcal{M}.
\]

More generally:

Let $E, F \to \mathcal{M}$ be vector bundles. A differential operator

\[
P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)
\]

is a $b$-differential operator of order $m$ if

\[
x^{-\nu} Px^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.
\]
**b-Differential operators**

A linear $b$-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial \mathcal{M}$ has the form

$$
\sum_{\alpha+k\leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial \mathcal{M}.
$$

$x$ is a defining function for $\partial \mathcal{M}$ with $x > 0$ in $\mathcal{M}$

More generally:

Let $E, F \to \mathcal{M}$ be vector bundles. A differential operator

$$
P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)
$$

is a $b$-differential operator of order $m$ if

$$
x^{-\nu} Px^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.
$$

$\text{Diff}^m(\mathcal{M}; E, F)$ is the space of linear differential operators of order $m$ with smooth coefficients.
\textbf{\textit{b-Differential operators}}

A linear \textit{b}-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial \mathcal{M}$ has the form

$$\sum_{\alpha + k \leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial \mathcal{M}.$$ 

More generally:

Let $E, F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

is a \textit{b}-differential operator of order $m$ if

$$x^{-\nu} P x^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.$$ 

Example: Let $P = a \partial_x$. Then

$$P xu = a \partial_x xu = x a \partial_x u + au$$
\textbf{\textit{b}}-Differential operators

A linear \textit{b}-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial \mathcal{M}$ has the form

$$\sum_{\alpha+k \leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial \mathcal{M}.$$  

More generally:

\textit{Let $E$, $F \to \mathcal{M}$ be vector bundles. A differential operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ is a \textit{b}-differential operator of order $m$ if}

$$x^{-\nu} Px^{\nu} \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.$$  

Example: Let $P = a \partial_x$. Then

$$x^{-1} Pxu = x^{-1} a \partial_x xu = x a \partial_x u + x^{-1} au.$$  

$\partial$ is a defining function for $\partial \mathcal{M}$ with $\partial > 0$ in $\mathcal{M}$.
**b-Differential operators**

A linear $b$-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial \mathcal{M}$ has the form

$$\sum_{\alpha+k \leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial \mathcal{M}.$$  

More generally:

*Let $E$, $F \to \mathcal{M}$ be vector bundles. A differential operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ is a $b$-differential operator of order $m$ if*

$$x^{-\nu} P x^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.$$  

**Example:** Let $P = a \partial_x$. Then

$$x^{-1} P x u = x^{-1} a \partial_x x u = x a \partial_x u + x^{-1} a u.$$  

If this has smooth coefficients, then $a = xa'$. 

$x$ is a defining function for $\partial \mathcal{M}$ with $x > 0$ in $\mathcal{M}$. 

\[ \text{Diff}^m(\mathcal{M}; E, F) \text{ is the space of linear differential operators of order } m \text{ with smooth coefficients.} \]
**b-Differential operators**

A linear $b$-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial \mathcal{M}$ has the form

$$\sum_{\alpha+k\leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial \mathcal{M}.$$ 

$x$ is a defining function for $\partial \mathcal{M}$ with $x > 0$ in $\mathcal{M}$.

More generally:

*Let $E, F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

*is a $b$-differential operator of order $m$ if

$$x^{-\nu} Px^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.$$*

Example: Let $P = a \partial_x$. Then

$$x^{-1} Pxu = x^{-1} a \partial_x xu = xa \partial_x u + x^{-1} au.$$  

If this has smooth coefficients, then $a = xa'$. So $P = a'x \partial_x$. 

\[\Box\]
A linear $b$-differential operator on $\mathcal{M}$ is a linear differential operator on $\mathcal{M}$ which near $\partial \mathcal{M}$ has the form

$$\sum_{\alpha + k \leq m} a_{k\alpha} (xD_x)^k D_y^\alpha, \quad a_{k\alpha} \in C^\infty \text{ up to } \partial \mathcal{M}.$$ 

More generally:

Let $E, F \to \mathcal{M}$ be vector bundles. A differential operator

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$$

is a $b$-differential operator of order $m$ if

$$x^{-\nu} Px^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \ldots, m.$$ 

Example: Let $P = a \partial_x$. Then

$$x^{-1} Pxu = x^{-1} a \partial_x xu = xa \partial_x u + x^{-1} au.$$ 

If this has smooth coefficients, then $a = xa'$. So $P = a'x \partial_x$. 

$\text{Diff}^m(\mathcal{M}; E, F)$ is the space of linear $b$-differential operators on $\mathcal{M}$ of order $m$ with smooth coefficients.

$x$ is a defining function for $\partial \mathcal{M}$ with $x > 0$ in $\mathcal{M}$. 

 Diff$^m_b(\mathcal{M}; E, F)$ is the space of linear $b$-differential operators on $\mathcal{M}$ of order $m$ with smooth coefficients.
The vector bundle map $\text{ev} : bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathring{\mathcal{M}}$, so the dual map $\text{ev}^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathring{\mathcal{M}}$.

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \to \pi^*F$$
The vector bundle map $ev : bTM \to TM$ is an isomorphism over $\tilde{M}$, so the dual map $ev^* : T^*M \to bT^*M$ is also an isomorphism over $\tilde{M}$.

Let $P \in Diff^m_b(M; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \to \pi^*F$$

$\sigma(P)$ is a section of $\pi^*\text{Hom}(E, F)$.
The vector bundle map $\text{ev} : \bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathring{\mathcal{M}}$, so the dual map $\text{ev}^* : T^*\mathcal{M} \to \bT^*\mathcal{M}$ is also an isomorphism over $\mathring{\mathcal{M}}$.

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^* E \to \pi^* F$$

$\sigma(P)$ is a section of $\pi^* \text{Hom}(E, F)$.
The vector bundle map \( ev : bT\mathcal{M} \to T\mathcal{M} \) is an isomorphism over \( \mathcal{M} \), so the dual map \( ev^* : T^*\mathcal{M} \to bT^*\mathcal{M} \) is also an isomorphism over \( \mathcal{M} \).

Let \( P \in \text{Diff}^m_b(\mathcal{M}; E, F) \). Its principal symbol is a homomorphism

\[
\sigma(P) : \pi^*E \to \pi^*F
\]

\( \sigma(P) \) is a section of \( \pi^*\text{Hom}(E, F) \)

This induces a homomorphism

\[
b\sigma(P) : b\pi^*E \to b\pi^*F
\]

by way of the following argument:
The vector bundle map $ev : bT^*M \rightarrow T^*M$ is an isomorphism over $M$, so the dual map $ev^* : T^*M \rightarrow bT^*M$ is also an isomorphism over $M$.

Let $P \in \text{Diff}^\text{m}_b(M; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \rightarrow \pi^*F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^*E \rightarrow b\pi^*F$$

by way of the following argument:

$$\pi^* \text{Hom}(E, F) \xrightarrow{\pi} T^*M$$
**b-Symbol**

The vector bundle map $\text{ev} : bTM \to TM$ is an isomorphism over $\mathring{\mathcal{M}}$, so the dual map $\text{ev}^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathring{\mathcal{M}}$.

Let $P \in \text{Diff}^m_b(M; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \to \pi^*F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^*E \to b\pi^*F$$

by way of the following argument:

$$\begin{array}{ccc}
\pi^*\text{Hom}(E, F) & \xrightarrow{\pi} & \sigma(P) \\
\pi & \downarrow & \\
T^*\mathcal{M} & & \\
\end{array}$$
The vector bundle map $ev : bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $ev^* : T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^* E \to \pi^* F$$

$\sigma(P)$ is a section of $\pi^* \text{Hom}(E, F)$

This induces a homomorphism

$$b\sigma(P) : b\pi^* E \to b\pi^* F$$

by way of the following argument:

$$
\begin{array}{ccc}
b\pi^* \text{Hom}(E, F) & \xrightarrow{b\pi} & bT^*\mathcal{M} \\
\downarrow & & \downarrow \\
\pi^* \text{Hom}(E, F) & \xrightarrow{\pi} & T^*\mathcal{M} \\
\end{array}
$$
The vector bundle map $\text{ev} : bT\mathcal{M} \rightarrow T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $\text{ev}^* : T^*\mathcal{M} \rightarrow bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \rightarrow \pi^*F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^*E \rightarrow b\pi^*F$$

by way of the following argument:

$$\begin{array}{ccc}
    b\pi^* \text{Hom}(E, F) & \xrightarrow{b\pi} & bT^*\mathcal{M}
    \\
    \downarrow & & \downarrow \text{ev}^*
    \\
    \pi^* \text{Hom}(E, F) & \xrightarrow{\pi} & T^*\mathcal{M}
\end{array}$$

isomorphism over $\mathcal{M}$, image of codimension 1 over $\partial\mathcal{M}$
The vector bundle map \( \text{ev} : bT^*M \to TM \) is an isomorphism over \( \partial M \), so
the dual map \( \text{ev}^* : T^*M \to bT^*M \) is also an isomorphism over \( \partial M \).

Let \( P \in \text{Diff}_b^m(M; E, F) \). Its principal symbol is a homomorphism

\[
\sigma(P) : \pi^* E \to \pi^* F
\]

This induces a homomorphism

\[
b\sigma(P) : b\pi^* E \to b\pi^* F
\]

by way of the following argument:
The vector bundle map $ev: bT\mathcal{M} \to T\mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $ev^*: T^*\mathcal{M} \to bT^*\mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P): \pi^*E \to \pi^*F$$

This induces a homomorphism

$$b\sigma(P): b\pi^*E \to b\pi^*F$$

by way of the following argument:

$$b\pi^*\text{Hom}(E, F) \xleftarrow{b\pi} \pi^*\text{Hom}(E, F) \xrightarrow{\pi} \pi^*\text{Hom}(E, F) \xrightarrow{\sigma(P)} \pi^*\text{Hom}(E, F)$$

$$bT^*\mathcal{M} \xleftarrow{ev^*} T^*\mathcal{M}$$

isomorphism over $\mathcal{M}$, image of codimension 1 over $\partial\mathcal{M}$
The vector bundle map \( \text{ev} : bT\mathcal{M} \rightarrow T\mathcal{M} \) is an isomorphism over \( \mathcal{M} \), so the dual map \( \text{ev}^* : T^*\mathcal{M} \rightarrow bT^*\mathcal{M} \) is also an isomorphism over \( \mathcal{M} \).

Let \( P \in \text{Diff}^m_b(\mathcal{M}; E, F) \). Its principal symbol is a homomorphism

\[
\sigma(P) : \pi^*E \rightarrow \pi^*F
\]

\( \sigma(P) \) is a section of \( \pi^*\text{Hom}(E, F) \)

This induces a homomorphism

\[
b\sigma(P) : b\pi^*E \rightarrow b\pi^*F
\]

by way of the following argument:

\[
\begin{array}{c}
b\pi^*\text{Hom}(E, F) \\
\downarrow b\pi \\
\pi^*\text{Hom}(E, F) \\
\downarrow \pi \\
T^*\mathcal{M} \\
\downarrow \text{ev}^*
\end{array}
\]

With frame \([x\partial_x], [\partial_y]\), dual frame \([\frac{dx}{x}], [dy_j]\), the dual of

\[
\text{ev} : a_0[x\partial_x] + \sum a_j[\partial_y] \mapsto a_0x\partial_x + \sum a_j\partial_y
\]

is

\[
\nu = \xi dx + \sum \eta_j dy_j \mapsto \\
\text{ev}^*(\nu) = \xi x\left[\frac{dx}{x}\right] + \sum \eta_j[dy_j].
\]
The vector bundle map $\text{ev} : bTM \to TM$ is an isomorphism over $\tilde{M}$, so the dual map $\text{ev}^* : T^*M \to bT^*M$ is also an isomorphism over $\tilde{M}$.

Let $P \in \text{Diff}^m_b(M; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \to \pi^*F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^*E \to b\pi^*F$$

by way of the following argument:

$$\begin{array}{ccc}
\pi^* \text{Hom}(E, F) & \xrightarrow{\pi^* \text{Hom}(E, F)} & \pi^* \text{Hom}(E, F) \\
\downarrow b\pi & & \downarrow b\pi \\
b\pi^* \text{Hom}(E, F) & \xleftarrow{b\pi} & b\pi^* \text{Hom}(E, F) \\
{\text{isomorphism over } \tilde{M},} & & {\text{isomorphism over } \tilde{M},} \\
{\text{image of codimension 1 over } \partial M} & & {\text{image of codimension 1 over } \partial M} \\
\end{array}$$

so if $\tilde{\nu} = \text{ev}^*\nu = \xi \left[ \frac{dx}{x} \right] + \sum \eta_j [dy_j]$
The vector bundle map \( \text{ev} : bT^*M \rightarrow TM \) is an isomorphism over \( \hat{M} \), so the dual map \( \text{ev}^* : T^*M \rightarrow bT^*M \) is also an isomorphism over \( \hat{M} \).

Let \( P \in \text{Diff}_b^m(\hat{M}; E, F) \). Its principal symbol is a homomorphism

\[
\sigma(P) : \pi^*E \rightarrow \pi^*F
\]

This induces a homomorphism

\[
b\sigma(P) : b\pi^*E \rightarrow b\pi^*F
\]

by way of the following argument:

\[
\begin{array}{ccc}
b\pi^* \text{Hom}(E, F) & \xrightarrow{b\pi} & b\pi^* \text{Hom}(E, F) \\
\pi^* \text{Hom}(E, F) & \xleftarrow{\pi} & \pi^* \text{Hom}(E, F)
\end{array}
\]

\[
\text{ev}^* \quad \text{isomorphism over } \hat{M}, \quad \text{image of codimension 1 over } \partial M
\]

so if \( \tilde{\nu} = \text{ev}^*\nu \)

\[
\tilde{\nu} = \xi \left( \sum_{j=1}^{\alpha} \eta_j dy_j \right)
\]

With frame \([x\partial_x], [\partial_y]\), dual frame \([dx], [dy]\), the dual of

\[
\text{ev} : a_0[x\partial_x] + \sum a_j[\partial_y] \mapsto a_0x\partial_x + \sum a_j\partial_y
\]

is

\[
\nu = \xi dx + \sum \eta_j dy_j \mapsto
\]

\[
\text{ev}^*(\nu) = \xi \left( \sum_{j=1}^{\alpha} \eta_j dy_j \right)
\]

If \( P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(xD_x)^k D_y^\alpha \)

then

\[
\sigma(P)(\nu) = \sum_{k+|\alpha| = m} a_{k\alpha}(x\xi)^k \eta^\alpha
\]
**$b$-Symbol**

The vector bundle map $\text{ev} : bT^*M \to TM$ is an isomorphism over $\mathcal{M}$, so the dual map $\text{ev}^* : T^*M \to bT^*M$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^*E \to \pi^*F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^*E \to b\pi^*F$$

by way of the following argument:

With frame $[x\partial_x], [\partial_y]$, dual frame $\left[\frac{dx}{x}\right], [dy_j]$, the dual of

$$\text{ev} : a_0[x\partial_x] + \sum a_j[\partial_y_j] \mapsto a_0x\partial_x + \sum a_j\partial_y_j$$

is

$$\nu = \xi dx + \sum \eta_j dy_j \mapsto$$

$$\text{ev}^*(\nu) = \xi x\left[\frac{dx}{x}\right] + \sum \eta_j [dy_j].$$

If $P = \sum k + |\alpha| \leq m a_k \alpha (x\xi)^k D_y^\alpha$

then

$$\sigma(P)(\nu) = \sum k + |\alpha| = m a_k \alpha (x\xi)^k \eta^\alpha$$

so if $\tilde{\nu} = \text{ev}^* \nu = \left[\xi\xi\right] \frac{dx}{x} + \sum \eta_j [dy_j]$ then

$$\sigma(P)((\text{ev}^*)^{-1} \tilde{\nu}) = \sum k + |\alpha| = m a_k \alpha \tilde{\xi}^k \eta^\alpha$$
**b-Symbol**

The vector bundle map $\ev : bT \mathcal{M} \to T \mathcal{M}$ is an isomorphism over $\mathcal{M}$, so the dual map $\ev^* : T^* \mathcal{M} \to bT^* \mathcal{M}$ is also an isomorphism over $\mathcal{M}$.

Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$. Its principal symbol is a homomorphism

$$\sigma(P) : \pi^* E \to \pi^* F$$

This induces a homomorphism

$$b\sigma(P) : b\pi^* E \to b\pi^* F$$

by way of the following argument:

$$\begin{align*}
\xymatrix{ \pi^* \text{Hom}(E, F) \ar[r] & \pi^* \text{Hom}(E, F) \ar[d]^\pi \\
b\pi^* \text{Hom}(E, F) \ar[u]_{b\pi} & b\sigma(P) \ar[l] \\
bT^* \mathcal{M} \ar[u]^{\text{isomorphism over } \mathcal{M}} \ar[r]_{\ev^*} & T^* \mathcal{M} \\
\mathcal{M} \ar[u]_{\text{image of codimension } 1 \text{ over } \partial \mathcal{M}} & \\
}
\end{align*}$$

With frame $[x \partial_x], [\partial_y]$, dual frame $[dx], [dy]$, the dual of

$$
\ev : a_0[x \partial_x] + \sum a_j[\partial_y] \mapsto a_0x \partial_x + \sum a_j \partial_y
$$

is

$$\nu = \xi dx + \sum \eta_j dy_j \mapsto \ev^*(\nu) = \xi [\frac{dx}{x}] + \sum \eta_j [dy_j].$$

If $P = \sum_{k+|\alpha|\leq m} a_k\alpha (x \xi)^k D_{\alpha}$, then

$$
\sigma(P)(\nu) = \sum_{k+|\alpha|=m} a_k\alpha (x \xi)^k \eta^\alpha
$$

so if $\tilde{\nu} = \ev^* \nu = \left(\xi \frac{dx}{x}\right) + \sum \eta_j [dy_j]$ then $\sigma(P)((\ev^*)^{-1} \tilde{\nu}) = \sum_{k+|\alpha|=m} a_k\alpha \tilde{\xi}^k \eta^\alpha$.
\textbf{b-ellipticity}

Naturally,

\begin{quote}
\textit{The operator }P \in \text{Diff}^m_b(M; E, F) \text{ is } b\text{-elliptic if } b\sigma(P) \text{ is invertible on } bT^*M \setminus 0.
\end{quote}

\textbf{Example}: Let \( g = \frac{dx^2}{x^2} + dy^2 \) on \([0, \infty) \times S^1\). Then

\[ \Delta = -(x\partial_x)^2 - \partial_y^2 \]
**b-ellipticity**

Naturally,

*The operator* \( P \in \text{Diff}^m_b(\mathcal{M}; E, F) \) *is b-elliptic if* \( ^b\sigma(P) \) *is invertible on* \( ^b\mathcal{T}^*\mathcal{M}\setminus0 \).

**Example:** Let \( g = \frac{dx^2}{x^2} + dy^2 \) on \([0, \infty) \times S^1\). Then

\[
\Delta = -(x\partial_x)^2 - \partial_y^2, \quad \sigma(\Delta) = (x\xi)^2 + \eta^2
\]

and so

\[
^b\sigma(\Delta) = \xi^2 + \eta^2.
\]

Thus \( \Delta \) is \( b \)-elliptic.
**$b$-ellipticity**

Naturally,

*The operator $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$ is $b$-elliptic if $^b\sigma(P)$ is invertible on $^bT^*\mathcal{M}\setminus 0$.***

**Example:** Let $g = \frac{dx^2}{x^2} + dy^2$ on $[0, \infty) \times S^1$. Then

$$\Delta = -(x\partial_x)^2 - \partial_y^2, \quad \sigma(\Delta) = (x\xi)^2 + \eta^2$$

and so

$$^b\sigma(\Delta) = \xi^2 + \eta^2.$$ 

Thus $\Delta$ is $b$-elliptic.

In general, if $P = \sum_{k+|\alpha|\leq m} a_{k\alpha}(xD_x)^k D_y^\alpha$, then

$$^b\sigma(P) = \sum_{k+|\alpha|=m} a_{k\alpha} \xi^k \eta^\alpha.$$
Mellin transform

If $\mathcal{M}$ is a manifold with boundary, then

$$\mathring{C}^\infty(\mathcal{M}) = \{ u \in C^\infty(\mathcal{M}) : u \text{ vanishes to infinite order on } \partial \mathcal{M} \}$$

Let $u \in \mathring{C}^\infty[0, \infty)$ be compactly supported. The Mellin transform of $u$ is

$$\mathcal{M}(u)(\sigma) = \int_0^\infty x^{-i\sigma} u(x) \frac{dx}{x}$$

The inverse Mellin transform is

$$\mathcal{M}^{-1}(v) = \frac{1}{2\pi i} \int_{\text{Re } \sigma = 0} x^{i\sigma} v(\sigma) d\sigma.$$ 

The Mellin transform is the Fourier transform with $e^t$ replaced by $x$. 
If $u \in L^2([0, \infty); \frac{dx}{x})$ then $\mathcal{M}(u) \in L^2(\mathbb{R})$.

Further, if $u \in L^2([0, \infty); \frac{dx}{x})$ has compact support, then $\mathcal{M}(u)$ is holomorphic in $\text{Im} \sigma > 0$: Since $|x^{-i\sigma}| = x^{\text{Im} \sigma}$,

$$\int_0^\infty x^{-i\sigma} u(x) \frac{dx}{x}$$

is integrable if $\text{Im} \sigma > 0$ and defines a holomorphic function of $\sigma$ there.
If \( u \in L^2([0, \infty); \frac{dx}{x}) \) then 
\[
\mathcal{M}(u) \in L^2(\mathbb{R}).
\]

Further, if \( u \in L^2([0, \infty); \frac{dx}{x}) \) has compact support, then \( \mathcal{M}(u) \) is holomorphic in \( \text{Im} \sigma > 0 \): Since \( |x^{-i\sigma}| = x^{\text{Im} \sigma} \),

\[
\int_0^\infty x^{-i\sigma} u(x) \frac{dx}{x}
\]

is integrable if \( \text{Im} \sigma > 0 \) and defines a holomorphic function of \( \sigma \) there.

A (smooth) \( b \)-density is a density on \( \hat{\mathcal{M}} \) of the form \( m_b = \frac{1}{x} m \) where \( m \) is a smooth positive density on \( \mathcal{M} \). Define

\[
L^2_b(\mathcal{M}) = L^2(\mathcal{M}, m_b)
\]

Let \( \mathcal{N} = \partial \mathcal{M} \), let \([0, \varepsilon) \times \mathcal{N}\) be a tubular neighborhood of \( \mathcal{N} \) in \( \mathcal{M} \). Let \( \omega \in C^\infty(\mathcal{M}), \omega = 1 \) near \( \mathcal{N} \), \( \omega \) compactly supported in \([0, \varepsilon) \times \mathcal{N}\). 

\[
\mathcal{M}(u)(\sigma, p) = \int_0^\infty x^{-i\sigma} \omega(x, p) u(x, p) \frac{dx}{x}, \quad p \in \mathcal{N}, \sigma \in \mathbb{C}, \text{Im} \sigma \geq 0.
\]
If $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u} \big|_{\mathcal{N}}$. 

Note that $P \in \text{Diff}_b^m(\mathcal{M}; E, F) \Rightarrow x^{-1}\sigma P x\sigma \in \text{Diff}_b^m(\mathcal{M}; E, F)$ for all $\sigma \in C^\infty(\mathbb{M})$.

The indicial operator of $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$ is $\hat{P}(\sigma) = (x^{-1}\sigma P x\sigma)_b$.

$\hat{P}(\sigma)$ is a polynomial in $\sigma \in C^\infty(\mathbb{M})$ with values in $\text{Diff}(\mathcal{N}; E, F)$. 

G. A. Mendoza (Temple University)
If $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

*Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u}\big|_{\mathcal{N}}$.  

$P\tilde{u}\big|_{\mathcal{N}}$ depends only on $u$. The difference between two extensions of $u$ is $xv$, $v \in C^\infty(\mathcal{M}; E)$. By definition of $b$ operator, $x^{-1}Pxv$ is smooth up to $\mathcal{N}$ if $v$ is. So $P(xv)\big|_{\mathcal{N}} = 0$. 

If $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

$P_b u = P\tilde{u}\big|_{\mathcal{N}}$.

$P\tilde{u}\big|_{\mathcal{N}}$ depends only on $u$. The difference between two extensions of $u$ is $xv, v \in C^\infty(\mathcal{M}; E)$. By definition of $b$ operator, $x^{-1}Pxv$ is smooth up to $\mathcal{N}$ if $v$ is. So $P(xv)\big|_{\mathcal{N}} = 0$.

Note that $P \in \text{Diff}_b^m \implies x^{-i\sigma}Px^{i\sigma} \in \text{Diff}_b^m$ for all $\sigma \in \mathbb{C}$:

$$(xD_x)^k(x^{i\sigma}u) = x^{i\sigma}(xD_x + \sigma)^k u$$
If $P \in \text{Diff}_b^m(M; E, F)$, then $P$ defines an operator $P_b$ on $\mathcal{N} = \partial \mathcal{M}$:

Given $u \in C^\infty(\mathcal{N}; E)$, let $\tilde{u}$ be a smooth extension of $u$, let $P_b u = P\tilde{u}|_{\mathcal{N}}$.

$P\tilde{u}|_{\mathcal{N}}$ depends only on $u$. The difference between two extensions of $u$ is $xv$, $v \in C^\infty(\mathcal{M}; E)$. By definition of $b$ operator, $x^{-1}Pxv$ is smooth up to $\mathcal{N}$ if $v$ is. So $P(xv)|_{\mathcal{N}} = 0$.

Note that $P \in \text{Diff}_b^m \implies x^{-i\sigma}Px^{i\sigma} \in \text{Diff}_b^m$ for all $\sigma \in \mathbb{C}$:

$$(xD_x)^k(x^{i\sigma} u) = x^{i\sigma}(xD_x + \sigma)^k u$$

$P = P(x, y, xD_x, D_y) \implies x^{-i\sigma}Px^{i\sigma} = P(x, y, xD_x + \sigma, D_y)$. 
If \( P \in \text{Diff}_b^m(\mathcal{M}; E, F) \), then \( P \) defines an operator \( P_b \) on \( \mathcal{N} = \partial \mathcal{M} \): 

\[ \text{Given } u \in C^\infty(\mathcal{N}; E), \text{ let } \tilde{u} \text{ be a smooth extension of } u, \text{ let } P_b u = P\tilde{u} \big|_{\mathcal{N}}. \]

\( P\tilde{u} \big|_{\mathcal{N}} \) depends only on \( u \). The difference between two extensions of \( u \) is \( xv, v \in C^\infty(\mathcal{M}; E) \). By definition of \( b \) operator, \( x^{-1}Pxv \) is smooth up to \( \mathcal{N} \) if \( v \) is. So \( P(xv) \big|_{\mathcal{N}} = 0 \).

Note that \( P \in \text{Diff}_b^m \implies x^{-i\sigma}Px^{i\sigma} \in \text{Diff}_b^m \) for all \( \sigma \in \mathbb{C} \):

\[
(xD_x)^k(x^{i\sigma}u) = x^{i\sigma}(xD_x + \sigma)^k u \quad \text{for all } \sigma \in \mathbb{C}.
\]

\[ P = P(x, y, xD_x, D_y) \implies x^{-i\sigma}Px^{i\sigma} = P(x, y, xD_x + \sigma, D_y). \]

The indicial operator of \( P \in \text{Diff}_b^m(\mathcal{M}; E, F) \) is 

\[ \hat{P}(\sigma) = (x^{-i\sigma}Px^{i\sigma})_b \]

\( \hat{P}(\sigma) \) is a polynomial in \( \sigma \in \mathbb{C} \) with values in \( \text{Diff}(\mathcal{N}; E, F) \).
Let $P \in \text{Diff}^m_b(M; E, F)$, so

$$P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha$$

locally. Then

$$x^{i\sigma} P x^{-i\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha$$

and

$$x^{i\sigma} P x^{-i\sigma} \big|_{\mathcal{N}} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.$$
Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$, so

$$P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha$$

locally. Then

$$x^{i\sigma} P x^{-i\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha$$

and

$$\hat{P}(\sigma) = x^{i\sigma} P x^{-i\sigma}\bigg|_{\mathcal{N}} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.$$
Let \( P \in \text{Diff}_b^m(\mathcal{M}; E, F) \), so
\[
P = \sum_{k + |\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha
\]
locally. Then
\[
x^{i\sigma} P x^{-i\sigma} = \sum_{k + |\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha
\]
and
\[
\hat{P}(\sigma) = x^{i\sigma} P x^{-i\sigma} \big|_{\mathcal{N}} = \sum_{k + |\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.
\]
The principal symbol of \( \hat{P}(\sigma) \) is
\[
\sigma(\hat{P}(\sigma)) = \sum_{|\alpha| = m} a_{0\alpha}(0, y)\eta^\alpha
\]
Let \( P \in \text{Diff}^m_b(\mathcal{M}; E, F) \), so
\[
P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha
\]
locally. Then
\[
x^i\sigma \, P^{x^{-i}\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha
\]
and
\[
\hat{P}(\sigma) = x^i\sigma \, P^{x^{-i}\sigma}|_{\mathcal{N}} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.
\]
The principal symbol of \( \hat{P}(\sigma) \) is
\[
\sigma(\hat{P}(\sigma)) = \sum_{|\alpha|=m} a_{0\alpha}(0, y)\eta^\alpha = b_\sigma(P)(0, y, 0, \eta)
\]
Let \( P \in \text{Diff}_b^m(\mathcal{M}; E, F) \), so
\[
P = \sum_{k + |\alpha| \leq m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha
\]
locally. Then
\[
x^{i\sigma} P x^{-i\sigma} = \sum_{k + |\alpha| \leq m} a_{k\alpha}(x, y)(xD_x + \sigma)^k D_y^\alpha
\]
and
\[
\hat{P}(\sigma) = x^{i\sigma} P x^{-i\sigma} \big|_{\mathcal{N}} = \sum_{k + |\alpha| \leq m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha.
\]
The principal symbol of \( \hat{P}(\sigma) \) is
\[
\sigma(\hat{P}(\sigma)) = \sum_{|\alpha| = m} a_{0\alpha}(0, y)\eta^\alpha = b_\sigma(P)(0, y, 0, \eta)
\]
If \( P \) is elliptic, then \( \hat{P}(\sigma) \) is elliptic.
Let \( P \in \text{Diff}_b^m(\mathcal{M}; E, F) \), so 
\[
P = \sum_{k + |\alpha| \leq m} a_{k\alpha}(x, y) (xD_x)^k D_y^\alpha
\]
locally. Then 
\[
\chi^i \sigma P \chi^{-i} \sigma = \sum_{k + |\alpha| \leq m} a_{k\alpha}(x, y) (xD_x + \sigma)^k D_y^\alpha
\]
and 
\[
\hat{P}(\sigma) = \chi^i \sigma P \chi^{-i} \sigma \big|_N = \sum_{k + |\alpha| \leq m} a_{k\alpha}(0, y) \sigma^k D_y^\alpha.
\]
The principal symbol of \( \hat{P}(\sigma) \) is 
\[
\sigma(\hat{P}(\sigma)) = \sum_{|\alpha| = m} a_{0\alpha}(0, y) \eta^\alpha = b_{\sigma}(P)(0, y, 0, \eta)
\]
if \( P \) is elliptic, then \( \hat{P}(\sigma) \) is elliptic.

More is true because \[
\sum_{k + |\alpha| = m} a_{k\alpha}(x, y) \sigma^k \eta^\alpha \neq 0 \text{ if } \Re \sigma \text{ is large relative to } |\Im \sigma|.
\]
Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$, so

$$P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y) (xD_x)^k D_y^\alpha$$

locally. Then

$$x^{i\sigma} P x^{-i\sigma} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x, y) (xD_x + \sigma)^k D_y^\alpha$$

and

$$\hat{P}(\sigma) = x^{i\sigma} P x^{-i\sigma} \bigg|_\mathcal{N} = \sum_{k+|\alpha| \leq m} a_{k\alpha}(0, y) \sigma^k D_y^\alpha.$$ 

The principal symbol of $\hat{P}(\sigma)$ is

$$\sigma(\hat{P}(\sigma)) = \sum_{|\alpha| = m} a_{0\alpha}(0, y) \eta^\alpha = b\sigma(P)(0, y, 0, \eta).$$

If $P$ is elliptic, then $\hat{P}(\sigma)$ is elliptic.

More is true because

$$\sum_{k+|\alpha| = m} a_{k\alpha}(x, y) \sigma^k \eta^\alpha \neq 0 \text{ if } \Re \sigma \text{ is large relative to } |\Im \sigma|.$$ 

$$\sum_{k+|\alpha| = m} a_{k\alpha}(0, y) \sigma^k \eta^\alpha = \sum_{k+|\alpha| = m} a_{k\alpha}(0, y) (\Re \sigma + i\Im \sigma)^k \eta^\alpha$$

$$= \sum_{k+|\alpha| = m} a_{k\alpha}(0, y) (\Re \sigma)^k (1 + i\Im \sigma/\Re \sigma)^k \eta^\alpha$$

$$= b\sigma(P)(0, y, \Re \sigma, \eta) (I + O(|\Re \sigma|^{-1}))$$

if $|\Im \sigma/\Re \sigma| < c$ with small enough $c > 0$. 

G. A. Mendoza (Temple University)
Using the invertibility of
\[ \sum_{k + |\alpha| = m} a_{k\alpha}(0, y) \sigma^k \eta^\alpha \]
(with \(|\text{Im} \sigma| < c|\text{Re} \sigma|\)) one finds a family of pseudodifferential operators \(Q(\sigma)\) on \(\mathcal{N}\) such that
\[ Q(\sigma) \hat{P}(\sigma) = I - R(\sigma) \]
with \(R(\sigma)\) of order \(-1\) and \(\|R(\sigma)\| \leq C/|\text{Re} \sigma|\).
Using the invertibility of
\[ \sum_{k+|\alpha|=m} a_{k\alpha}(0, y)\sigma^k \eta^\alpha \]
(with \(|\text{Im} \sigma| < c|\text{Re} \sigma|\)) one finds a family of pseudodifferential operators \(Q(\sigma)\) on \(\mathcal{N}\) such that
\[ Q(\sigma) \hat{P}(\sigma) = I - R(\sigma) \]
with \(R(\sigma)\) of order \(-1\) and \(\|R(\sigma)\| \leq C/|\text{Re} \sigma|\). Consequently:

\[ \hat{P}(\sigma) \text{ is invertible in } |\text{Im} \sigma| < c|\text{Re} \sigma|. \]
Using the invertibility of
\[ \sum_{k+|\alpha|=m} a_{k\alpha}(0, y) \sigma^k \eta^\alpha \]
(with \(|\text{Im} \sigma| < c|\text{Re} \sigma|\)) one finds a family of pseudodifferential operators \( Q(\sigma) \) on \( \mathcal{N} \) such that
\[ Q(\sigma)\hat{P}(\sigma) = I - R(\sigma) \]
with \( R(\sigma) \) of order \(-1\) and \( \| R(\sigma) \| \leq C/|\text{Re} \sigma| \). Consequently:
\( \hat{P}(\sigma) \) is invertible in \(|\text{Im} \sigma| < c|\text{Re} \sigma|\). Since
\[ \hat{P}(\sigma) : H^m(\mathcal{N}) \subset L^2(\mathcal{N}) \to L^2(\mathcal{N}) \]
is a holomorphic Fredholm family which is invertible at some point, the set
\[ \text{spec}_b(P) = \{ \sigma \in \mathbb{C} : \hat{P}(\sigma) \text{ is not invertible} \} \]
is discrete without points of accumulation.
Using the invertibility of
\[ \sum_{k+|\alpha|=m} a_{k\alpha}(0, y)\sigma^k \eta^\alpha \]
(with \(|\text{Im } \sigma| < c|\text{Re } \sigma|\)) one finds a family of pseudodifferential operators \(Q(\sigma)\) on \(\mathcal{N}\) such that
\[ Q(\sigma)\hat{P}(\sigma) = I - R(\sigma) \]
with \(R(\sigma)\) of order \(-1\) and \(\|R(\sigma)\| \leq C/|\text{Re } \sigma|\). Consequently:

\(\hat{P}(\sigma)\) is invertible in \(|\text{Im } \sigma| < c|\text{Re } \sigma|\). Since
\[ \hat{P}(\sigma) : H^m(\mathcal{N}) \subset L^2(\mathcal{N}) \to L^2(\mathcal{N}) \]
is a holomorphic Fredholm family which is invertible at some point,
the set
\[ \text{spec}_b(P) = \{ \sigma \in \mathbb{C} : \hat{P}(\sigma) \text{ is not invertible} \} \]
is discrete without points of accumulation. For any \(a > 0\), the set
\[ \text{spec}_b(P) \cap \{ \sigma : |\text{Im } \sigma| < a \} \] is finite.
Sobolev spaces

1. If $s$ is a nonnegative integer, then $H^s_b(M)$ consists of all $u \in L^2_b(M)$ such that

$$X_1 \ldots X_k u \in L^2_b(M) \text{ for all } X_1, \ldots, X_k \in C^\infty(M; bT^*M), \ k \leq s$$

2. The space $H^{-s}_b(M)$ is the dual of $H^s_b(M)$
3. If $s$ is not an integer, then $H^s_b(M)$ is defined by interpolation.
4. If $s$ and $\nu$ are real numbers, then $x^\nu H^s_b(M) = \{x^\nu u : u \in H^s_b(M)\}$.

If $P \in \text{Diff}^m_b(M; E, F)$, then $P : x^\nu H^s_b(M; E) \rightarrow x^\nu H^{s-m}_b(M; F)$ is continuous.
Elliptic regularity

Let $P \in \text{Diff}^m_b(\mathcal{M}; E, F)$ be $b$-elliptic. If $u \in x^\nu H^s_b(\mathcal{M}; E)$ and $Pu \in x^\nu H^s_b(\mathcal{M}; F)$, then $u \in x^\nu H^{s+m}_b(\mathcal{M}; E)$.

The proof is by construction of an operator $Q$ such that

$$QP = I - R$$

with $Q : x^\nu H^t_b \to x^\nu H^{t+m}_b$ and $R : x^\nu H^t_b \to x^\nu H^\infty_b$ for any $t$. 
Parametrices

The operator $Q$ is obtained by constructing its parametrix. Suppose for the moment that

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$$

is an operator on an open set $U \subset \mathbb{R}^n$. If $u \in C_c^\infty(U)$ then

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left[ \int_{U} e^{-ix' \cdot \xi} u(x') \, dx' \right] d\xi$$

so

$$Pu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \left[ \int_{U} e^{-ix' \cdot \xi} u(x') \, dx' \right] d\xi$$

Let $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$. If $P$ is elliptic, that is,

$$\sigma(P)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

is invertible, then $p(x, \xi)$ is invertible for large $\xi$. 
Parametrices

The operator $Q$ is obtained by constructing its parametrix. Suppose for the moment that

$$ P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha $$

is an operator on an open set $U \subset \mathbb{R}^n$. If $u \in C_c^\infty(U)$ then

$$ u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left[ \int_U e^{-ix' \cdot \xi} u(x') \, dx' \right] \, d\xi $$

so

$$ Pu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \left[ \int_U e^{-ix' \cdot \xi} u(x') \, dx' \right] \, d\xi $$

Let $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$. If $P$ is elliptic, that is,

$$ \sigma(P)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha $$

$$ Pu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} p(x, \xi) u(x') \, dx' \, d\xi $$

is invertible, then $p(x, \xi)$ is invertible for large $\xi$. 
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. 
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. 
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}$$

So if

$$Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} q(x, \xi)u(x') \, dx' \, d\xi$$

then

$$PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} \chi(\xi)u(x') \, dx' \, d\xi$$

$$+ \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \binom{\alpha}{\beta} a_\alpha(x)\xi^\beta D_x^{\alpha-\beta} q(x, \xi)u(x') \, dx' \, d\xi$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}$$

So if

$$Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} q(x, \xi) u(x') \, dx' \, d\xi$$

then

$$PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi + \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) a_\alpha(x) \xi^\beta D_x^{\alpha-\beta} q(x, \xi) u(x') \, dx' \, d\xi$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}$$

So if

$$Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} q(x, \xi) u(x') \, dx' \, d\xi$$

then

$$PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi + \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) a_\alpha(x) \xi^\beta D_x^{\alpha-\beta} q(x, \xi) u(x') \, dx' \, d\xi$$

$$= u(x) + \int \check{\chi}(x-x') u(x') \, dx' \, d\xi = u(x) + (\check{\chi} * u)(x)$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x, \xi) = \chi(\xi) p(x, \xi)^{-1}$$

So if

$$Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} q(x, \xi) u(x') \, dx' \, d\xi$$

then

$$PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi$$

$$+ \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) a_{\alpha}(x) \xi^\beta D_x^{\alpha-\beta} q(x, \xi) u(x') \, dx' \, d\xi$$

$$\frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} u(x') \, dx' \, d\xi + \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} (\chi(\xi) - 1) u(x') \, dx' \, d\xi$$

$$= u(x) + \int \check{\chi}(x-x') u(x') \, dx' \, d\xi = u(x) + (\check{\chi} * u)(x)$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}$$

So if

$$Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} q(x, \xi) u(x') \, dx' \, d\xi$$

then

$$PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi$$

$$+ \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \binom{\alpha}{\beta} a_\alpha(x) \xi^\beta D^{\alpha-\beta}_x q(x, \xi) u(x') \, dx' \, d\xi$$

$$+ \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi$$

$$= u(x) + \int \tilde{\chi}(x-x') u(x') \, dx' \, d\xi = u(x) + (\tilde{\chi} * u)(x)$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}$$

So if

$$Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} q(x, \xi) u(x') \, dx' \, d\xi$$

then

$$PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi$$

$$+ \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) a_{\alpha}(x) \xi^\beta D_x^{\alpha-\beta} q(x, \xi) u(x') \, dx' \, d\xi$$

$$\frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi = \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} u(x') \, dx' \, d\xi + \frac{1}{(2\pi)^n} \int e^{i(x-x') \cdot \xi} (\chi(\xi) - 1) u(x') \, dx' \, d\xi$$

$$= u(x) + \int \tilde{\chi}(x - x') u(x') \, dx' \, d\xi = u(x) + (\tilde{\chi} \ast u)(x)$$
Suppose $P = p(x, D_x)$ is elliptic. Then $p(x, \xi)$ is invertible for large $|\xi|$, say if $|\xi| > C$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ if $|\xi| < 2C$ and $\chi(\xi) = 1$ if $|\xi| > 3C$. Let

$$q(x, \xi) = \chi(\xi)p(x, \xi)^{-1}$$

So if

$$Qu(x) = \frac{1}{(2\pi)^n} \int e^{i(\xi-x') \cdot \xi} q(x, \xi) u(x') \, dx' \, d\xi$$

then

$$PQu(x) = \frac{1}{(2\pi)^n} \int e^{i(\xi-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi$$

$$+ \frac{1}{(2\pi)^n} \int e^{i(\xi-x') \cdot \xi} \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \binom{\alpha}{\beta} a_\alpha(x) \xi^\beta D_x^{\alpha-\beta} q(x, \xi) u(x') \, dx' \, d\xi$$

$$\frac{1}{(2\pi)^n} \int e^{i(\xi-x') \cdot \xi} \chi(\xi) u(x') \, dx' \, d\xi = \frac{1}{(2\pi)^n} \int e^{i(\xi-x') \cdot \xi} u(x') \, dx' \, d\xi + \frac{1}{(2\pi)^n} \int e^{i(\xi-x') \cdot \xi} (\chi(\xi) - 1) u(x') \, dx' \, d\xi$$

$$= u(x) + \int \tilde{\chi}(x-x') u(x') \, dx' \, d\xi = u(x) + (\tilde{\chi} \ast u)(x)$$

$$PQ = I - R$$
where $R$ improves Sobolev regularity by 1.
Suppose \( q(x, \xi) \) is smooth and
\[
|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C(1 + |\xi|)^{m-|\beta|}
\]
for some \( m \). Then
\[
\int e^{i(x-x')\cdot \xi} q(x, \xi) d\xi
\]
has singularities only on \( x = x' \):
\[
(x - x')^\beta e^{i(x-x')\cdot \xi} = D_\xi^\beta e^{i(x-x')\cdot \xi}
\]
so
\[
K_\beta(x, x') = (x - x')^\beta \int e^{i(x-x')\cdot \xi} q(x, \xi) d\xi
\]
Suppose $q(x, \xi)$ is smooth and

$$|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C(1 + |\xi|)^{m-|\beta|}$$

for some $m$. Then

$$\int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi$$

has singularities only on $x = x'$:

$$(x - x')^\beta e^{i(x-x') \cdot \xi} = D_\xi^\beta e^{i(x-x') \cdot \xi}$$

so

$$K_\beta(x, x') = (x - x')^\beta \int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi$$

$$= (-1)^{|\beta|} \int e^{i(x-x') \cdot \xi} D_\xi^\beta q(x, \xi) \, d\xi$$

If $m - |\beta| < -n$ then $D_\xi^\beta q(x, \xi)$ is integrable, so $K_\beta$ is continuous.
Suppose \( q(x, \xi) \) is smooth and
\[
|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C(1 + |\xi|)^{m-|\beta|}
\]
for some \( m \). Then
\[
\int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi
\]
has singularities only on \( x = x' \):
\[
(x - x')^\beta e^{i(x-x') \cdot \xi} = D_\xi^\beta e^{i(x-x') \cdot \xi}
\]
so
\[
K_\beta(x, x') = (x - x')^\beta \int e^{i(x-x') \cdot \xi} q(x, \xi) \, d\xi
= (-1)^{|\beta|} \int e^{i(x-x') \cdot \xi} D_\xi^\beta q(x, \xi) \, d\xi
\]
If \( m - |\beta| < -n \) then \( D_\xi^\beta q(x, \xi) \) is integrable, so \( K_\beta \) is continuous.
If \( m - |\beta| < -n - k \) then \( K_\beta \in C^k \).
Suppose $q(x, \xi)$ is smooth and
\[
|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C(1 + |\xi|)^{m-|\beta|}
\]
for some $m$. Then
\[
\int e^{i(x-x')\cdot\xi} q(x, \xi) \, d\xi
\]
has singularities only on $x = x'$:
\[
(x - x')^\beta e^{i(x-x')\cdot\xi} = D_\xi^\beta e^{i(x-x')\cdot\xi}
\]
so
\[
K_\beta(x, x') = (x - x')^\beta \int e^{i(x-x')\cdot\xi} q(x, \xi) \, d\xi
\]
\[
= (-1)^{|\beta|} \int e^{i(x-x')\cdot\xi} D_\xi^\beta q(x, \xi) \, d\xi
\]
If $m - |\beta| < -n$ then $D_\xi^\beta q(x, \xi)$ is integrable, so $K_\beta$ is continuous.
If $m - |\beta| < -n - k$ then $K_\beta \in C^k$. 

Suppose $P \in \text{Diff}^{m}_b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\dot{\mathcal{M}} \times \dot{\mathcal{M}}$ we know that $K_Q$ is locally given by

$$\int \, e^{i(z(p) - z(p'))} \cdot q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. 
Suppose \( P \in \text{Diff}^m_b \) is \( b \)-elliptic. To construct \( Q \) such that \( PQ = I - R \) with “good” \( R \) we try to find \( K_Q \) defined on \( \mathcal{M} \times \mathcal{M} \) such that

\[
Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')
\]

In \( \mathcal{M} \times \mathcal{M} \) we know that \( K_Q \) is locally given by

\[
\int e^{i(z(p) - z(p')) \cdot \xi} q(p, p', \xi) \, d\xi.
\]

This has singularities on the diagonal in \( \mathcal{M} \times \mathcal{M} \).
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\hat{\mathcal{M}} \times \hat{\mathcal{M}}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p')) \cdot \xi} q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. 
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\mathring{\mathcal{M}} \times \mathring{\mathcal{M}}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p'))} \cdot \xi \, q(p, p', \xi) \, d\xi.$$

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. Resolve the singularity.
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p')) \cdot \xi} q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. Resolve the singularity: Let

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}, \quad r \geq 0, \ s \in [-1, 1]$$
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p'))} \xi q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. Resolve the singularity: Let

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}, \quad r \geq 0, \quad s \in [-1, 1]$$
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p'))} \xi q(p, p', \xi) \, d\xi. $$

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. Resolve the singularity: Let

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}, \quad r \geq 0, \; s \in [-1, 1]$$

This results in a new smooth manifold with corners, $\mathcal{M} \tilde{\times} \mathcal{M}$. 
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p'))} \cdot \xi \, q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. Resolve the singularity: Let

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}, \quad r \geq 0, \quad s \in [-1, 1]$$

This results in a new smooth manifold with corners, $\mathcal{M} \tilde{\times} \mathcal{M}$. It has a left face,
Suppose $P \in \text{Diff}^b_m$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\tilde{\mathcal{M}} \times \tilde{\mathcal{M}}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p'))} \cdot \xi \, q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. Resolve the singularity: Let

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}, \quad r \geq 0, \quad s \in [-1, 1]$$

This results in a new smooth manifold with corners, $\mathcal{M} \tilde{x} \mathcal{M}$. It has a left face, a right face,
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p'))} \xi \, q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. Resolve the singularity: Let

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}, \quad r \geq 0, \quad s \in [-1, 1]$$

This results in a new smooth manifold with corners, $\mathcal{M} \times \mathcal{M}$. It has a left face, a right face, and a “new” face, the “front face” $r = 0$. 

\[\text{Diagram showing the lifting of the diagonal and the intersecting faces.}\]
Suppose $P \in \text{Diff}^m_b$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\hat{M} \times \hat{M}$ such that

$$Qf(p) = \int_{\hat{M}} K(p, p') \, dm(p')$$

In $\hat{M} \times \hat{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p) - z(p')) \cdot \xi} q(p, p', \xi) \, d\xi.$$ 

This has singularities on the diagonal in $\hat{M} \times \hat{M}$. The diagonal intersects the corner of $\hat{M} \times \hat{M}$. Resolve the singularity: Let

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}, \quad r \geq 0, \; s \in [-1, 1]$$

This results in a new smooth manifold with corners, $\hat{M} \check{\times} \hat{M}$. It has a left face, a right face, and a “new” face, the “front face” $r = 0$. The lifting of the diagonal in $\hat{M} \times \hat{M}$ has closure

$$\check{\Delta} = \{ s = 0, \; y = y' \}.$$
Suppose $P \in \text{Diff}_b^m$ is $b$-elliptic. To construct $Q$ such that $PQ = I - R$ with “good” $R$ we try to find $K_Q$ defined on $\mathcal{M} \times \mathcal{M}$ such that

$$Qf(p) = \int_{\mathcal{M}} K(p, p') \, dm(p')$$

In $\mathcal{M} \times \mathcal{M}$ we know that $K_Q$ is locally given by

$$\int e^{i(z(p)-z(p'))} \cdot \xi \, q(p, p', \xi) \, d\xi.$$  

This has singularities on the diagonal in $\mathcal{M} \times \mathcal{M}$. The diagonal intersects the corner of $\mathcal{M} \times \mathcal{M}$. Resolve the singularity: Let

$$r = \frac{x + x'}{2}, \quad s = \frac{x - x'}{x + x'}, \quad r \geq 0, \quad s \in [-1, 1]$$

This results in a new smooth manifold with corners, $\mathcal{M} \tilde{\times} \mathcal{M}$. It has a left face, a right face, and a “new” face, the “front face” $r = 0$. The lifting of the diagonal in $\mathcal{M} \times \mathcal{M}$ has closure

$$\tilde{\Delta} = \{s = 0, \ y = y'\}.$$  

It intersects $\partial_{ff} \mathcal{M} \tilde{\times} \mathcal{M}$ transversally.

\[\text{(Diagram of manifold)}\]
Let $\varphi : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ be the blow-down map. View the operator $P$ as acting in the first factor in $\mathcal{M} \times \mathcal{M}$. 

The resulting lifted operator is $\tilde{\pi}^* LP = \sum \alpha \left((1 + s) r, y\right) \left(\frac{1}{2}\left((1 + s) r \partial r + (1 - s) \partial s\right)\right)^k D^\alpha y$. 

\[x \partial x = \varphi^* \left(\frac{1}{2}\left((1 + s) r \partial r + (1 - s) \partial s\right)\right),\]

\[x' \partial x' = \varphi^* \left(\frac{1}{2}\left((1 - s) r \partial r - (1 - s) \partial s\right)\right),\]

etc.
Let \( \varphi : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M} \) be the blow-down map. View the operator \( P \) as acting in the first factor in \( \mathcal{M} \times \mathcal{M} \). Let

\[
\begin{array}{c}
\pi_L \downarrow \\
\mathcal{M} \\
\mathcal{M} \times \mathcal{M}
\end{array}
\]
Let $\varphi : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ be the blow-down map. View the operator $P$ as acting in the first factor in $\mathcal{M} \times \mathcal{M}$. Let

$$
\begin{array}{c}
\mathcal{M} \\
\mathcal{M} \times \mathcal{M} \\
\mathcal{M}
\end{array}
\begin{array}{c}
\xrightarrow{\pi_L} \\
\xrightarrow{\pi_R} \\
\xrightarrow{\pi_L}
\end{array}
\begin{array}{c}
\mathcal{M}
\end{array}
$$

The resulting lifted operator is $\tilde{\pi}^* L P = \sum_{k=0}^{m} \alpha_k ((1 + s) r, y) \left( \frac{1}{2} \left( (1 + s) r \partial r + (1 - s) \partial s \right) \right)^k \partial y^j$, etc.
Let \( \wp : \mathcal{M} \tilde{\times} \mathcal{M} \to \mathcal{M} \times \mathcal{M} \) be the blow-down map. View the operator \( P \) as acting in the first factor in \( \mathcal{M} \times \mathcal{M} \). Let

\[
\mathcal{M} \tilde{\times} \mathcal{M} \xrightarrow{\wp} \mathcal{M} \times \mathcal{M}
\]

\[
\begin{array}{c}
\mathcal{M} \tilde{\times} \mathcal{M} \\
\xrightarrow{\wp} \\
\mathcal{M} \times \mathcal{M} \\
\end{array}
\]

\[
\begin{array}{c}
\pi_L \\
\mathcal{M} \\
\pi_R \\
\mathcal{M} \\
\end{array}
\]

Lift \( P \) through the left factor. Since \( x = r(1 + s) \), \( x' = r(1 - s) \), we have

\[
x \partial x = \wp^* \left( \frac{1}{2} \left[ (1 + s)r \partial r + (1 - s)^2 \partial s \right] \right),
\]

\[
x' \partial x' = \wp^* \left( \frac{1}{2} \left[ (1 - s)r \partial r - (1 - s)^2 \partial s \right] \right)
\]

and

\[
\wp^* (\partial y_j) = \partial y_j,
\]

etc.

The resulting lifted operator is

\[
\hat{\pi}^* L P = \sum_{k + |\alpha| \leq m} a_k \alpha ((1 + s)r, y) \left( \frac{1}{2} \left[ (1 + s)r \partial r + (1 - s)^2 \partial s \right] \right)^k \partial^\alpha y
\]
Let $\varphi : \tilde{M} \times \tilde{M} \to M \times M$ be the blow-down map. View the operator $P$ as acting in the first factor in $M \times M$. Let

\[
\begin{array}{ccc}
\tilde{\pi}_L & \xrightarrow{\varphi} & \pi_L \\
M \times M & \xrightarrow{\varphi} & M \\
\tilde{\pi}_R & \xrightarrow{\varphi} & \pi_R \\
\end{array}
\]

Lift $P$ through the left factor. Since

\[
x = \frac{r(1 + s)}{2}, \quad x' = \frac{r(1 - s)}{2},
\]

we have

\[
x \partial_x = \varphi_*(\frac{1}{2}[(1 + s)r \partial_r + (1 - s^2)\partial_s]),
\]

\[
x' \partial_{x'} = \varphi_*(\frac{1}{2}[(1 - s)r \partial_r - (1 - s^2)\partial_s])
\]

and $\varphi_*(\partial y^j) = \partial y^j$, etc.
Let \( \varphi : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M} \) be the blow-down map. View the operator \( P \) as acting in the first factor in \( \mathcal{M} \times \mathcal{M} \). Let 

\[
\begin{array}{c}
\mathcal{M} \times \mathcal{M} \\
\downarrow \varphi \downarrow \pi_L \downarrow \\
\mathcal{M} \times \mathcal{M} \\
\downarrow \pi_L \downarrow \\
\mathcal{M} \\
\downarrow \pi_R \\
\mathcal{M} \times \mathcal{M} \\
\end{array}
\]

Lift \( P \) through the left factor. Since 

\[
x = \frac{r(1 + s)}{2}, \quad x' = \frac{r(1 - s)}{2},
\]

we have 

\[
x \partial_x = \varphi_* \left( \frac{1}{2} [(1 + s) r \partial_r + (1 - s^2) \partial_s] \right),
\]

\[
x' \partial_{x'} = \varphi_* \left( \frac{1}{2} [(1 - s) r \partial_r - (1 - s^2) \partial_s] \right)
\]

and \( \varphi_*(\partial y^j) = \partial y^j \), etc. The resulting lifted operator is 

\[
\tilde{\pi}_L^* P = \sum_{k+|\alpha| \leq m} a_{k,\alpha} ((1 + s) r, y) \left( \frac{1}{2} [(1 + s) r D_r + (1 - s^2) D_s] \right)^k D_y^\alpha
\]
From previous slide,

\[ \tilde{\pi}_L^* P = \sum_{k + |\alpha| \leq m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} \left[ (1 + s)rD_r + (1 - s^2)D_s \right] \right)^k D_y^\alpha \]
From previous slide,

\[ \tilde{\pi}_L^* P = \sum_{k+|\alpha| \leq m} a_{k\alpha}((1 + s) r, y) \left( \frac{1}{2} \left[ (1 + s) r D_r + (1 - s^2) D_s \right] \right)^k D_y^\alpha \]

The principal symbol of \( \tilde{\pi}_L^* P \) is

\[ \sum_{k+|\alpha| = m} a_{k\alpha}((1 + s) r, y) \left( \frac{1}{2} \left[ (1 + s) \rho + (1 - s^2) \sigma \right] \right)^k \eta^\alpha \]

Setting \( s = 0 \) and \( \rho = 0 \) (to see what happens on the conormal bundle of the lifted diagonal) we get:

\[ \sum_{k+|\alpha| = m} a_{k\alpha}(r, y) \left( \frac{1}{2} \sigma \right)^k \eta^\alpha \]
From previous slide,

\[ \tilde{\pi}_L^* P = \sum_{k+|\alpha| \leq m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2}[(1 + s)rD_r + (1 - s^2)D_s] \right)^{k} D_y^{\alpha} \]

The principal symbol of \( \tilde{\pi}_L^* P \) is

\[ \sum_{k+|\alpha| = m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2}[(1 + s)\rho + (1 - s^2)\sigma] \right)^{k} \eta^{\alpha} \]

Setting \( s = 0 \) and \( \rho = 0 \) (to see what happens on the conormal bundle of the lifted diagonal) we get:

\[ \sum_{k+|\alpha| = m} a_{k\alpha}(r, y) \left( \frac{1}{2}\sigma \right)^{k} \eta^{\alpha} \]

This is

\[ b_\sigma(P)(r, y, \frac{1}{2}\sigma, \eta) \]

which is invertible (\( b \)-ellipticity).
From previous slide,

\[ \tilde{\pi}_L^* P = \sum_{k+|\alpha| \leq m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} [(1 + s)rD_r + (1 - s^2)D_s] \right)^k D_y^\alpha \]

The principal symbol of $\tilde{\pi}_L^* P$ is

\[ \sum_{k+|\alpha| = m} a_{k\alpha}((1 + s)r, y) \left( \frac{1}{2} [(1 + s)\rho + (1 - s^2)\sigma] \right)^k \eta^\alpha \]

Setting $s = 0$ and $\rho = 0$ (to see what happens on the conormal bundle of the lifted diagonal) we get:

\[ \sum_{k+|\alpha| = m} a_{k\alpha}(r, y) \left( \frac{1}{2} \sigma \right)^k \eta^\alpha \]

This is

\[ b_\sigma(P)(r, y, \frac{1}{2} \sigma, \eta) \]

which is invertible ($b$-ellipticity).
Further,
\[
\tilde{\pi}_L^* P = \sum_{k + |\alpha| \leq m} a_{k\alpha}((1 + s)r, y)\left(\frac{1}{2}[(1 + s)rD_r + (1 - s^2)D_s]\right)^k D^\alpha_y
\]
extends smoothly across \( s = 0 \). So there is a distribution \( \tilde{\mathcal{K}}_Q \) on \( \tilde{\mathcal{M}} \tilde{\times} \mathcal{M} \) which is the restriction of a distribution on an extension of \( \tilde{\mathcal{M}} \tilde{\times} \mathcal{M} \) across \( \partial_{ff} \mathcal{M} \tilde{\times} \mathcal{M} \) supported near \( \tilde{\Delta} \) such that
\[
\tilde{\pi}_L^* P \tilde{\mathcal{K}}_Q = \delta(s)\delta(y - y') - \tilde{\mathcal{K}}_R
\]
where \( \tilde{\mathcal{K}}_R \) is smooth. The distribution \( \tilde{\mathcal{K}}_Q \) and function \( \tilde{\mathcal{K}}_R \) descend form the interior of \( \mathcal{M} \tilde{\times} \mathcal{M} \) to \( \tilde{\mathcal{M}} \times \tilde{\mathcal{M}} \), give a distribution and function \( \mathcal{K}_Q, K_R \) such that
\[
\pi_L^* P K_Q = \delta_\Delta - K_R.
\]
This gives
\[
P Q = I - R
\]
\[
Qf(p) = \int_{\mathcal{M}} K_Q(p, p')u(p')
\]
likewise \( R \) with \( K_R \).
Further,
\[ \tilde{\pi}_L^*P = \sum_{k+|\alpha| \leq m} a_{k\alpha}((1+s)r, y)\left(\frac{1}{2}[(1+s)rD_r + (1-s^2)D_s]\right)^kD_y^\alpha \]
extends smoothly across \( s = 0 \). So there is a distribution \( \tilde{\psi}_Q \) on \( \tilde{\mathcal{M}} \) which is the restriction of a distribution on an extension of \( \tilde{\mathcal{M}} \) across \( \partial \tilde{\mathcal{M}} \) supported near \( \tilde{\Delta} \) such that
\[ \tilde{\pi}_L^*P\tilde{\psi}_Q = \delta(s)\delta(y - y') - \tilde{\psi}_R \]
where \( \tilde{\psi}_R \) is smooth. The distribution \( \tilde{\psi}_Q \) and function \( \tilde{\psi}_R \) descend form the interior of \( \tilde{\mathcal{M}} \) to \( \tilde{\mathcal{M}} \times \tilde{\mathcal{M}} \), give a distribution and function \( \psi_Q, \psi_R \) such that
\[ \pi_L^*P\psi_Q = \delta_\Delta - \psi_R. \]
This gives
\[ PQ = I - R \]
likewise \( R \) with \( \psi_R \)

The operator \( x^\nu Qx^{-\nu} \) maps \( x^\nu H_b^t \) to \( x^\nu H_b^{t+m} \) for arbitrary \( t \) and \( \nu \) \( x^\nu Rx^{-\nu} \) maps \( x^\nu H_b^t \) to \( x^\nu H_b^\infty \).
Asymptotics

Suppose $P$ is $b$-elliptic and $u \in \times^\nu H^s_b$ is such that $Pu = 0$ (or $Pu \in \times^\infty H^\infty_b$). Then $u \in \times^\nu H^\infty_b$: With $Q$ and $R$ as just constructed,

$$Pu = f,$$
Asymptotics

Suppose $P$ is $b$-elliptic and $u \in x^\nu H^s_b$ is such that $Pu = 0$ (or $Pu \in x^\infty H^\infty_b$). Then $u \in x^\nu H^\infty_b$: With $Q$ and $R$ as just constructed,

$$Pu = f, \quad Qf = QPu = u - Ru$$
Asymptotics

Suppose $P$ is $b$-elliptic and $u \in x^\nu H^s_b$ is such that $Pu = 0$ (or $Pu \in x^\infty H^\infty_b$). Then $u \in x^\nu H^\infty_b$: With $Q$ and $R$ as just constructed,

$$Pu = f, \quad Qf = QPu = u - Ru$$

The operator $x^{-\mu}Qx^\mu$ is of the same kind as $Q$, so

$$x^{-\nu}Qf = x^{-\nu}Qx^\nu(x^{-\nu}f) \in H^\infty_b \quad \text{since} \quad x^{-\nu}f \in H^\infty_b$$

Since $Ru \in x^\nu H^\infty_b$,

$$u = Qf + Ru \in x^\nu H^\infty_b.$$
Suppose $u \in x^{\nu}L^2_b$ and $Pu \in x^\infty H^\infty_b$. Then

$\mathcal{M}(u)(\sigma, y) = \int x^{-i\sigma} u(x, y) \frac{dx}{x}$

is holomorphic in $\text{Im} \sigma > -\nu$. (assume $u$ supported near $\partial \mathcal{M}$)
Suppose \( u \in x^\nu L_b^2 \) and \( Pu \in x^\infty H_b^\infty \). Then

\[
\mathcal{M}(u)(\sigma, y) = \int x^{-i\sigma} u(x, y) \frac{dx}{x}
\]

is holomorphic in \( \text{Im} \sigma > -\nu \). Near \( \partial \mathcal{M} \),

\[
u(x, y) = \frac{1}{2\pi} \int_{\text{Im} \sigma = t} x^{i\sigma} \mathcal{M}(u)(\sigma, y) d\sigma
\]

with \( t \geq -\nu \).
Suppose \( u \in x^{\nu}L^2_b \) and \( Pu \in x^\infty H^\infty_b \). Then
\[
\mathcal{M}(u)(\sigma, y) = \int x^{-i\sigma} u(x, y) \frac{dx}{x}
\]
is holomorphic in \( \text{Im} \sigma > -\nu \). Near \( \partial \mathcal{M} \),
\[
u(x, y) = \frac{1}{2\pi} \int_{\text{Im} \sigma = t} x^{i\sigma} \mathcal{M}(u)(\sigma, y) d\sigma
\]
with \( t \geq -\nu \). We have \( P(u) \in \dot{C}^\infty(M) \) so \( \mathcal{M}(Pu) \) is entire. Using the
Taylor expansion of \( P \) at \( x = 0 \):
\[
P = \sum_{\ell=0}^{N} x^{\ell} P_\ell(y, xD_x, D_y) + x^{N+1} \tilde{P}_{N+1}(x, y, xD_x, D_y)
\]
we get
\[
\mathcal{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} \int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x, y) \frac{dx}{x}
\]
\[
+ \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x}
\]
The left hand side of
\[
\mathcal{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} \int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x, y) \frac{dx}{x} 
\]
\[
+ \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x}
\]

is entire. Since \( P_\ell(y, xD_x, D_y) u \in x^\nu H^\infty_b \), the term
\[
\int x^{-i(\sigma+i\ell)} P_\ell(y, xD_x, D_y) u(x, y) \frac{dx}{x} = P_\ell(y, \sigma, D_y) \int x^{-i(\sigma+i\ell)} u(x, y) \frac{dx}{x}
\]
is holomorphic in \( \text{Im} (\sigma + i\ell) > -\nu \), that is, \( \text{Im} \sigma > -\nu - \ell \). Likewise the reminder gives a term holomorphic in \( \text{Im} \sigma > -\nu - N - 1 \).
The left hand side of
\[
\mathcal{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} \int x^{-i(\sigma+i\ell)} P_{\ell}(y, xD_x, D_y) u(x, y) \frac{dx}{x} + \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x}
\]
is entire. Since \(P_{\ell}(y, xD_x, D_y) u \in x^\nu H^\infty_b\), the term
\[
\int x^{-i(\sigma+i\ell)} P_{\ell}(y, xD_x, D_y) u(x, y) \frac{dx}{x} = P_{\ell}(y, \sigma, D_y) \int x^{-i(\sigma+i\ell)} u(x, y) \frac{dx}{x}
\]
is holomorphic in \(\text{Im} (\sigma + i\ell) > -\nu\), that is, \(\text{Im} \sigma > -\nu - \ell\). Likewise the reminder gives a term holomorphic in \(\text{Im} \sigma > -\nu - N - 1\). So (with \(N = 0\)):
\[
P_0(y, \sigma, D_y) \mathcal{M}(u)(\sigma) = \mathcal{M}(Pu)(\sigma) - \int x^{-i(\sigma+i)} \tilde{P}_1(x, y, xD_x, D_y) u(x, y) \frac{dx}{x}
\]
is holomorphic in \(\text{Im} \sigma > -\nu - 1\). But \(P_0(y, \sigma, D_y) = \hat{P}(\sigma)\).
From previous slide:

\[ P_0(y, \sigma, D_y) \mathcal{M}(u)(\sigma) = \mathcal{M}(Pu)(\sigma) - \int x^{-i(\sigma+i)} \tilde{P}_1(x, y, xD_x, D_y)u(x, y) \frac{dx}{x} \]

is holomorphic in \( \text{Im} \sigma > -\nu - 1 \) and \( P_0(y, \sigma, D_y) = \hat{P}(\sigma) \). So

\[ \mathcal{M}(u)(\sigma) = \hat{P}(\sigma)^{-1} \left[ \mathcal{M}(Pu)(\sigma) - \int x^{-i(\sigma+i)} \tilde{P}_1(x, y, xD_x, D_y)u(x, y) \frac{dx}{x} \right] \]

which is holomorphic in \( \text{Im} \sigma > -\nu \), extends as a meromorphic function to \( \text{Im} \sigma > -\nu - 1 \) with poles in \( \text{spec}_b(P) \cap \{-\nu - 1 < \text{Im} \sigma < \nu\} \).
In general,

\[ \mathcal{M}(Pu)(\sigma) = \sum_{\ell=0}^{N} P_\ell(y, \sigma, D_y) \mathcal{M}(u)(\sigma + i\ell) \]

\[ + \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x} \]

gives

\[ \hat{P}(\sigma) \mathcal{M}(u)(\sigma) = \mathcal{M}(Pu)(\sigma) - \sum_{\ell=1}^{N} P_\ell(y, \sigma, D_y) \mathcal{M}(u)(\sigma + i\ell) \]

\[ - \int x^{-i(\sigma+i(N+1))} \tilde{P}_{N+1}(x, y, xD_x, D_y) u(x, y) \frac{dx}{x} \]

Once one has shown that \( \mathcal{M}(u)(\sigma) \) has a meromorphic extension to \( \text{Im} \sigma > -\nu - N \), the right hand side is meromorphic in \( \text{Im} \sigma > -\nu - N - 1 \), and so \( \mathcal{M}(u)(\sigma) \) has a meromorphic extension to \( \text{Im} \sigma > -\nu - N - 1 \).
The poles of $\mathcal{M}(u)$ are contained in

$$\{\sigma - i\ell : \sigma \in \text{spec}_b(P), \ \text{Im} \sigma < -\nu, \ k = 0, 1, \ldots \}$$

Using the Mellin inversion formula:

$$u(x, y) = \frac{1}{2\pi} \int_{\text{Im} \sigma = t} x^{i\sigma} \mathcal{M}(u)(\sigma, y) \, d\sigma$$

(with some $t > -\nu$) one gets

$$u(x, y) = \sum_{\sigma \in \text{spec}_b(P)} \sum_{k=0}^{N_{\sigma, \ell}} x^{i\sigma + \ell} u_{\sigma, \ell, k}(y) \log^k x + \frac{1}{2\pi} \int_{\text{Im} \sigma = t-s} x^{i\sigma} \mathcal{M}(u)(\sigma, y) \, d\sigma$$

($t - s \notin \text{spec}_b(P)$ and $\ell$ runs through nonnegative integers). The reminder is $O(x^{-\nu-s+\epsilon})$ and the $u_{\sigma, \ell, k}(y)$ are smooth.
Thus:

\( P \in \text{Diff}_b^m(\mathcal{M}; E, F) \) is \( b \)-elliptic and \( u \in \nu H_b^s(\mathcal{M}; E) \) is such that \( Pu \in \mathcal{C}_b(\mathcal{M}; F) \). Then \( u \in \nu H_b^\infty(\mathcal{M}; E) \) and

\[
 u(x, y) \sim \sum_{\sigma \in \text{spec}_b(P)} \sum_{\ell = 0}^{\infty} \sum_{k = 0}^{N_{\sigma, \ell}} x^{i\sigma + \ell} u_{\sigma, \ell, k}(y) \log^k x
\]

with \( u_{\sigma, \ell, k} \in C^\infty(\mathcal{N}; E) \).


