Classification theorems for manifolds with $\mathbb{R}$-action

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Geometric Analysis Seminar
Humbold University
March 2013
Overview

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$$bT^{0,1}\mathcal{M} \subset \mathbb{C}^bT\mathcal{M}$$

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There is also an analogue of the classification of holomorphic line bundles by the Picard group, but I’ll not discuss this here.

These classification theorems permit the construction of new complex $b$-manifolds out of a given one.

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...we recall, is a vector bundle 

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$$bT\mathcal{M} \to \mathcal{M} \quad \text{ev} : bT\mathcal{M} \to T\mathcal{M}$$

such that the induced map

$$\text{ev}_* : C^\infty(\mathcal{M}; bT\mathcal{M}) \to C^\infty(\mathcal{M}; T\mathcal{M})$$

is an isomorphism onto $C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M})$, the space of vector fields on $\mathcal{M}$ which are tangential over $\partial \mathcal{M}$.
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Since \( C^\infty_{\text{tan}}(\mathcal{M}; T\mathcal{M}) \) is a Lie algebra and \( \text{ev}^* \) an isomorphism,

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[X, Y] = \text{ev}^{-1}_{*}[\text{ev}^*_*X, \text{ev}^*_*Y]
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Since $C^\infty_{\tan}(M; TM)$ is a Lie algebra and $\text{ev}_*$ an isomorphism,

$$C^\infty(M; bTM)$$

is a Lie algebra and

$$[X, Y] = \text{ev}_*^{-1}[\text{ev}_*X, \text{ev}_*Y]$$

Also $C^\infty(M; \mathbb{C}bTM)$ admits a Lie bracket.
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Also $C^\infty(M; C^bTM)$ admits a Lie bracket. of course!
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Also $C^\infty(M; C^bTM)$ admits a Lie bracket. of course!

Consequently, it makes sense to speak about involutive subbundle of $C^bTM$. 

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Complex $b$-structures

Because $C^\infty(\mathcal{M}; \mathbb{C}^bT\mathcal{M})$ admits a Lie bracket, it makes sense to talk about involutive sub-bundles of $\mathbb{C}^bT\mathcal{M}$.

Definition: A complex $b$-structure on $\mathcal{M}$ is a sub-bundle $b\mathcal{T}_0^1 \mathcal{M} \subset \mathbb{C}^bT\mathcal{M}$ such that

- $b\mathcal{T}_0^1 \mathcal{M}$ is involutive;
- $b\mathcal{T}_0^1 \mathcal{M} + b\mathcal{T}_0^1 \mathcal{M} = \mathbb{C}^bT\mathcal{M}$ as a direct sum.

The interior of $\mathcal{M}$ is a complex manifold.

The complex $b$-structure gives a differential complex:

$$\cdots \rightarrow C^\infty(\mathcal{M}; \mathbb{C}^b\bigwedge^p \mathcal{M}) \overset{b\partial}{\rightarrow} C^\infty(\mathcal{M}; \mathbb{C}^b\bigwedge^{p+1} \mathcal{M}) \rightarrow \cdots$$

One is interested in the behavior near the boundary of $b\partial$-closed forms, in particular, of solutions of $b\partial u = 0$ when $u$ is a function.
Complex $b$-structures

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**Definition:** A complex $b$-structure on $M$ is a sub-bundle

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The interior of $M$ is a complex manifold.

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**Complex $b$-structures**

Because $\mathcal{C}^\infty(M; \mathbb{C}^bTM)$ admits a Lie bracket, it makes sense to talk about involutive sub-bundles of $\mathbb{C}^bTM$.

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$$\cdots \rightarrow \mathcal{C}^\infty(M; b\wedge^{p,q}M) \xrightarrow{\overline{\partial b}} \mathcal{C}^\infty(M; b\wedge^{p,q+1}M) \rightarrow \cdots$$

One is interested in the behavior near the boundary of $\overline{\partial b}$-closed forms, in particular, of solutions of $\overline{\partial b}u = 0$ when $u$ is a function.
Example

View $S^2 \subset \mathbb{C}^2$, $\omega = (\omega_1, \omega_2) \in S^2$, $(z, w) \in \mathbb{C}$. Let

$\varphi : [0, \infty) \times S^3 \to \mathbb{C}^2$, \quad (r, \omega) \mapsto (r^3 \omega_1, r^2 \omega_2) = (z, w)$. 

This is an anisotropic blowup of $\mathbb{C}^2$ (as opposed to an isotropic [spherical] blowup).

We have $\varphi^* (\partial r) = 3 r^2 \omega_1 \partial z + 3 r^2 \omega_1 \partial z + 2 r \omega_2 \partial w + 2 r \omega_2 \partial w$. 

$J(\varphi^* (r \partial r)) = 3 i (z \partial z - z \partial z) + 2 i (w \partial w - w \partial w)$. 

Using polar coordinates on two copies of $\mathbb{C}$, $z = re^{i \theta}$, $w = \rho e^{i \psi}$, write

$S^3 = \{(r, \rho, e^{i \theta}, e^{i \psi}) : r^2 + \rho^2 = 1, r, \rho \geq 0\}$

Then $T = 3 \partial \theta + 2 \partial \psi$ is tangent to $S^2 (r = \text{constant})$ and $\varphi^* (T) = 3 i (z \partial z - z \partial z) + 2 i (w \partial w - w \partial w) = J(\varphi^* (r \partial r))$. 

So $\varphi^* (r \partial r + i T)$ and the CR vector field $w \partial z - z \partial w$ span $T_0, 1 \mathbb{C}^2$. 

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We have

\[ \varphi_* \left( \frac{\partial}{\partial r} \right) = 3r^2 \omega_1 \partial_z + 3r^2 \overline{\omega}_1 \partial_{\overline{z}} + 2r \omega_2 \partial_w + 2r \overline{\omega}_2 \partial_{\overline{w}} \]
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$$\varphi_*(r \partial_r) = 3r^2 \omega_1 \partial_z + 3r^2 \omega_1 \partial_{\overline{z}} + 2r^2 \omega_2 \partial_w + 2r^2 \omega_2 \partial_{\overline{w}}$$
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$$J(\varphi_*(r \partial_r)) = 3i(z \partial_z - \bar{z} \partial \bar{z}) + 2i(w \partial_w - \bar{w} \partial \bar{w}).$$

Using polar coordinates on two copies of $\mathbb{C}$, $z = re^{i\theta}$, $w = \rho e^{i\psi}$, write

$$S^3 = \{(r, \rho, e^{i\theta}, e^{i\psi}) : r^2 + \rho^2 = 1, \ r, \rho \geq 0\}$$

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So $\varphi_*(r \partial_r + iT) = 6\bar{z} \partial_z + 4\bar{w} \partial_w \in T^{0,1} \mathbb{C}^2$. 

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Then $\mathcal{T} = 3\partial_\theta + 2\partial_\psi$ is tangent to $S^2$ ($r =$ constant) and

$$\varphi_*(\mathcal{T}) = 3i(z \partial_z - \overline{z} \partial_{\overline{z}}) + 2i(w \partial_w - \overline{w} \partial_{\overline{w}}) = J(\varphi_*(r \partial_r))$$

So $\varphi_*(r \partial_r + i\mathcal{T}) = 6z \partial_z + 4w \partial_w \in T^{0,1}\mathbb{C}^2$.

$r \partial_r + i\mathcal{T}$ and the CR vector field $w \partial_z - z \partial_{\overline{w}}$ span $bT^{0,1}([0, \infty) \times S^3)$.
\[ \varphi : [0, \infty) \times S^3 \rightarrow \mathbb{C}^2, \]
\[ \varphi(t, \omega) = (t^3 \omega_1, t^2 \omega_2) \]
Let $\mathcal{V} = \{ z^2 - w^3 = 0 \} \subset \mathbb{C}^2$. The closure of $\varphi^{-1}(\mathcal{V} \setminus 0) \subset [0, \infty) \times S^3$ is

$$\mathcal{M} = \{ (r, \omega) \in [0, \infty) \times S^3 : \omega_1^2 - \omega_2^3 = 0 \}.$$
Let $\mathcal{V} = \{ z^2 - w^3 = 0 \} \subset \mathbb{C}^2$. The closure of $\varphi^{-1}(\mathcal{V}\setminus 0) \subset [0, \infty) \times S^3$ is
\[
\mathcal{M} = \{ (r, \omega) \in [0, \infty) \times S^3 : \omega_2^2 - \omega_3^3 = 0 \}.
\]

$\varphi : [0, \infty) \times S^3 \to \mathbb{C}^2$, $\varphi(t, \omega) = (t^3 \omega_1, t^2 \omega_2)$. This is a smooth manifold with boundary. The complex structure of $\mathcal{V}$ reg lifts and extends to a complex $b$-structure on $\mathcal{M}$. In fact, the complex structure is spanned by $r \partial_r + iJ(r \partial_r) = x \partial_x + i T$. 

(Temple University)
Let \( \mathcal{V} = \{ z^2 - w^3 = 0 \} \subset \mathbb{C}^2 \). The closure of \( \varphi^{-1}(\mathcal{V} \setminus 0) \subset [0, \infty) \times S^3 \) is
\[
\mathcal{M} = \{ (r, \omega) \in [0, \infty) \times S^3 : \omega_1^2 - \omega_2^3 = 0 \}.
\]

(Proposition 1) The complex structure of \( \mathcal{V} \) lifts and extends to a complex \( b \)-structure on \( \mathcal{M} \).
Let \( \mathcal{V} = \{ z^2 - w^3 = 0 \} \subset \mathbb{C}^2 \). The closure of \( \varphi^{-1}(\mathcal{V} \setminus 0) \subset [0, \infty) \times S^3 \) is
\[
\mathcal{M} = \{ (r, \omega) \in [0, \infty) \times S^3 : \omega_2^2 - \omega_3^2 = 0 \}.
\]
This is a smooth manifold with boundary.

\[
\varphi : [0, \infty) \times S^3 \rightarrow \mathbb{C}^2, \\
\varphi(t, \omega) = (t^3 \omega_1, t^2 \omega_2)
\]

\[
(r^3 \omega_1)^2 - (r^2 \omega_2)^3 = r^6 (\omega_1^2 - \omega_2^2)
\]
Let $\mathcal{V} = \{ z^2 - w^3 = 0 \} \subset \mathbb{C}^2$. The closure of $\varphi^{-1}(\mathcal{V} \setminus 0) \subset [0, \infty) \times S^3$ is

$$\mathcal{M} = \{ (r, \omega) \in [0, \infty) \times S^3 : \omega_1^2 - \omega_2^3 = 0 \}.$$ 

This is a smooth manifold with boundary. The complex structure of $\mathcal{V}_{\text{reg}}$ lifts and extends to a complex $b$-structure on $\mathcal{M}$.

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This is a smooth manifold with boundary. The complex structure of $\mathcal{V}_{\text{reg}}$ lifts and extends to a complex $b$-structure on $\mathcal{M}$. In fact, the complex structure is spanned by

$$r \partial_r + i J(r \partial_r)$$
Let \( \mathcal{V} = \{ z^2 - w^3 = 0 \} \subset \mathbb{C}^2 \). The closure of 
\( \varphi^{-1}(\mathcal{V} \setminus 0) \subset [0, \infty) \times S^3 \) is 
\[ M = \{(r, \omega) \in [0, \infty) \times S^3 : \omega_1^2 - \omega_2^3 = 0 \}. \]

This is a smooth manifold with boundary. The complex structure of \( \mathcal{V}_{\text{reg}} \) lifts and extends to a complex \( b \)-structure on \( M \). In fact, the complex structure is spanned by 
\[ r \partial_t + iJ(r \partial_t) = x \partial_x + iT. \]
Structure of the boundary

Recall:

$$bT \mathcal{M} \xrightarrow{ev} T \mathcal{M}$$

with kernel \( \text{span}_{\mathbb{R}}(x \partial_x) \)

along \( \mathcal{N} = \partial \mathcal{M} \).

\( \partial \mathcal{M} = \{ x = 0 \} \) (\( x \) rather than \( t \))

Back to the general setup...
Structure of the boundary

\[ \partial \mathcal{M} = \{ x = 0 \} \ (x \text{ rather than } t) \]

Back to the general setup...

\[ \mathcal{C}^b \mathcal{T} \mathcal{M} \xrightarrow{\text{ev}} \mathcal{C} \mathcal{T} \mathcal{M} \]

with kernel span_{\mathcal{C}}(x \partial_x)

along \( \mathcal{N} = \partial \mathcal{M} \).
Structure of the boundary

Back to the general setup...
Structure of the boundary

Back to the general setup...

\[
b^\mathcal{N}_{0,1} \mathcal{M} \xrightarrow{\iota} \mathbb{C}^{b^\mathcal{N}} \mathcal{M} \xrightarrow{\text{ev}} \mathbb{C}^\mathcal{T}_\mathcal{N} \mathcal{M}, \quad \ker \text{ev} = \text{span}_\mathbb{C}(x\partial_x|_\mathcal{N})
\]
Structure of the boundary

Recall:

$$\partial M = \{x = 0\} \quad (x \text{ rather than } t)$$

Back to the general setup...

$$\begin{align*}
\mathbb{C}TN & \hookrightarrow C^0 TN \\
bT_{N}^0,1M & \xrightarrow{\iota} C^0 bT_{N}M & \xrightarrow{\text{ev}} C T_{N}M, \\
\ker \text{ev} & = \text{span}_{\mathbb{C}}(x\partial_x|_N)
\end{align*}$$
Structure of the boundary

Back to the general setup...

\[ \partial \mathcal{M} = \{ x = 0 \} \ (x \text{ rather than } t) \]

\[ b^{T^0_1} \mathcal{M} \xrightarrow{\iota} \mathbb{C} b^{T_N} \mathcal{M} \xrightarrow{\text{ev}} \mathbb{C} T_N \mathcal{M}, \quad \ker \text{ev} = \text{span}_\mathbb{C} (x \partial_x |_N) \]
Structure of the boundary

Back to the general setup...

\[ bT_{\mathcal{N}}^{0,1} \mathcal{M} \xrightarrow{\iota} \mathbb{C}^bT_{\mathcal{N}}\mathcal{M} \xrightarrow{ev} \mathbb{C}T_{\mathcal{N}}\mathcal{M}, \]

\[ \ker ev = \text{span}_{\mathbb{C}}(x\partial_x|_\mathcal{N}) \]

\[ \partial \mathcal{M} = \{ x = 0 \} \text{ (x rather than } \tau) \]
Structure of the boundary

Back to the general setup...

\[ \partial \mathcal{M} = \{ x = 0 \} \] (x rather than τ)

\[ \text{injective} \]

\[ \text{ev}_\mathcal{N} = \text{ev}|_\mathcal{N} \circ \iota \]

\[ bT^0,1_\mathcal{N}; \mathcal{M} \xrightarrow{\iota} \mathbb{C}^bT_\mathcal{N}\mathcal{M} \xrightarrow{\text{ev}} \mathbb{C}T_\mathcal{N}\mathcal{M}, \]

\[ \ker \text{ev} = \text{span}_{\mathbb{C}}(x\partial_x|_\mathcal{N}) \]

\[ \ker(\text{ev}_\mathcal{N}) = 0 \]
Structure of the boundary

Back to the general setup...

\[ \text{ev}_N \left( bT^0,1_N^{-1}M \right) \hookrightarrow \mathbb{C} T_N \]

\[ bT^0,1_N^{-1} M \xhookrightarrow{\iota} \mathbb{C} bT^0,1_N^{-1} M \xrightarrow{\text{ev}} \mathbb{C} T_N M, \]

\[ \ker \text{ev} = \text{span}_{\mathbb{C}} (x\partial_x \big|_N) \]

\[ \ker (\text{ev}_N) = 0 \]
Structure of the boundary

Back to the general setup...

\[ \bar{\mathcal{V}} = \text{ev}_\mathcal{N}(bT^{0,1}_\mathcal{N} \mathcal{M}) \hookrightarrow \mathbb{C}T\mathcal{N}, \text{ involutive, } \bar{\mathcal{V}} + \mathcal{V} = \mathbb{C}T\mathcal{N} \]

\[ bT^{0,1}_\mathcal{N} \mathcal{M} \hookrightarrow \mathbb{C}bT_\mathcal{N}\mathcal{M} \xrightarrow{\text{ev}} \mathbb{C}T_\mathcal{N}\mathcal{M}, \]

ker ev = span_{\mathbb{C}}(x\partial_x|_\mathcal{N})

ker(ev_\mathcal{N}) = 0

\[ \partial \mathcal{M} = \{x = 0\} \text{ (} x \text{ rather than } t \)
Structure of the boundary

\[ \nabla = \text{ev}_{\mathcal{N}}(bT^{0,1}_{\mathcal{N}}\mathcal{M}) \]

\[ bT^{0,1}_{\mathcal{N}}\mathcal{M} \leftarrow \mathbb{C}bT_{\mathcal{N}}\mathcal{M} \xrightarrow{\text{ev}} \mathbb{C}T_{\mathcal{N}}\mathcal{M}, \]

\[ \ker \text{ev} = \text{span}_{\mathbb{C}}(x\partial_x|_{\mathcal{N}}) \]

\[ \ker(\text{ev}_{\mathcal{N}}) = 0 \]

\[ \partial \mathcal{M} = \{x = 0\} \text{ (}x\text{ rather than } t) \]

Associated complex:

\[ \cdots \rightarrow C^\infty(\mathcal{N}, \wedge^q \nabla^*) \xrightarrow{D} C^\infty(\mathcal{N}, \wedge^{q+1} \nabla^*) \rightarrow \cdots \]

\[ \mathbb{C}T\mathcal{N}, \text{ involutive, } \nabla + \mathcal{V} = \mathbb{C}T\mathcal{N} \]

\[ \downarrow \]

a real element
Structure of the boundary

\[ \mathcal{V} = \text{ev}_\mathcal{N}(bT^0,1_\mathcal{M}) \]

\[ bT^0,1_\mathcal{M} \hookrightarrow \mathbb{C}bT_\mathcal{N}\mathcal{M} \xrightarrow{\text{ev}} \mathbb{C}T_\mathcal{N}\mathcal{M}, \]

\[ \ker(\text{ev}_\mathcal{N}) = 0 \]

\[ \overline{b_{\partial x}} \text{ vanishes on } \mathcal{N}. \]

\[ \partial \mathcal{M} = \{ x = 0 \} \text{ (} x \text{ rather than } t \text{)} \]

Associated complex:

\[ \cdots \to C^\infty(\mathcal{N}, \Lambda^q \overline{\mathcal{V}}^*) \overset{\partial}{\to} C^\infty(\mathcal{N}, \Lambda^{q+1} \overline{\mathcal{V}}^*) \to \cdots \]

\[ \mathbb{C}T\mathcal{N}, \text{ involutive, } \mathcal{V} + \mathcal{V} = \mathbb{C}T\mathcal{N} \]

\[ \text{a real element} \]

\[ \ker(\text{ev}_\mathcal{N}) = \text{span}_\mathbb{C}(x\partial_x|_\mathcal{N}) \]
Structure of the boundary

\[ \bar{V} = \text{ev}_N(\mathcal{N}, b^{T^0,1}M) \mapsto \mathbb{C}T\mathcal{N}, \text{ involutive, } \overline{V} + V = \mathbb{C}T\mathcal{N} \]

\[ b\partial x \text{ vanishes on } \mathcal{N}. \text{ To see this, use } b\partial x = j^* dx: \]

\[ \partial M = \{ x = 0 \} (x \text{ rather than } t) \]

\[ \ldots \to C^\infty(\mathcal{N}, \wedge^q \overline{V}^*) \xrightarrow{\partial} C^\infty(\mathcal{N}, \wedge^{q+1} \overline{V}^*) \to \ldots \]

\[ \ker \text{ev} = \text{span}_\mathbb{C}(x\partial x|_\mathcal{N}) \]

\[ \ker(\text{ev}_N) = 0 \]

\[ j : T^{0,1}M \mapsto \mathbb{C}T\mathcal{M} \]
Structure of the boundary

\[ \partial \mathcal{M} = \{ x = 0 \} \] (\( x \) rather than \( t \))

Associated complex:

\[ \cdots \to C^\infty(\mathcal{N}, \wedge^q \overline{\nu}^*) \xrightarrow{\overline{\nu}} C^\infty(\mathcal{N}, \wedge^{q+1} \overline{\nu}^*) \to \cdots \]

\[ \overline{\nu} = \text{ev}_\mathcal{N}(bT^0,1_\mathcal{N} \mathcal{M}) \hookrightarrow \mathbb{C} T\mathcal{N}, \text{ involutive, } \overline{\nu} + \nu = \mathbb{C} T\mathcal{N} \]

\[ b\overline{\partial} x \] vanishes on \( \mathcal{N} \). To see this, use \( b\overline{\partial} x = j^* dx \):

\[ \langle b\overline{\partial} x, x\partial x + iJ(x\partial x) \rangle = \langle j^* dx, x\partial x + iJ(x\partial x) \rangle \]
Structure of the boundary

\[ \overline{\nabla} = \text{ev}_N (bT^0_{\nabla} M) \]

Associated complex:

\[ \cdots \to C^\infty (\mathcal{N}, \wedge^q \overline{\nabla}^*) \xrightarrow{\overline{\nabla}} C^\infty (\mathcal{N}, \wedge^{q+1} \overline{\nabla}^*) \to \cdots \]

\[ \nabla = \text{ev}_N (bT^0_{\nabla} M) \hookrightarrow \mathbb{C} T \mathcal{N}, \text{ involutive, } \overline{\nabla} + \nabla = \mathbb{C} T \mathcal{N} \]

\[ b\overline{\partial}x \text{ vanishes on } \mathcal{N}. \text{ To see this, use } b\overline{\partial}x = j^* dx: \]

\[ \langle b\overline{\partial}x, x\partial_x + iJ(x\partial_x) \rangle = \langle \cdot dx, j(x\partial_x + iJ(x\partial_x)) \rangle \]
Structure of the boundary

\( \overline{\nabla} = \text{ev}_N(bT^0_1N) \)

\( bT^0_1N \hookrightarrow \mathbb{C}bT_NN \xrightarrow{\text{ev}} \mathbb{C}T_NN, \text{ involutive, } \overline{\nabla} + \nabla = \mathbb{C}T_NN \)

\( b\overline{\partial}x \) vanishes on \( N \). To see this, use \( b\overline{\partial}x = j^* dx \):

\[
\langle b\overline{\partial}x, x\partial_x + iJ(x\partial_x) \rangle = \langle \langle dx, j(x\partial_x + iJ(x\partial_x)) \rangle \rangle = (x\partial_x + i\text{ev}(J(x\partial_x)))x
\]
Structure of the boundary

\[ \mathcal{V} = \text{ev}_\mathcal{N}(bT^0,1_N \mathcal{M}) \]

\[ bT^0,1_N \mathcal{M} \hookrightarrow \mathbb{C} bT_N \mathcal{M} \xrightarrow{\text{ev}} \mathbb{C} T_N \mathcal{M}, \quad \ker \text{ev} = \text{span}_\mathbb{C}(x \partial_x \big|_{\mathcal{N}}) \]

\[ \ker(\text{ev}_\mathcal{N}) = 0 \]

\( b\bar{\partial}x \) vanishes on \( \mathcal{N} \). To see this, use \( b\bar{\partial}x = j^* dx \):

\[ \langle b\bar{\partial}x, x \partial_x + iJ(x \partial_x) \rangle = \langle dx, j(x \partial_x + iJ(x \partial_x)) \rangle \]

\[ = (x \partial_x + i\text{ev}(J(x \partial_x))) x = 0 \text{ on } \mathcal{N} \]
Structure of the boundary

\[ \overline{\nabla} = \text{ev}_\mathcal{N}(bT^0,1_\mathcal{N}M) \]

\[ bT^0,1_\mathcal{N}M \hookrightarrow \mathcal{C}bT^*\mathcal{N}M \xrightarrow{\text{ev}} \mathcal{C}T_\mathcal{N}M, \quad \ker \text{ev} = \text{span}_\mathbb{C}(x\partial_x|_\mathcal{N}) \]

\[ b\overline{\partial}_x \text{ vanishes on } \mathcal{N}. \] To see this, use \[ b\overline{\partial}_x = j^*dx: \]

\[ \langle b\overline{\partial}_x, \nu + iJ(\nu) \rangle = (\text{ev}(\nu) + i\text{ev}J(\nu))x = 0 \text{ on } \mathcal{N} \]
Structure of the boundary

\[ \partial M = \{ x = 0 \} \ (x \text{ rather than } t) \]

\[ \mathcal{V} = \text{ev}_\mathcal{N}(bT^0_1 \mathcal{M}) \]

\[ bT^0_1 \mathcal{M} \xrightarrow{\mathcal{L}} \mathbb{C}^bT\mathcal{N} \mathcal{M} \xrightarrow{\text{ev}} \mathbb{C}T\mathcal{N} \mathcal{M}, \quad \ker \text{ev} = \text{span}_\mathbb{C}(x\partial_x|_\mathcal{N}) \]

\[ \ker(\text{ev}_\mathcal{N}) = 0 \]

\[ j : T^{0,1} \mathcal{M} \xhookrightarrow{} \mathbb{C}T\mathcal{M} \]

\( b\overline{\partial}x \) vanishes on \( \mathcal{N} \).

Since \( b\overline{\partial}(x\phi) = b\overline{\partial}x \wedge \phi + x b\overline{\partial}\phi = 0 \) on \( \mathcal{N} \) get induced operators

\[ C^\infty(\mathcal{N}; \wedge^0_\mathcal{N} \mathcal{M}) \xrightarrow{b\overline{\partial}_b} C^\infty(\mathcal{N}; \wedge^0_\mathcal{N}^{q+1} \mathcal{M}) \]
Structure of the boundary

\[ \overline{\mathcal{V}} = \text{ev}_\mathcal{N}( \mathcal{M} ) \]

\[ \mathcal{M} \overset{b}{\mapsto} b^{\mathcal{N}} \mathcal{T} \overset{\text{ev}}{\mapsto} \mathcal{N}, \quad \text{ker}\, \text{ev} = \text{span}_\mathbb{C}( x\partial_x |_\mathcal{N} ) \]

\[ b\overline{\partial}_x \text{ vanishes on } \mathcal{N}. \]

Since \( b\overline{\partial}(x\phi) = b\overline{\partial}_x \wedge \phi + x \overline{\partial}_x \phi = 0 \) on \( \mathcal{N} \) get induced operators

\[ C^\infty(\mathcal{N}; \bigwedge^{0,q}_{\mathcal{N}} \mathcal{M}) \overset{b\overline{\partial}_b}{\twoheadrightarrow} C^\infty(\mathcal{N}; \bigwedge^{0,q+1}_{\mathcal{N}} \mathcal{M}) \]

\[ \bigwedge^q \overline{\mathcal{V}}^* \quad \bigwedge^{q+1} \overline{\mathcal{V}}^* \]
Structure of the boundary

Associated complex:

\[ \cdots \to C^\infty(\mathcal{N}, \wedge q \mathcal{V}^*) \xrightarrow{\overline{b}} C^\infty(\mathcal{N}, \wedge^{q+1} \mathcal{V}^*) \to \cdots \]

\[ \mathcal{V} = \text{ev}_\mathcal{N}(b^{T^0,1}_N \mathcal{M}) \]

\[ b^{T^0,1}_N \mathcal{M} \xrightarrow{\iota} \mathbb{C} b^{T^0}_N \mathcal{M} \xrightarrow{\text{ev}} \mathbb{C} T\mathcal{N} \mathcal{M}, \quad \ker \text{ev} = \text{span}_\mathbb{C} (x \partial_x |_\mathcal{N}) \]

\[ b \partial x \text{ vanishes on } \mathcal{N}. \]

Since \( b \overline{\partial}(x \phi) = b \overline{\partial} x \wedge \phi + x b \overline{\partial} \phi = 0 \) on \( \mathcal{N} \) get induced operators

\[ C^\infty(\mathcal{N}; \wedge^{0,q}_\mathcal{N} \mathcal{M}) \xrightarrow{b \overline{\partial}_b} C^\infty(\mathcal{N}; \wedge^{0,q+1}_\mathcal{N} \mathcal{M}) \]

\[ C^\infty(\mathcal{N}; \wedge^q \mathcal{V}^*) \quad C^\infty(\mathcal{N}; \wedge^{q+1} \mathcal{V}^*) \]

\[ \partial \mathcal{M} = \{ x = 0 \} \text{ (} x \text{ rather than } t \text{)} \]

\[ b \mathcal{M} \]
Structure of the boundary

\[ \overline{\mathcal{V}} = \text{ev}\mathcal{N}(bT_{\mathcal{N}}^{0,1} \mathcal{M}) \]

\[ \partial \mathcal{M} = \{ x = 0 \} \ (x \text{ rather than } \tau) \]

Associated complex:

\[ \cdots \rightarrow C^\infty(\mathcal{N}, \Lambda^q \overline{\mathcal{V}}^*) \xrightarrow{\overline{\mathcal{D}}} C^\infty(\mathcal{N}, \Lambda^{q+1} \overline{\mathcal{V}}^*) \rightarrow \cdots \]

\[ \mathbb{C} T\mathcal{N}, \text{ involutive, } \overline{\mathcal{V}} + \mathcal{V} = \mathbb{C} T\mathcal{N} \]

\[ \bigwedge^q \mathcal{V} \]

\[ b\overline{\partial} \text{ vanishes on } \mathcal{N}. \]

Since \[ b\overline{\partial}(x\phi) = b\overline{\partial}x \wedge \phi + x b\overline{\partial}\phi = 0 \text{ on } \mathcal{N} \] get induced operators

\[ C^\infty(\mathcal{N}; \Lambda^0_{\mathcal{N}} \mathcal{M}) \xrightarrow{b\overline{\partial}b} C^\infty(\mathcal{N}; \Lambda^0_{\mathcal{N}} \mathcal{M}) \]

\[ C^\infty(\mathcal{N}; \Lambda^0_{\mathcal{N}} \mathcal{M}) \xrightarrow{\overline{\mathcal{D}}} C^\infty(\mathcal{N}; \Lambda^0_{\mathcal{N}} \mathcal{M}) \]

\[ \text{ker } \text{ev} = \text{span}_\mathbb{C}(x\partial_x |_{\mathcal{N}}) \]

\[ \text{ker}(\text{ev}_\mathcal{N}) = 0 \]

\[ j : T^{0,1} \mathcal{M} \hookrightarrow \mathbb{C} T\mathcal{M} \]
Associated complex: $\overline{\partial}_b$  \[ \cdots \rightarrow C^\infty(\mathcal{N}, \wedge^q \overline{\nu}^*) \overset{\overline{\partial}_b}{\longrightarrow} C^\infty(\mathcal{N}, \wedge^{q+1} \overline{\nu}^*) \rightarrow \cdots \]

Structure of the boundary

\[ \overline{\nu} = \mathrm{ev}_\mathcal{N}(b\mathcal{T}_\mathcal{N}^{0,1} \mathcal{M}) \]

$\mathcal{T}_\mathcal{N},$ involutive, $\overline{\nu} + \nu = \mathcal{T}_\mathcal{N}$

$\partial \mathcal{M} = \{ x = 0 \}$ (x rather than t)

\[ b\mathcal{T}_\mathcal{N}^{0,1} \mathcal{M} \leftarrow \mathbb{C} b\mathcal{T}_\mathcal{N} \mathcal{M} \overset{\text{ev}}{\rightarrow} \mathbb{C} \mathcal{T}_\mathcal{N} \mathcal{M}, \quad \ker \text{ev} = \text{span}_\mathbb{C}(x \partial_x \big|_\mathcal{N}) \]

$\partial \mathcal{M}$

$\overline{\partial}_b$ vanishes on $\mathcal{N}.$

Since $\overline{\partial}_b(x \phi) = \overline{\partial}_b x \wedge \phi + x \overline{\partial}_b \phi = 0$ on $\mathcal{N}$ get induced operators

\[ C^\infty(\mathcal{N}; \wedge^0_\mathcal{N} \mathcal{M}) \overset{b\overline{\partial}_b}{\longrightarrow} C^\infty(\mathcal{N}; \wedge^0_\mathcal{N} \mathcal{M}) \]

\[ C^\infty(\mathcal{N}; \wedge^q_\mathcal{N} \overline{\nu}^*) \overset{\overline{\partial}}{\longrightarrow} C^\infty(\mathcal{N}; \wedge^{q+1}_\mathcal{N} \overline{\nu}^*) \]
Structure of the boundary

$\overline{\mathcal{V}} = \text{ev}_\mathcal{N}(bT_{\mathcal{N}}^{0,1} \mathcal{M}) \hookrightarrow \mathbb{C} T \mathcal{N},$ involutive, $\overline{\mathcal{V}} + \mathcal{V} = \mathbb{C} T \mathcal{N}$

$\partial \mathcal{M} = \{ x = 0 \} \ (x \text{ rather than } t)$

$\overline{\partial} x \text{ vanishes on } \mathcal{N}.$

$\overline{\partial} x \overline{\partial} \left( \frac{b\overline{\partial} x}{x} \right) = 0$

$\overline{\partial} x$ is smooth and $\overline{\partial} \left( \frac{b\overline{\partial} x}{x} \right) = 0$
Structure of the boundary

\[ \overline{\mathcal{V}} = \text{ev}_\mathcal{N}(bT^0,1_N) \]

Associated complex: \( b\overline{\partial}_b \)

\[ \cdots \rightarrow C^\infty(\mathcal{N}, \wedge^q \overline{\mathcal{V}}^*) \overset{\overline{D}}{\rightarrow} C^\infty(\mathcal{N}, \wedge^{q+1} \overline{\mathcal{V}}^*) \rightarrow \cdots \]

\( \mathcal{N} \) is such that \( \overline{\mathcal{V}} + \mathcal{V} = \mathbb{C} T\mathcal{N} \)

\[ \text{ker ev} = \text{span}_\mathbb{C}(x\partial_x|_\mathcal{N}) \]

\[ \text{ker}(\text{ev}_\mathcal{N}) = 0 \]

\( j : T^{0,1}M \rightarrow \mathbb{C} T M \)

\[ b\overline{\partial}_x \text{ vanishes on } \mathcal{N}. \]

\[ b\overline{\partial}_x \frac{x}{x} \text{ is smooth and } b\overline{\partial} \left( \frac{b\overline{\partial}_x}{x} \right) = 0 \]

\[ \left. b\overline{\partial}_x \frac{x}{x} \right|_\mathcal{N} \]
Structure of the boundary

\[ \bar{\mathcal{V}} = \text{ev}_\mathcal{N}(bT_0^1 \mathcal{M}) \hookrightarrow \mathbb{C}T\mathcal{N}, \text{ involutive, } \bar{\mathcal{V}} + \mathcal{V} = \mathbb{C}T\mathcal{N} \]

\[ bT_0^1 \mathcal{M} \hookrightarrow \mathbb{C}bT_\mathcal{N}\mathcal{M} \xrightarrow{\text{ev}} \mathbb{C}T\mathcal{N}\mathcal{M}, \]

\[ \ker \text{ev} = \text{span}_\mathbb{C} (x\partial_x|_\mathcal{N}) \]

\[ \ker(\text{ev}_\mathcal{N}) = 0 \]

\[ j : T^{0,1}\mathcal{M} \hookrightarrow \mathbb{C}T\mathcal{M} \]

\[ \mathcal{M} = \{x = 0\} \ (x \text{ rather than } \tau) \]

\[ \partial \mathcal{M} \]

Associated complex: \( \partial_b \)

\[ \cdots \xrightarrow{\text{D}} C^\infty(\mathcal{N}, \wedge^q \bar{\mathcal{V}}^* ) \xrightarrow{b\partial} C^\infty(\mathcal{N}, \wedge^{q+1} \bar{\mathcal{V}}^* ) \xrightarrow{\partial} \cdots \]

\[ b\partial x \text{ vanishes on } \mathcal{N}. \]

\[ \frac{b\partial x}{x} \text{ is smooth and } b\partial \left( \frac{b\partial x}{x} \right) = 0 \]

\[ \left. \frac{b\partial x}{x} \right|_\mathcal{N} = \beta_x \text{ as section of } \bar{\mathcal{V}}^* \]
Structure of the boundary

\[ \bar{V} = \text{ev}_\mathcal{N}(b^T) \]

Associated complex: \( \overline{b\partial_b} \)

\[ \cdots \to C^\infty(\mathcal{N}, \bigwedge^q \bar{V}^*) \to \cdots \]

\[ \bar{V} = \nabla = \nabla_x + \nabla_y = \nabla_{\text{struct.}} \]

\[ b^T \mathcal{M} \xhookrightarrow{\text{ev}} \mathbb{C} b^T \mathcal{M} \xrightarrow{\text{ev}} \mathbb{C} T \mathcal{N}, \text{ involutive, } \bar{V} + V = \mathbb{C} T \mathcal{N} \]

\( \text{ker } \text{ev} = \text{span}_\mathbb{C} \left( x\partial_x \big|_\mathcal{N} \right) \)

\[ \text{ker}(\text{ev}_\mathcal{N}) = 0 \]

\[ j : T^{0,1} \mathcal{M} \hookrightarrow \mathbb{C} T \mathcal{M} \]

\( b\overline{\partial} x \) vanishes on \( \mathcal{N} \).

\[ \frac{b\overline{\partial} x}{x} \text{ is smooth and } \frac{b\overline{\partial} \left( \frac{b\overline{\partial} x}{x} \right)}{x} = 0 \]

\[ \frac{b\overline{\partial} x}{x} \bigg|_{\mathcal{N}} = \beta_x \text{ as section of } \bar{V}^* \text{ is such that } \overline{D} \beta_x = 0. \]
Structure of the boundary

\[ \nabla = \text{ev}_N(b^T_0 M) \]

Associated complex: \( b\bar{\partial}_b \)

\[ \cdots \rightarrow C^\infty(N, \wedge^q \nabla^*) \rightarrow C^\infty(N, \wedge^{q+1} \nabla^*) \rightarrow \cdots \]

\[ \nabla = \text{ev}_N, \text{ involutive, } \nabla + \nabla = C T N \]

\[ b^T_0 M \hookrightarrow \mathbb{C} b^T_0 M \xrightarrow{\text{ev}} \mathbb{C} T N M, \quad \ker \text{ev} = \text{span}_{\mathbb{C}}(x \partial_x|_N) \]

\[ b\bar{\partial}_x \text{ vanishes on } N. \]

\[ \beta = \{ \beta_x : x \text{ is defining function for } N \} \]

Let \( J : bT M \rightarrow bT M \) be the almost complex structure.
Structure of the boundary

\( \overline{\mathcal{V}} = \text{ev}_\mathcal{N}(bT_{\mathcal{N}}^{0,1} M) \)

\( bT_{\mathcal{N}}^{0,1} M \hookrightarrow \mathbb{C} bT_\mathcal{N} M \xrightarrow{\text{ev}} \mathbb{C} T_\mathcal{N} M, \quad \ker \text{ev} = \text{span}_\mathbb{C}(x \partial_x |_{\mathcal{N}}) \)

\( \partial \mathcal{M} = \{ x = 0 \} \) (x rather than \( t \))

Associated complex: \( b\overline{\partial}_b \)

\[ \cdots \rightarrow C^\infty(\mathcal{N}, \wedge^q \overline{\mathcal{V}}^*) \xrightarrow{\overline{D}} C^\infty(\mathcal{N}, \wedge^{q+1} \overline{\mathcal{V}}^*) \rightarrow \cdots \]

\( J : bT^0,1 M \hookrightarrow \mathbb{C} T M \)

Let \( J : bT M \rightarrow bT M \) be the almost complex structure.

\( x \partial_x |_{\mathcal{N}} \) is a canonical section of \( bT_\mathcal{N} M \)

\( x' = e^u x : \quad \beta_{x'} - \beta_x = \overline{D} u \)
Structure of the boundary

\[ \overline{V} = \text{ev}_N(bT^{0,1}_N \mathcal{M}) \]

\[ bT^{0,1}_N \mathcal{M} \xrightarrow{\iota} \mathbb{C}bT_N \mathcal{M} \xrightarrow{\text{ev}} \mathbb{C}T_N \mathcal{M}, \]

\[ \text{ker ev} = \text{span}_\mathbb{C}(x\partial_x|_N) \]

\[ \text{ker}(\text{ev}_N) = 0 \]

\[ \partial \mathcal{M} = \{x = 0\} \ (x \text{ rather than } t) \]

\[ \cdots \rightarrow C^\infty(\mathcal{N}, \wedge^q \overline{V}^*) \xrightarrow{\overline{D}} C^\infty(\mathcal{N}, \wedge^{q+1} \overline{V}^*) \rightarrow \cdots \]

\[ b\overline{\partial}_x \text{ vanishes on } \mathcal{N}. \]

\[ \beta = \{ \beta_x : x \text{ is defining function for } \mathcal{N} \} \]

Let \( J : bT \mathcal{M} \rightarrow bT \mathcal{M} \) be the almost complex structure.

\[ x\partial_x|_\mathcal{N} \text{ is a canonical section of } bT_N \mathcal{M} \text{ and so is } J(x\partial_x|_\mathcal{N}). \]
Structure of the boundary

\[ \overline{V} = \text{ev}_{\mathcal{N}}(bT_{\mathcal{N}}^{0,1} M) \]

\[ bT_{\mathcal{N}}^{0,1} M \xleftarrow{L} \mathbb{C}^bT_{\mathcal{N}}M \xrightarrow{ev} \mathbb{C}T_{\mathcal{N}}M, \]

ker ev = span_{\mathbb{C}}(x\partial_x|_{\mathcal{N}})

\[ x' = e^u x : \quad \beta_{x'} - \beta_x = \overline{D}u \]

Let \( J : bT M \rightarrow bT M \) be the almost complex structure.

\[ x\partial_x|_{\mathcal{N}} \text{ is a canonical section of } bT_{\mathcal{N}}M \text{ and so is } J(x\partial_x|_{\mathcal{N}}). \]

Therefore \( \mathcal{T} = \text{ev}(J(x\partial_x|_{\mathcal{N}})) \) is a canonical vector field on \( \mathcal{N}. \)
Structure of the boundary

\[ \mathcal{V} = \text{ev}_\mathcal{N}(bT^0,1_{\mathcal{N}}\mathcal{M}) \]

\[ \mathcal{V} = \text{ev}_\mathcal{N}(bT^0,1_{\mathcal{N}}\mathcal{M}) \]

\[ \partial \mathcal{M} = \{ x = 0 \} \quad (x \text{ rather than } \tau) \]

Associated complex: \( \overline{\partial}_b \)

\[ \cdots \rightarrow C^\infty(\mathcal{N}, \wedge^q \mathcal{V}^*) \rightarrow C^\infty(\mathcal{N}, \wedge^{q+1} \mathcal{V}^*) \rightarrow \cdots \]

\[ bT^0,1_{\mathcal{N}}\mathcal{M} \hookrightarrow \mathbb{C} bT_{\mathcal{N}}\mathcal{M} \xrightarrow{\text{ev}} \mathbb{C} T_{\mathcal{N}}\mathcal{M}, \quad \ker \text{ev} = \text{span}_\mathbb{C}(x\partial_x|_{\mathcal{N}}) \]

\[ b\overline{\partial}_x \text{ vanishes on } \mathcal{N}. \]

\[ \beta = \{ \beta_x : x \text{ is defining function for } \mathcal{N} \} \]

Let \( J : bT\mathcal{M} \rightarrow bT\mathcal{M} \) be the almost complex structure.

\[ x\partial_x|_{\mathcal{N}} \text{ is a canonical section of } bT_{\mathcal{N}}\mathcal{M} \text{ and so is } J(x\partial_x|_{\mathcal{N}}). \]

Therefore \( \mathcal{T} = \text{ev}(J(x\partial_x|_{\mathcal{N}})) \) is a canonical vector field on \( \mathcal{N}. \)

\[ x\partial_x|_{\mathcal{N}} + iJ(x\partial_x|_{\mathcal{N}}) \text{ is a nonzero section of } bT^0,1_{\mathcal{N}}\mathcal{M}. \]
Structure of the boundary

\[ \overline{V} = \text{ev}_\mathcal{N}(bT^0,1_N \mathcal{M}) \hookrightarrow \mathbb{C} T\mathcal{N}, \text{ involutive, } \overline{V} + V = \mathbb{C} T\mathcal{N} \]

\[ bT^0,1_N \mathcal{M} \xhookrightarrow{\iota} \mathbb{C} bT_N \mathcal{M} \xrightarrow{\text{ev}} \mathbb{C} T_N \mathcal{M}, \quad \ker \text{ev} = \text{span}_\mathbb{C}(x\partial_x|_\mathcal{N}) \]

\[ \overline{\partial}x \text{ vanishes on } \mathcal{N}. \]

\[ \beta = \{ \beta_x : x \text{ is defining function for } \mathcal{N} \} \]

Let \( J : bT \mathcal{M} \rightarrow bT \mathcal{M} \) be the almost complex structure.

\( x\partial_x|_\mathcal{N} \) is a canonical section of \( bT_N \mathcal{M} \) and so is \( J(x\partial_x|_\mathcal{N}) \).

Therefore \( T = \text{ev}(J(x\partial_x|_\mathcal{N})) \) is a canonical vector field on \( \mathcal{N} \).

\[ \text{ev}_\mathcal{N}(x\partial_x|_\mathcal{N} + iJ(x\partial_x|_\mathcal{N})) \text{ is a nonzero section of } \mathbb{C} T\mathcal{N}. \]

\( \partial \mathcal{M} = \{ x = 0 \} (x \text{ rather than } t) \)
Structure of the boundary

Fractions of text are emphasized:

\[ \overline{\mathcal{V}} = \text{ev}_\mathcal{N}(^{bT}_{\mathcal{N}}^{0,1} \mathcal{M}) \}

\[ \mathcal{N} \xrightarrow{\partial} \mathcal{C} T\mathcal{N}, \text{ involutive, } \overline{\mathcal{V}} + \mathcal{V} = \mathcal{C} T\mathcal{N} \]

\[ b^{T_{0,1}} \mathcal{M} \xrightarrow{\text{ev}} \mathcal{C} b^{T_{\mathcal{N}}} \mathcal{M} \]

\[ \ker \text{ev} = \text{span}_\mathbb{C}(x\partial_x|_\mathcal{N}) \]

\[ \ker(\text{ev}_\mathcal{N}) = 0 \]

\[ j : T^{0,1} \mathcal{M} \xleftarrow{\text{ev}} \mathbb{C} T\mathcal{M} \]

\[ b\overline{\partial}x \text{ vanishes on } \mathcal{N}. \]

\[ \beta = \{ \beta_x : x \text{ is defining function for } \mathcal{N} \} \]

Let \( J : bT \mathcal{M} \rightarrow bT \mathcal{M} \) be the almost complex structure.

\[ x\partial_x|_\mathcal{N} \text{ is a canonical section of } b^{T_{\mathcal{N}}} \mathcal{M} \text{ and so is } J(x\partial_x|_\mathcal{N}). \]

Therefore \( \mathcal{T} = \text{ev}(J(x\partial_x|_\mathcal{N})) \) is a canonical vector field on \( \mathcal{N}. \)

\[ \text{ev}_\mathcal{N}(iJ(x\partial_x|_\mathcal{N})) \text{ is a nonzero section of } \mathcal{C} T\mathcal{N}. \]

\[ \partial \mathcal{M} = \{ x = 0 \} (x \text{ rather than } r) \]

Associated complex: \( b\overline{\partial}_b \)

\[ \ldots \rightarrow C^\infty(\mathcal{N}, \wedge^q \overline{\mathcal{V}}^*) \overset{\overline{\partial}}{\rightarrow} C^\infty(\mathcal{N}, \wedge^{q+1} \overline{\mathcal{V}}^*) \rightarrow \ldots \]
Structure of the boundary

\[ \nabla = \text{ev}_N(b\mathcal{T}_N^{0,1}\mathcal{M}) \hookrightarrow \mathbb{C}T\mathcal{N}, \text{ involutive, } \nabla + \mathcal{V} = \mathbb{C}T\mathcal{N} \]

\[ b\mathcal{T}_N^{0,1}\mathcal{M} \hookrightarrow \mathbb{C}b\mathcal{T}_N\mathcal{M} \xrightarrow{\text{ev}} \mathbb{C}T\mathcal{N}\mathcal{M}, \quad \ker\text{ev} = \text{span}_\mathbb{C}(x\partial_x|_\mathcal{N}) \]

\[ \ker(\text{ev}_\mathcal{N}) = 0 \]

\[ b\overline{\partial}x \text{ vanishes on } \mathcal{N}. \]

\[ \beta = \{\beta_x : x \text{ is defining function for } \mathcal{N}\} \]

Let \( J : b\mathcal{T}\mathcal{M} \to b\mathcal{T}\mathcal{M} \) be the almost complex structure.

\[ x\partial_x|_\mathcal{N} \text{ is a canonical section of } b\mathcal{T}_\mathcal{N}\mathcal{M} \text{ and so is } J(x\partial_x|_\mathcal{N}). \]

Therefore \( \mathcal{T} = \text{ev}(J(x\partial_x|_\mathcal{N})) \) is a canonical vector field on \( \mathcal{N} \).

\[ i\mathcal{T} = \text{ev}_\mathcal{N}(iJ(x\partial_x|_\mathcal{N})) \text{ is a nonzero section of } \mathbb{C}T\mathcal{N}. \]
Structure of the boundary

Associated complex: $b\overline{\partial}_b \xrightarrow{\phi} \cdots \xrightarrow{\partial} C^\infty(N, \Lambda^q \overline{V}^*) \xrightarrow{D} C^\infty(N, \Lambda^{q+1} \overline{V}^*) \xrightarrow{\partial} \cdots$

$\overline{V} = \text{ev}_N(bT_{N}^{0,1}M) \hookrightarrow \mathbb{C}T\mathcal{N}$, involutive, $\overline{V} + V = \mathbb{C}T\mathcal{N}$

$bT_{N}^{0,1}M \xrightarrow{\phi} \mathbb{C}bT_{N}M \xrightarrow{\text{ev}} \mathbb{C}T_{N}M$, ker ev = span$_\mathbb{C}(x\partial_x|_N)$

$\overline{b\partial}_x$ vanishes on $\mathcal{N}$.

$\beta = \{\beta_x : x \text{ is defining function for } \mathcal{N}\}$

Let $J : bT\mathcal{M} \rightarrow bT\mathcal{M}$ be the almost complex structure.

$x\partial_x|_\mathcal{N}$ is a canonical section of $bT_{N}M$ and so is $J(x\partial_x|_N)$.

Therefore $\mathcal{T} = \text{ev}(J(x\partial_x|_N))$ is a canonical vector field on $\mathcal{N}$.
Structure of the boundary

Associated complex: $b\overset{\bar{\partial}}{\partial}_b \dashv \cdots \to C^\infty(\mathcal{N}, \wedge^q \overline{V}^*) \overset{\bar{D}}{\to} C^\infty(\mathcal{N}, \wedge^{q+1} \overline{V}^*) \to \cdots$

$b\overline{\partial}x$ vanishes on $\mathcal{N}$.

$\beta = \{ \beta_x : x \text{ is defining function for } \mathcal{N} \}$

Let $J : bT\mathcal{M} \to bT\mathcal{M}$ be the almost complex structure.

$x\partial_x|_{\mathcal{N}}$ is a canonical section of $bT_{\mathcal{N}}\mathcal{M}$ and so is $J(x\partial_x|_{\mathcal{N}})$.

$T = ev(J(x\partial_x|_{\mathcal{N}}))$ is a canonical vector field on $\mathcal{N}$. 

nowhere zero real
Structure of the boundary

\[ \nabla = \text{ev}_N(b^{T^0,1}_N M) \hookrightarrow \mathbb{C} T N, \text{ involutive, } \nabla + \nabla = \mathbb{C} T N \]

\[ b^{T^0,1}_N M \xrightarrow{\text{ev}} \mathbb{C} b^{T^0,1}_N M \to \mathbb{C} T N M, \quad \ker \text{ev} = \text{span}_\mathbb{C}(x \partial_x|_N) \]

\[ \beta = \{ \beta_x : x \text{ is defining function for } N \} \]

Let \( J : bT M \to bT M \) be the almost complex structure.

\[ x \partial_x|_N \text{ is a canonical section of } bT N M \text{ and so is } J(x \partial_x|_N). \]

\[ T = \text{ev}(J(x \partial_x|_N)) \text{ is a canonical vector field on } N. \]

\[ \beta \text{ nowhere zero real } \quad \nabla \cap \nabla = \text{span}_\mathbb{C}(T) \]

\[ \partial M = \{ x = 0 \} \text{ (} x \text{ rather than } t \)
Classification

I’ll focus first on the family $\mathcal{F}$ consisting of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The effect of (3) is to make $p \sim p'$ $\iff$ $p' \in O_p$ the closure of orbit $T$ through $p$. Here is an example where $\sim$ is not a relation of equivalence: (No symmetry) $q \not\sim p \not\in O_p(q \not\in O_{p'}).$

The set of closures of orbits, $B_N$, is a Hausdorff space, $\pi_N : N \to B_N$.

I’ll write $a_t$ for the action of a general $T$. In general the orbits of $T$ need not be periodic (need not be compact).
Classification

I’ll focus first on the family $\mathcal{F}$ consisting of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The effect of (3) is to make $p \sim p' \iff p' \in \overline{O}_p$ a relation of equivalence.
Classification

I’ll focus first on the family $\mathcal{F}$ consisting of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
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The set of closures of orbits, $\mathcal{B}_\mathcal{N}$, is a Hausdorff space.

Here is an example where $\sim$ is not a relation of equivalence: (No symmetry)

$p' \in \overline{O}_p$ but $p \notin \overline{O}_{p'}$

$\cos^2 \theta \partial_\theta + 5 \partial_\psi$
Classification

I’ll focus first on the family $\mathcal{F}$ consisting of pairs $(\mathcal{N}, T)$ such that

1) $\mathcal{N}$ is compact;
2) $T$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $T$-invariant Riemannian metric on $\mathcal{N}$.

The set of closures of orbits, $\mathcal{B}_N$, is a Hausdorff space,

$$\pi_N : N \rightarrow \mathcal{B}_N$$

is like the circle bundle of a Hermitian line bundle with $T$ as the infinitesimal generator of the action

$$(e^{it}, p) \mapsto e^{it} \cdot p = e^{it} p.$$
Classification

I’ll focus first on the family $\mathcal{F}$ consisting of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The set of closures of orbits, $B_\mathcal{N}$, is a Hausdorff space,

$$\pi_\mathcal{N} : \mathcal{N} \to B_\mathcal{N}.$$ 

is like the circle bundle of a Hermitian line bundle with $\mathcal{T}$ as the infinitesimal generator of the action

$$(e^{it}, p) \mapsto e^{it} \cdot p = e^{it} p.$$

I’ll write $\alpha_t$ for the action of a general $\mathcal{T}$. In general the orbits of $\mathcal{T}$ need not be periodic (need not be compact).
Example:

\[ \mathcal{F} = \{ (\mathcal{N}, \mathcal{T}) \text{ such that} \]
\[ 1) \mathcal{N} \text{ is compact} \]
\[ 2) \mathcal{T} \text{ is a nowhere zero real vector field} \]
\[ 3) \text{there is a } \mathcal{T} \text{-invariant metric}\}. \]

Let \( \mathcal{N} = S^2_{N-1} \subset C^N \). Let \( \tau_1, \ldots, \tau_N \) be positive, and let \( \mathcal{T} = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - z_j \partial z_j) \).

Then \( a_t(z_1, \ldots, z_N) = (e^{i \tau_1 t} z_1, \ldots, e^{i \tau_N t} z_N) \).

Since \( a^* t (dz_j \otimes dz_j) = dz_j \otimes dz_j \), the standard metric of \( S^2_{N-1} \) is \( \mathcal{T} \)-invariant.

The closure of each orbit is a torus. If \( p = (z_1, \ldots, z_N) \in S^2_{N-1} \) and \( d_p = \dim \text{span} \{ \tau_j / 2\pi : z_j \neq 0 \} \), then \( \dim O_p = d_p \).

If \( d_p > 1 \) then \( O_p \) is not closed.

The closure of \( \{ a_t : t \in \mathbb{R} \} \) in \( \text{Homeo}(S^2_{N-1}) \) can be identified with \( \text{Homeo}(N) = \text{group of homeomorphisms } N \rightarrow N \) with compact-open topology.

\( \mathcal{G}_0 = \text{closure of } \{ (e^{i \tau_1 t}, \ldots, e^{i \tau_N t}) : t \in \mathbb{R} \} \).

\[ \vdots \]
Example: Let $\mathcal{N} = S^{2N-1} \subset \mathbb{C}^N$. Let $\tau_1, \ldots, \tau_N$ be positive, and let

$$\mathcal{T} = i \sum_{j=1}^N \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

$\mathcal{F} = \{(\mathcal{N}, \mathcal{T}) \text{ such that} \newline
1) \mathcal{N} \text{ is compact} \\
2) \mathcal{T} \text{ is a nowhere zero real vector field} \\
3) \text{there is a } \mathcal{T} \text{-invariant metric}\}.$
Example: Let $\mathcal{N} = S^{2N-1} \subset \mathbb{C}^N$. Let $\tau_1, \ldots, \tau_N$ be positive, and let

$$\mathcal{T} = i \sum_{j=1}^N \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

Then

$$a_t(z_1, \ldots, z_N) = (e^{i\tau_1 t} z_1, \ldots, e^{i\tau_N t} z_N)$$
Example: Let $\mathcal{N} = S^{2N-1} \subset \mathbb{C}^N$. Let $\tau_1, \ldots, \tau_N$ be positive, and let

$$\mathcal{T} = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \overline{z}_j \partial \overline{z}_j).$$

Then

$$a_t(z_1, \ldots, z_N) = (e^{i\tau_1 t} z_1, \ldots, e^{i\tau_N t} z_N)$$

Since $a^*_t dz_j = e^{i\tau_j} dz_j$, the standard metric of $S^{2N-1}$ is $\mathcal{T}$-invariant.
**Example:** Let $\mathcal{N} = S^{2N-1} \subset \mathbb{C}^N$. Let $\tau_1, \ldots, \tau_N$ be positive, and let

$$\mathcal{T} = i \sum_{j=1}^N \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

Then

$$a_t(z_1, \ldots, z_N) = (e^{i\tau_1 t} z_1, \ldots, e^{i\tau_N t} z_N).$$

Since $a^*_t dz_j = e^{i\tau_j} dz_j$, the standard metric of $S^{2N-1}$ is $\mathcal{T}$-invariant.

The closure of each orbit is a torus. If $p = (z_1, \ldots, z_N) \in S^{2N-1}$ and

$$d_p = \dim \text{span}_Q \{\tau_j/2\pi : z_j \neq 0\},$$

then

$$\dim \overline{O}_p = d_p.$$ 

If $d_p > 1$ then $O_p$ is not closed.
Example: Let \( \mathcal{N} = S^{2N-1} \subset \mathbb{C}^N \). Let \( \tau_1, \ldots, \tau_N \) be positive, and let

\[
\mathcal{T} = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \overline{z}_j \partial \overline{z}_j).
\]

Then

\[
a_t(z_1, \ldots, z_N) = (e^{i\tau_1 t} z_1, \ldots, e^{i\tau_N t} z_N)
\]

Since \( a_t^* dz_j = e^{i\tau_j} dz_j \), the standard metric of \( S^{2N-1} \) is \( \mathcal{T} \)-invariant.

The closure of each orbit is a torus. If \( p = (z_1, \ldots, z_N) \in S^{2N-1} \) and

\[
d_p = \dim \text{span}_Q \{ \tau_j/2\pi : z_j \neq 0 \},
\]

then

\[
\dim \overline{O}_p = d_p.
\]

If \( d_p > 1 \) then \( O_p \) is not closed.

The closure of \( \{ a_t : t \in \mathbb{R} \} \) in \( \text{Homeo}(S^{2N-1}) \) can be identified with

\[
G_0 = \text{closure of } \{(e^{i\tau_1 t}, \ldots, e^{i\tau_N t}) : t \in \mathbb{R}\}.
\]

“Structure group” of \( (S^{2N-1}, \mathcal{T}) \).
For every \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) there is \(N \in \mathbb{N}\), an embedding
\[
F : \mathcal{N} \to \mathbb{C}^N,
\]
and positive numbers \(\tau_1, \ldots, \tau_N\) such that
\[
F^* \mathcal{T} = i \sum_{j=1}^N \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j)
\]

\[\mathcal{F} = \{(\mathcal{N}, \mathcal{T}) \text{ such that} \]
1) \(\mathcal{N}\) is compact
2) \(\mathcal{T}\) is a nowhere zero real vector field
3) there is a \(\mathcal{T}\)-invariant metric\}. 

For every \((N, T) \in F\) there is \(N \in \mathbb{N}\), an embedding
\[
F : N \rightarrow \mathbb{C}^N,
\]
and positive numbers \(\tau_1, \ldots, \tau_N\) such that
\[
F_\ast T = i \sum_{j=1}^N \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j)
\]
This is similar to the embedding used to define the first Chern class of a complex line bundle over a compact base:

\[
\mathcal{F} = \{(N, T) \text{ such that} \\
1) N \text{ is compact} \\
2) T \text{ is a nowhere zero real vector field} \\
3) \text{there is a } T\text{-invariant metric}\}.
\]
For every \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) there is \(\mathcal{N} \in \mathbb{N}\), an embedding

\[
F : \mathcal{N} \rightarrow \mathbb{C}^N,
\]

and positive numbers \(\tau_1, \ldots, \tau_N\) such that

\[
F^* \mathcal{T} = i \sum_{j=1}^N \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j)
\]

This is similar to the embedding used to define the first Chern class of a complex line bundle over a compact base:

\[
\begin{array}{c}
\text{A complex line bundle} \\
E \\
\pi \\
B
\end{array}
\]

\[
\mathcal{F} = \{(\mathcal{N}, \mathcal{T}) \text{ such that} \\
1) \mathcal{N} \text{ is compact} \\
2) \mathcal{T} \text{ is a nowhere zero real vector field} \\
3) \text{there is a } \mathcal{T}-\text{invariant metric}\}.
\]
For every \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) there is \(N \in \mathbb{N}\), an embedding

\[ F : \mathcal{N} \rightarrow \mathbb{C}^N, \]

and positive numbers \(\tau_1, \ldots, \tau_N\) such that

\[ F^* \mathcal{T} = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j) \]

This is similar to the embedding used to define the first Chern class of a complex line bundle over a compact base:

A complex line bundle

\[ E \]

\[ \pi \]

\[ \mathcal{B} \]

the tautological line bundle

\[ \Gamma \]

\[ \pi \]

\[ \mathbb{C}P^{N-1} \]

the fiber at \(q \in \mathbb{C}P^{N-1}\) is \(q\)
For every $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$ there is $\mathcal{N} \in \mathbb{N}$, an embedding

$$ F : \mathcal{N} \rightarrow \mathbb{C}^N, $$

and positive numbers $\tau_1, \ldots, \tau_N$ such that

$$ F_* \mathcal{T} = i \sum_{j=1}^N \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j) $$

This is similar to the embedding used to define the first Chern class of a complex line bundle over a compact base:

A complex line bundle $E \xrightarrow{\pi} \mathcal{B}$

the tautological line bundle $\Gamma \xrightarrow{\pi} \mathbb{CP}^{N-1}$

homeomorphism onto image $(N \text{ large})$
For every \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) there is \(\mathcal{N} \in \mathbb{N}\), an embedding

\[
F : \mathcal{N} \to \mathbb{C}^N,
\]

and positive numbers \(\tau_1, \ldots, \tau_N\) such that

\[
F^* \mathcal{T} = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \overline{z}_j \partial \overline{z}_j)
\]

This is similar to the embedding used to define the first Chern class of a complex line bundle over a compact base:

\[
\begin{array}{ccc}
E & \approx & \Phi^* \Gamma \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{B} & \rightarrow & \mathbb{C}P^{N-1}
\end{array}
\]

A complex line bundle

the tautological line bundle

homeomorphism onto image (\(N\) large)
For every \( (\mathcal{N}, \mathcal{T}) \in \mathcal{F} \) there is \( \mathcal{N} \in \mathbb{N} \), an embedding

\[ F : \mathcal{N} \rightarrow \mathbb{C}^N, \]

and positive numbers \( \tau_1, \ldots, \tau_N \) such that

\[ F^* \mathcal{T} = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j) \]

This is similar to the embedding used to define the first Chern class of a complex line bundle over a compact base:

\[ \pi \]

passing to circle bundles:

\[ \Phi \]

equivariant diffeomorphism onto image

\[ \mathcal{F} = \{(\mathcal{N}, \mathcal{T}) \text{ such that} \]

1) \( \mathcal{N} \) is compact
2) \( \mathcal{T} \) is a nowhere zero real vector field
3) there is a \( \mathcal{T} \)-invariant metric\}. 

\[ \Phi \]

\[ \mathbb{CP}^{N-1} \]
For every \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) there is \(\mathcal{N} \in \mathbb{N}\), an embedding

\[
F : \mathcal{N} \to \mathbb{C}^N,
\]

and positive numbers \(\tau_1, \ldots, \tau_N\) such that

\[
F_* \mathcal{T} = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j)
\]

This is similar to the embedding used to define the first Chern class of a complex line bundle over a compact base:

\[
\mathcal{N} \xrightarrow{F} S^{2N-1}, \mathcal{T}'
\]

\[
\pi \downarrow \quad \Phi \downarrow
\]

\[
\mathcal{B}_N \xrightarrow{\Phi} \mathcal{B}_{S^{2N-1}, \mathcal{T}}
\]
Equivalence

\( F = \{ (\mathcal{N}, \mathcal{T}) \text{ such that} \)

1) \( \mathcal{N} \) is compact
2) \( \mathcal{T} \) is a nowhere zero real vector field
3) there is a \( \mathcal{T} \)-invariant metric\}

(\( \mathcal{N}', \mathcal{T}' \)) is globally equivalent to (\( \mathcal{N}, \mathcal{T} \)) if there is a global equivariant diffeomorphism \( \mathcal{N}' \rightarrow \mathcal{N} \).
Equivalence

\((\mathcal{N}', \mathcal{T}')\) is globally equivalent to \((\mathcal{N}, \mathcal{T})\) if there is a global equivariant diffeomorphism \(\mathcal{N}' \to \mathcal{N}\).

\(\mathcal{F} = \{ (\mathcal{N}, \mathcal{T}) \text{ such that } \)

1) \(\mathcal{N}\) is compact
2) \(\mathcal{T}\) is a nowhere zero real vector field
3) there is a \(\mathcal{T}\)-invariant metric \}.

\((\mathcal{N}', \mathcal{T}')\) is locally equivalent to \((\mathcal{N}, \mathcal{T})\) if there are open covers \(\{ U'_a \}_{a \in A}\) of \(\mathcal{N}'\) and \(\{ U_a \}_{a \in A}\) of \(\mathcal{N}\) by invariant sets and equivariant diffeomorphisms

\[ h_a : U'_a \to U_a \]

such that

\[ h_a h_b^{-1}(p) \in \overline{\mathcal{O}}_p \text{ for all } a, b \in A \text{ and } p \in U_a \cap U_b. \]
Equivalence

$\mathcal{F} = \{(\mathcal{N}, \mathcal{T}) \text{ such that} \\\n1) \mathcal{N} \text{ is compact} \\
2) \mathcal{T} \text{ is a nowhere zero real vector field} \\
3) \text{there is a } \mathcal{T} \text{-invariant metric}\}.$

$(\mathcal{N}', \mathcal{T}')$ is globally equivalent to $(\mathcal{N}, \mathcal{T})$ if there is a global equivariant diffeomorphism $\mathcal{N}' \to \mathcal{N}$.

$(\mathcal{N}', \mathcal{T}')$ is locally equivalent to $(\mathcal{N}, \mathcal{T})$ if there are open covers $\{U'_a\}_{a \in A}$ of $\mathcal{N}'$ and $\{U_a\}_{a \in A}$ of $\mathcal{N}$ by invariant sets and equivariant diffeomorphisms $h_a : U'_a \to U_a$ such that

$$h_a h_b^{-1}(p) \in \overline{\mathcal{O}}_p \text{ for all } a, b \in A \text{ and } p \in U_a \cap U_b.$$

If $(\mathcal{N}', \mathcal{T}')$ is locally equivalent to $(\mathcal{N}, \mathcal{T})$, then $\mathcal{B}_{\mathcal{N}'}$ is canonically diffeomorphic to $\mathcal{B}_{\mathcal{N}}$. Details for this and the following can be found in G. M., Characteristic classes of the boundary of a complex b-manifold, in Complex Analysis: Several complex Variables and Connections with PDE Theory (Trends in Mathematics), 245–262, P. Ebenfelt et al (Eds.), Birkhäuser, 2010.
Fix some \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\), let \(\mathcal{B}\) be its base space, 
\[\pi : \mathcal{N} \to \mathcal{B}\]
the projection map, \(G\) the closure of \(\{a_t : t \in \mathbb{R}\}\) in \(\text{Homeo}(\mathcal{N})\), a compact abelian group, 
\(\mathfrak{g}\) its Lie algebra, \(\mathfrak{z}\) the kernel of \(\exp : \mathfrak{g} \to G\).

\[\mathcal{F} = \{(\mathcal{N}, \mathcal{T}) \text{ such that}\]
\[\begin{align*}
1) & \mathcal{N} \text{ is compact} \\
2) & \mathcal{T} \text{ is a nowhere zero real vector field} \\
3) & \text{there is a } \mathcal{T}\text{-invariant metric}\end{align*}\]

Homeo\((\mathcal{N})\) is the group of homeos \(\mathcal{N} \to \mathcal{N}\) with the compact-open topology, and \(a_t\) is an isometry for some metric.
Fix some \((N, T) \in F\), let \(B\) be its base space, 
\[ \pi : N \to B \]
the projection map, \(G\) the closure of \(\{a_t : t \in \mathbb{R}\}\)
in \(\text{Homeo}(N)\), a compact abelian group, 
\(g\) its Lie algebra, \(z\) the kernel of \(\exp : g \to G\).
\[ \exists! \hat{T} \in g : a_t = \exp(t\hat{T}). \]

\[ F = \{(N, T) \text{ such that} \]
1) \(N\) is compact
2) \(T\) is a nowhere zero real vector field
3) there is a \(T\)-invariant metric\}.

\(\text{Homeo}(N)\) is the group of homeos \(N \to N\) with the compact-open topology, and \(a_t\) is an isometry for some metric.
Fix some \((\mathcal{N}, T) \in \mathcal{F}\), let \(B\) be its base space, 
\[\pi : \mathcal{N} \to B\]
the projection map, \(G\) the closure of \(\{a_t : t \in \mathbb{R}\}\) in \(\text{Homeo}(\mathcal{N})\), a compact abelian group, 
\(\mathfrak{g}\) its Lie algebra, \(\mathfrak{z}\) the kernel of \(\exp : \mathfrak{g} \to G\).
\[\exists ! \hat{T} \in \mathfrak{g} : a_t = \exp(t\hat{T}).\]
Let \(\mathcal{L} = \text{sheaf of locally constant } \mathfrak{z}\text{-valued functions on } B\).

\[\mathcal{F} = \{(\mathcal{N}, T) \text{ such that}\]
\[1) \mathcal{N} \text{ is compact}\]
\[2) T \text{ is a nowhere zero real vector field}\]
\[3) \text{there is a } T\text{-invariant metric}\}\]
Fix some \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\), let \(\mathcal{B}\) be its base space,  
\[ \pi : \mathcal{N} \to \mathcal{B} \]
the projection map, \(G\) the closure of \(\{a_t : t \in \mathbb{R}\}\)  
in Homeo(\(\mathcal{N}\)), a compact abelian group,  
\(\mathfrak{g}\) its Lie algebra, \(\mathfrak{z}\) the kernel of \(\exp : \mathfrak{g} \to G\).  
\[ \exists! \hat{T} \in \mathfrak{g} : a_t = \exp(t\hat{T}). \]
Let \(\mathcal{L} = \text{sheaf of locally constant } \mathfrak{z}\text{-valued functions on } \mathcal{B}\).

The set of equivalence classes of elements of \(\mathcal{F}\)  
which are locally equivalent to \((\mathcal{N}, \mathcal{T})\) modulo global equivalence is in  
one to one correspondence with \(\check{H}^2(\mathcal{B}, \mathcal{L})\), the second \(\check{\text{C}}\)ech cohomology  
group of \(\mathcal{B}\) with coefficients in \(\mathcal{L}\).
Fix some \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\), let \(\mathcal{B}\) be its base space, 
\[ \pi : \mathcal{N} \to \mathcal{B} \]
the projection map, \(G\) the closure of \(\{a_t : t \in \mathbb{R}\}\) in \(\text{Homeo}(\mathcal{N})\), a compact abelian group, 
\(\mathfrak{g}\) its Lie algebra, \(\mathfrak{z}\) the kernel of \(\exp : \mathfrak{g} \to G\). 
\[ \exists! \hat{T} \in \mathfrak{g} : a_t = \exp(t\hat{T}). \]
Let \(\mathcal{L} = \text{sheaf of locally constant } \mathfrak{z}\text{-valued functions on } \mathcal{B}. \)

The set of equivalence classes of elements of \(\mathcal{F}\) which are locally equivalent to \((\mathcal{N}, \mathcal{T})\) modulo global equivalence is in one to one correspondence with \(\check{H}^2(\mathcal{B}, \mathcal{L})\), the second \(\check{\text{Čech}}\) cohomology group of \(\mathcal{B}\) with coefficients in \(\mathcal{L}\).

The classification is relative to a fixed, but arbitrarily chosen, element of \(\mathcal{F}\) because there is no canonical element to which to compare. As with complex line bundles, this follows from the exactness of the long sequence in cohomology associated with a short sequence of sheafs.
Some sheaves \((\mathcal{N}, \mathcal{T})\) such that
1) \(\mathcal{N}\) is compact
2) \(\mathcal{T}\) is a nowhere zero real vector field
3) there is a \(\mathcal{T}\)-invariant metric.

\((\mathcal{N}, \mathcal{T})\) under addition
\((\mathcal{N}, \mathcal{T})\) under composition

\(C^{\infty}(\mathcal{B}, g)\) and \(I^{\infty}(\mathcal{B}, \mathcal{N})\) be the respective sheaves.

\(C^{\infty}(\mathcal{B}, g)\) is fine.

The set of equivalence classes of elements of \(F\) which are locally equivalent to \((\mathcal{N}, \mathcal{T})\) modulo global equivalence is in one to one correspondence with \(\check{\mathcal{H}}_{1}(\mathcal{B}, I^{\infty}(\mathcal{B}, \mathcal{N}))\).

\((\mathcal{N}', \mathcal{T}') \sim (\mathcal{N}, \mathcal{T})\) if there are open covers \(\{U_{a}'\}_{a \in A}\) of \(\mathcal{N}'\) and \(\{U_{a}\}_{a \in A}\) of \(\mathcal{N}\) by invariant sets and for each \(a \in A\) equivariant diffeomorphisms \(h_{a}: U_{a}' \to U_{a}\) such that \(h_{ab} = h_{a} h_{b}^{-1}\) \(\in I^{\infty}(U_{a} \cap U_{b})\) for all \(a, b \in A\).

\(\{h_{ab}\}_{a, b \in A}\) defines a \(\check{\mathcal{C}}ech\) cocycle.

There is an exact sequence

\(0 \to \mathbb{Z}(\mathcal{B}) \to C^{\infty}(\mathcal{B}, g) \xrightarrow{\text{Exp}} I^{\infty}(\mathcal{B}, \mathcal{N}) \to 0\).

\(K\) sheaf of locally constant \(z\)-valued functions on \(\mathcal{B}\).

Consequence: \(\check{\mathcal{H}}_{1}(\mathcal{B}, I^{\infty}(\mathcal{B})) \approx \check{\mathcal{H}}_{2}(\mathcal{B}, \mathbb{Z}(\mathcal{B}))\) using (*) and the long exact sequence in cohomology.
Some sheaves

$B =$ base space of $\mathcal{N}$, $U \subset B$ open

$(\mathcal{N}, \mathcal{T})$ such that
1) $\mathcal{N}$ is compact
2) $\mathcal{T}$ is a nowhere zero real vector field
3) there is a $\mathcal{T}$-invariant metric.
Some sheaves

\( B = \text{base space of } \mathcal{N}, \ U \subset B \text{ open} \)

\( (\mathcal{N}, \mathcal{T}) \text{ such that} \)
1) \( \mathcal{N} \text{ is compact} \)
2) \( \mathcal{T} \text{ is a nowhere zero real vector field} \)
3) there is a \( \mathcal{T} \)-invariant metric.

\( C^\infty(U, g) \) is the space of smooth functions \( f : \pi^{-1}(U) \rightarrow g \) such that \( f \) is constant on orbits of \( \mathcal{T} \).
Some sheaves

\( B = \text{base space of } \mathcal{N}, \ U \subset B \text{ open} \)

\((\mathcal{N}, \mathcal{T})\) such that

1) \(\mathcal{N}\) is compact
2) \(\mathcal{T}\) is a nowhere zero real vector field
3) there is a \(\mathcal{T}\)-invariant metric.

\(C^\infty(U, g)\) is the space of smooth functions \(f : \pi^{-1}(U) \to g\) such that \(f\) is constant on orbits of \(\mathcal{T}\).

\(I^\infty(U)\) is the space of smooth equivariant maps \(h : \pi^{-1}(U) \to \pi^{-1}(U)\) such that \(h(p) \in \overline{O}_p\) for each \(p \in \pi^{-1}(U)\).
Some sheaves

\( B = \text{base space of } \mathcal{N}, U \subset B \text{ open} \)

\( \mathcal{N}, \mathcal{T} \) such that

1) \( \mathcal{N} \) is compact
2) \( \mathcal{T} \) is a nowhere zero real vector field
3) there is a \( \mathcal{T} \)-invariant metric.

\( C^\infty(U, g) \) is the space of smooth functions \( f : \pi^{-1}(U) \to g \) such that \( f \) is constant on orbits of \( \mathcal{T} \).

\( \{ C^\infty(U, g) \}_{U \text{ open}} \) is an abelian presheaf over \( B \) under addition

\( \mathcal{I}^\infty(U) \) is the space of smooth equivariant maps \( h : \pi^{-1}(U) \to \pi^{-1}(U) \) such that \( h(p) \in \overline{O}_p \) for each \( p \in \pi^{-1}(U) \).
Some sheaves

\[ B = \text{base space of } \mathcal{N}, \; U \subset B \text{ open} \]

\((\mathcal{N}, \mathcal{T})\) such that

1) \(\mathcal{N}\) is compact
2) \(\mathcal{T}\) is a nowhere zero real vector field
3) there is a \(\mathcal{T}\)-invariant metric.

\(C^\infty(U, g)\) is the space of smooth functions \(f : \pi^{-1}(U) \rightarrow g\) such that \(f\) is constant on orbits of \(\mathcal{T}\).

\(\mathcal{I}^\infty(U)\) is the space of smooth equivariant maps \(h : \pi^{-1}(U) \rightarrow \pi^{-1}(U)\) such that \(h(p) \in \overline{\mathcal{O}}_p\) for each \(p \in \pi^{-1}(U)\).

\(\{C^\infty(U, g)\}_{\{U \text{ open}\}}\) is an abelian presheaf over \(B\) under addition

\(\{\mathcal{I}^\infty(U)\}_{\{U \text{ open}\}}\) is an abelian presheaf under composition

[Classification theorems]

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Some sheaves

$B =$base space of $\mathcal{N}$, $U \subset B$ open

$(\mathcal{N}, \mathcal{T})$ such that
1) $\mathcal{N}$ is compact
2) $\mathcal{T}$ is a nowhere zero real vector field
3) there is a $\mathcal{T}$-invariant metric.

$\mathcal{C}^\infty(U, \mathfrak{g})$ is the space of smooth functions $f : \pi^{-1}(U) \to \mathfrak{g}$ such that $f$ is constant on orbits of $\mathcal{T}$.

$\mathcal{I}^\infty(U)$ is the space of smooth equivariant maps $h : \pi^{-1}(U) \to \pi^{-1}(U)$ such that $h(p) \in \overline{O}_p$ for each $p \in \pi^{-1}(U)$.

$\{\mathcal{C}^\infty(U, \mathfrak{g})\}_{U \text{ open}}$ is an abelian presheaf over $B$ under addition

$\{\mathcal{I}^\infty(U)\}_{U \text{ open}}$ is an abelian presheaf under composition

Let $\mathcal{C}^\infty(B, \mathfrak{g})$ and $\mathcal{I}^\infty(B, \mathcal{N})$ be the respective sheafs.

$\mathcal{C}^\infty(B, \mathfrak{g})$ is fine.
Some sheaves

\( B = \text{base space of } \mathcal{N}, \ U \subset B \text{ open} \)

\( \mathcal{F} = \{ (\mathcal{N}, \mathcal{T}) \text{ such that} \)

1) \( \mathcal{N} \text{ is compact} \)
2) \( \mathcal{T} \text{ is a nowhere zero real vector field} \)
3) there is a \( \mathcal{T} \)-invariant metric.

\( \{ C^\infty(U, \mathfrak{g}) \}_U \text{ is an abelian presheaf over } B \text{ under addition} \)

\( \{ \mathcal{I}^\infty(U) \}_U \text{ is an abelian presheaf under composition} \)

\( \mathcal{C}^\infty(\mathcal{B}, \mathfrak{g}) \) and \( \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \) be the respective sheafs. \( \mathcal{C}^\infty(\mathcal{B}, \mathfrak{g}) \) is fine.

The set of equivalence classes of elements of \( \mathcal{F} \) which are locally equivalent to \( (\mathcal{N}, \mathcal{T}) \) modulo global equivalence is in one to one correspondence with \( \check{H}^1(\mathcal{B}, \mathcal{I}^\infty(\mathcal{B}, \mathcal{N})) \).

\( (\mathcal{N}', \mathcal{T}') \sim (\mathcal{N}, \mathcal{T}) \) if there are open covers \( \{ U'_a \}_{a \in A} \) of \( \mathcal{N}' \) and \( \{ U_a \}_{a \in A} \) of \( \mathcal{N} \) by invariant sets and for each \( a \in A \) equivariant diffeomorphisms \( h_a : U'_a \to U_a \) such that \( h_{ab} = h_a h^{-1}_b \in \mathcal{I}^\infty(U_a \cap U_b) \) for all \( a, b \in A \).
Some sheaves

$\mathcal{B} = \text{base space of } \mathcal{N}, \ U \subset \mathcal{B} \text{ open}$

$(\mathcal{N}, \mathcal{T}) \text{ such that}$

1) $\mathcal{N}$ is compact
2) $\mathcal{T}$ is a nowhere zero real vector field
3) there is a $\mathcal{T}$-invariant metric.

$C^\infty(U, \mathfrak{g})$ is the space of smooth functions $f : \pi^{-1}(U) \rightarrow \mathfrak{g}$ such that $f$ is constant on orbits of $\mathcal{T}$.

$\{C^\infty(U, \mathfrak{g})\}_{\{U \text{ open}\}}$ is an abelian presheaf over $\mathcal{B}$ under addition

$I^\infty(U)$ is the space of smooth equivariant maps $h : \pi^{-1}(U) \rightarrow \pi^{-1}(U)$ such that $h(p) \in \overline{O}_p$ for each $p \in \pi^{-1}(U)$.

$\{I^\infty(U)\}_{\{U \text{ open}\}}$ is an abelian presheaf under composition

Let $C^\infty(\mathcal{B}, \mathfrak{g})$ and $I^\infty(\mathcal{B}, \mathcal{N})$ be the respective sheafs.

$C^\infty(\mathcal{B}, \mathfrak{g})$ is fine.

The set of equivalence classes of elements of $\mathcal{F}$ which are locally equivalent to $(\mathcal{N}, \mathcal{T})$ modulo global equivalence is in one to one correspondence with $\check{H}^1(\mathcal{B}, I^\infty(\mathcal{B}, \mathcal{N}))$.

$(\mathcal{N}', \mathcal{T}') \sim (\mathcal{N}, \mathcal{T})$ if there are open covers $\{U'_a\}_{a \in A}$ of $\mathcal{N}'$ and $\{U_a\}_{a \in A}$ of $\mathcal{N}$ by invariant sets and for each $a \in A$ equivariant diffeomorphisms $h_a : U'_a \rightarrow U_a$ such that

$h_{ab} = h_a h_b^{-1} \in I^\infty(U_a \cap U_b)$ for all $a, b \in A$. 

$\{h_{ab}\}_{a, b \in A}$ defines a Čech cocycle.

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Some sheaves

\( B = \text{base space of } \mathcal{N}, \ U \subset B \text{ open} \)

\( C^\infty(U, g) \) is the space of smooth functions \( f : \pi^{-1}(U) \to g \) such that \( f \) is constant on orbits of \( T \).

\( \{ C^\infty(U, g) \}_{\{ U \text{ open} \}} \) is an abelian presheaf over \( B \) under addition

\( \mathcal{I}^\infty(U) \) is the space of smooth equivariant maps \( h : \pi^{-1}(U) \to \pi^{-1}(U) \) such that \( h(p) \in \overline{O}_p \) for each \( p \in \pi^{-1}(U) \).

\( \{ \mathcal{I}^\infty(U) \}_{\{ U \text{ open} \}} \) is an abelian presheaf under composition

Let \( C^\infty(B, g) \) and \( \mathcal{I}^\infty(B, \mathcal{N}) \) be the respective sheaves. \( C^\infty(B, g) \) is fine.

The set of equivalence classes of elements of \( \mathcal{F} \) which are locally equivalent to \( (\mathcal{N}, T) \) modulo global equivalence is in one to one correspondence with \( \check{H}^1(B, \mathcal{I}^\infty(B, \mathcal{N})) \).

There is an exact sequence

\[ 0 \to \mathcal{L}(B) \to C^\infty(B, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(B, \mathcal{N}) \to 0. \]
Some sheaves

$B =$ base space of $\mathcal{N}$, $U \subset B$ open

$(\mathcal{N}, \mathcal{T})$ such that
1) $\mathcal{N}$ is compact
2) $\mathcal{T}$ is a nowhere zero real vector field
3) there is a $\mathcal{T}$-invariant metric.

$C^\infty(U, \mathfrak{g})$ is the space of smooth functions$f : \pi^{-1}(U) \to \mathfrak{g}$ such that $f$ is constant on orbits of $\mathcal{T}$. 

$\{C^\infty(U, \mathfrak{g})\}_{\{U \text{ open}\}}$ is an abelian presheaf over $B$ under addition

$I^\infty(U)$ is the space of smooth equivariant maps $h : \pi^{-1}(U) \to \pi^{-1}(U)$ such that $h(p) \in \overline{O}_p$ for each $p \in \pi^{-1}(U)$.

$\{I^\infty(U)\}_{\{U \text{ open}\}}$ is an abelian presheaf under composition

Let $C^\infty(B, \mathfrak{g})$ and $I^\infty(B, \mathcal{N})$ be the respective sheafs.

$C^\infty(B, \mathfrak{g})$ is fine.

The set of equivalence classes of elements of $\mathcal{F}$ which are locally equivalent to $(\mathcal{N}, \mathcal{T})$ modulo global equivalence is in one to one correspondence with $\check{H}^1(B, I^\infty(B, \mathcal{N}))$.

There is an exact sequence

\[
0 \to \mathcal{L}(B) \xrightarrow{i} C^\infty(B, \mathfrak{g}) \xrightarrow{\text{Exp}} I^\infty(B, \mathcal{N}) \to 0.
\]

sheaf of locally constant $\mathfrak{g}$-valued functions on $B$. 

\[\text{(Temple University)}\] Classification theorems

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Some sheaves

\( C^\infty(U, g) \) is the space of smooth functions \( f: \pi^{-1}(U) \rightarrow g \) such that \( f \) is constant on orbits of \( T \).

\( \mathcal{I}^\infty(U) \) is the space of smooth equivariant maps \( h: \pi^{-1}(U) \rightarrow \pi^{-1}(U) \) such that \( h(p) \in \overline{O}_p \) for each \( p \in \pi^{-1}(U) \).

Let \( C^\infty(B, g) \) and \( \mathcal{I}^\infty(B, \mathcal{N}) \) be the respective sheafs.

\( \{C^\infty(U, g)\}_{U \text{ open}} \) is an abelian presheaf over \( B \) under addition.

\( \{\mathcal{I}^\infty(U)\}_{U \text{ open}} \) is an abelian presheaf under composition.

\( \mathcal{B} \) is the base space of \( \mathcal{N} \), \( U \subset \mathcal{B} \) open.

\( (\mathcal{N}, T) \) such that:
1) \( \mathcal{N} \) is compact
2) \( T \) is a nowhere zero real vector field
3) there is a \( T \)-invariant metric.

The set of equivalence classes of elements of \( \mathcal{F} \) which are locally equivalent to \( (\mathcal{N}, T) \) modulo global equivalence is in one to one correspondence with \( \check{H}^1(B, \mathcal{I}^\infty(B, \mathcal{N})) \).

Consequence:
\( \check{H}^1(B, \mathcal{I}^\infty(B)) \approx \check{H}^2(B, \mathcal{L}(B)) \) using \((*)\) and the long exact sequence in cohomology.

There is an exact sequence
\[
0 \rightarrow \mathcal{L}(B) \rightarrow C^\infty(B, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(B, \mathcal{N}) \rightarrow 0.
\]

sheaf of locally constant \( \mathbb{Z} \)-valued functions on \( B \).
$$0 \to \mathcal{L}(\mathcal{B}) \xrightarrow{\varepsilon} \mathcal{C}^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \to 0.$$
\[ 0 \to \mathcal{L}(B) \xrightarrow{\iota} \mathcal{C}^\infty(B, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(B, \mathcal{N}) \to 0. \]
$0 \to \mathcal{L}(\mathcal{B}) \xrightarrow{i} C^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \to 0.$

constant functions $f: \pi^{-1}(U) \to \mathfrak{g}$

$\mathcal{I}^\infty(\mathcal{B}, \mathcal{N})$ is the map obtained from inclusion $\pi^{-1}(U) \to \mathcal{I}^\infty(\mathcal{B}, \mathcal{N})$.

$\mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) = \{h \in \text{Homeo}(U) : h^*T = h\}$

Let $A_g$ be the action of $g \in G$ on $\mathcal{N}$.

$\exists^! \hat{T} \in g : \forall p : O_p = \{A_g p : g \in G\}$

$\exp \circ f : \pi^{-1}(U) \to G$ is constant on $O_p$ and $A_{\exp(f(p))} : O_p \to O_p$.

$\exp \circ f \cdot \alpha_t = \alpha_t \cdot \exp(f(a_t)) : A_{\exp(f(a_t))} \circ \alpha_t = A_{\exp(t \hat{T})} \circ \alpha_t = \alpha_t$. 

$\exp \circ f \in I^\infty(U)$; define $C^\infty(B, g)$ as the induced map.

The only nontrivial part about the exactness of the sequence is the proof of surjectivity of $\text{Exp}$.
\[ \text{constant functions } f : \pi^{-1}(U) \to \mathfrak{g} \]

\[ C^\infty(U, g) = \{ f : \pi^{-1}(U) \to g : f \text{ smooth, constant on orbits} \} \]

\[ 0 \to \mathcal{L}(\mathcal{B}) \xrightarrow{i} C^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \to 0. \]

\[ \text{Exp}^{-\to} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \to 0. \]
\[ 0 \to \mathcal{L}(\mathcal{B}) \overset{\imath}{\to} C^\infty(\mathcal{B}, g) \overset{\text{Exp}}{\to} \mathcal{I}\infty(\mathcal{B}, \mathcal{N}) \to 0. \]

\( C^\infty(U, g) = \{ f : \pi^{-1}(U) \to g : f \text{ smooth, constant on orbits} \} \)

\( \mathcal{L} \overset{\imath}{\to} C^\infty(\mathcal{B}, g) \) is the map obtained from inclusion \( \mathfrak{z} \hookrightarrow g \)

\( \mathcal{I}\infty(\mathcal{B}, \mathcal{N}) \) is the map obtained from inclusion \( \mathfrak{z} \hookrightarrow g \)

\( \pi^{-1}(U) \to g \) defined by presheaf \( \{ C^\infty(U, g) \} \)
\[ C^\infty(U, g) = \{ f : \pi^{-1}(U) \to g : f \text{ smooth, constant on orbits} \} \]

\[ 0 \to \mathcal{L}(\mathcal{B}) \xrightarrow{\iota} C^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \to 0. \]

\( \mathcal{L} \xrightarrow{\iota} C^\infty(\mathcal{B}, g) \) is the map obtained from inclusion \( \iota : \mathcal{Z} \hookrightarrow \mathcal{G} \)

Let \( \mathcal{A}_g \) be the action of \( g \in G \) on \( \mathcal{N} \).

\[ \exists! \hat{T} \in g : a_t = \exp(t \hat{T}), \mathcal{A}_{\exp(t \hat{T})} = a_t. \quad \forall p : \overline{O}_p = \{ \mathcal{A}_g p : g \in G \} \]
\[ 0 \to \mathcal{L}(\mathcal{B}) \overset{l}{\to} C^\infty(\mathcal{B}, \mathfrak{g}) \overset{\text{Exp}}{\to} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \to 0. \]

Constant functions \( f : \pi^{-1}(U) \to \mathfrak{z} \)
defined by presheaf \( \{ C^\infty(U, \mathfrak{g}) \} \)

\( \mathcal{L} \overset{l}{\to} C^\infty(\mathcal{B}, \mathfrak{g}) \) is the map obtained from inclusion \( \mathfrak{z} \hookrightarrow \mathfrak{g} \)

Let \( \mathcal{A}_g \) be the action of \( g \in G \) on \( \mathcal{N} \). If \( f \in C^\infty(U, \mathfrak{g}) \),
define

\[ \text{Exp}(f) : \pi^{-1}(U) \to \pi^{-1}(U) \]

by:

\[ C^\infty(U, \mathfrak{g}) = \{ f : \pi^{-1}(U) \to \mathfrak{g} : f \text{ smooth, constant on orbits} \} \]
\[ C^\infty(U, g) = \{ f : \pi^{-1}(U) \to g : f \text{ smooth, constant on orbits} \} \]

\[ 0 \to \mathcal{L}(\mathcal{B}) \xrightarrow{\iota} C^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \to 0. \]

\( \mathcal{L} \xrightarrow{\iota} C^\infty(\mathcal{B}, g) \) is the map obtained from inclusion \( \mathfrak{g} \hookrightarrow g \)

Let \( \mathfrak{U}_g \) be the action of \( g \in G \) on \( \mathcal{N} \). If \( f \in C^\infty(U, g) \),

\[ \exists! \hat{T} \in g : a_t = \exp(t\hat{T}), \mathfrak{U}_{\exp(t\hat{T})} = a_t. \quad \forall p : \overline{O}_p = \{ \mathfrak{U}_g p : g \in G \} \]

define

\[ \text{Exp}(f) : \pi^{-1}(U) \to \pi^{-1}(U) \]

by:

\[ \text{Exp}(f)(p) = \mathfrak{U}_{\exp(f(p))}(p) \]
$C^\infty(U, g) = \{ f : \pi^{-1}(U) \rightarrow g : f \text{ smooth, constant on orbits} \}$

\[ 0 \rightarrow \mathcal{L}(\mathcal{B}) \xrightarrow{\iota} C^\infty(\mathcal{B}, g) \xrightarrow{\operatorname{Exp}} \mathcal{I}(\mathcal{B}, \mathcal{N}) \rightarrow 0. \]

constant functions \( f : \pi^{-1}(U) \rightarrow \mathfrak{g} \) is the map obtained from inclusion \( \mathfrak{g} \hookrightarrow \mathfrak{g} \)

Let \( \mathcal{A}_g \) be the action of \( g \in G \) on \( \mathcal{N} \). If \( f \in C^\infty(U, g) \), define

\[ \operatorname{Exp}(f) : \pi^{-1}(U) \rightarrow \pi^{-1}(U) \]

by:

\[ \operatorname{Exp}(f)(p) = \mathcal{A}_\exp(f(p))(p) \]
\[ 0 \to L(B) \xrightarrow{\lambda} C^\infty(B, g) \xrightarrow{\text{Exp}} I^\infty(B, N) \to 0. \]

\( L \xrightarrow{\lambda} C^\infty(B, g) \) is the map obtained from inclusion \( \mathfrak{z} \hookrightarrow g \)

Let \( \mathcal{A}_g \) be the action of \( g \in G \) on \( N \). If \( f \in C^\infty(U, g) \),

\[ \exists! \hat{T} \in g : a_t = \exp(t \hat{T}), \mathcal{A}_{\exp(t \hat{T})} = a_t. \]

\[ \forall p : \overline{O}_p = \{ \mathcal{A}_g p : g \in G \} \]

\( \exp f : \pi^{-1}(U) \to G \) is constant on \( \overline{O}_p \) and

\( \mathcal{A}_{\exp(f(p))} : \overline{O}_p \to \overline{O}_p. \)[/p]
\[ \mathcal{L} \overset{l}{\longrightarrow} C^\infty(B, g) \overset{\text{Exp}}{\longrightarrow} \mathcal{I}^\infty(B, \mathcal{N}) \rightarrow 0. \]

\(0 \rightarrow \mathcal{L}(B) \overset{l}{\rightarrow} C^\infty(B, g) \overset{\text{Exp}}{\rightarrow} \mathcal{I}^\infty(B, \mathcal{N}) \rightarrow 0.\)

constant functions
\(f : \pi^{-1}(U) \rightarrow \mathcal{N}\)
defined by presheaf
\(\{C^\infty(U, g)\}\)

\(\mathcal{I}^\infty(U) = \{h \in \text{Homeo}(U) : h_* \mathcal{T} = h\}\)

\(\mathcal{L} \overset{l}{\rightarrow} C^\infty(B, g)\) is the map obtained from inclusion \(\mathcal{N} \hookrightarrow g\)

Let \(\mathcal{A}_g\) be the action of \(g \in G\) on \(\mathcal{N}\). If \(f \in C^\infty(U, g)\), define
\[ \text{Exp}(f) : \pi^{-1}(U) \rightarrow \pi^{-1}(U) \]

by:
\[ \text{Exp}(f)(p) = \mathcal{A}_{\text{exp}(f(p))}(p) \in G \]

\(\text{Exp}(f) \in \mathcal{I}^\infty(U)\); define
\[ C^\infty(B, g) \overset{\text{Exp}}{\longrightarrow} \mathcal{I}^\infty(B, \mathcal{N}) \]

as the induced map.
\[ C^\infty(U, g) = \{ f : \pi^{-1}(U) \to g : f \text{ smooth, constant on orbits} \} \]

\[ 0 \to \mathcal{L}(\mathcal{B}) \xrightarrow{\iota} C^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \to 0. \]

\[ \mathcal{L} \xrightarrow{\iota} C^\infty(\mathcal{B}, g) \] is the map obtained from inclusion \( \iota \xrightarrow{} g \)

Let \( \mathcal{A}_g \) be the action of \( g \in G \) on \( \mathcal{N} \). If \( f \in C^\infty(U, g) \), define

\[ \text{Exp}(f) : \pi^{-1}(U) \to \pi^{-1}(U) \]

by:

\[ \text{Exp}(f)(p) = \mathcal{A}_{\text{exp}(f(p))}(p) \]

\( \text{Exp}(f) \in \mathcal{I}^\infty(U) \); define

\[ C^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \]

as the induced map.

The only nontrivial part about the exactness of the sequence is the proof of surjectivity of \( \text{Exp} \).
Surjectivity of $\mathcal{C}^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N})$

Let $U \subset \mathcal{B}$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in \mathcal{C}^\infty(U', \mathfrak{g})$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$. 

Finish proof.
Surjectivity of $C^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N})$

Let $U \subset \mathcal{B}$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in C^\infty(U', \mathfrak{g})$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$.

**Proof:** There is $F : \mathcal{N} \to S^{2N-1}$ such that

$$F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^N \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

All $\tau_j > 0$. 

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Surjectivity of $\mathcal{C}^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N})$

Let $U \subset \mathcal{B}$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in \mathcal{C}^\infty(U', \mathfrak{g})$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$.

**Proof:** There is $F : \mathcal{N} \to S^{2N-1}$ such that

$$F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^N \tau_j(z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

If $p \in \pi^{-1}(U)$, then $h(p) \in \overline{O}_p$, so there is $\omega(p) \in G_0$ such that $F(h(p)) = \mathcal{A}'_{\omega(p)} F(p)$.
Surjectivity of $\mathcal{C}^\infty(B, g) \xrightarrow{\exp} \mathcal{I}^\infty(B, \mathcal{N})$

Let $U \subset B$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in C^\infty(U', g)$ s.t. $h = \exp(f)$ in $\pi^{-1}(U')$.

**Proof:** There is $F : \mathcal{N} \to S^{2N-1}$ such that

$$F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

If $p \in \pi^{-1}(U)$, then $h(p) \in \overline{O}_p$, so there is $\omega(p) \in G_0$ such that $F(h(p)) = \mathcal{A}'_{\omega(p)} F(p)$. All $\tau_j > 0$.

$\mathcal{A}'_g = \text{action of } G_0 = \text{closure of } \{e^{i\tau_1 t}, \ldots, e^{i\tau_N t}\}$ in $S^{2N-1}$. 

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Surjectivity of \( \mathcal{C}^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \)

Let \( U \subset \mathcal{B} \) be open, \( h \in \mathcal{I}^\infty(\pi^{-1}(U)), x_0 \in U \). There is a nbd \( U' \subset U \) of \( x_0 \) and \( f \in \mathcal{C}^\infty(U', \mathfrak{g}) \) s.t. \( h = \text{Exp}(f) \) in \( \pi^{-1}(U') \).

**Proof:** There is \( F : \mathcal{N} \to S^{2N-1} \) such that
\[
F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).
\]
If \( p \in \pi^{-1}(U) \), then \( h(p) \in \overline{O}_p \), so there is \( \omega(p) \in G_0 \) such that \( F(h(p)) = \mathcal{A}'(\omega(p))F(p) \).
Componentwise: \( F_\ell(h(p)) = \omega_\ell(p) F_\ell(p) \).

All \( \tau_j > 0 \).
\( \mathcal{A}_g' = \text{action of } G_0 = \text{closure of } \{ e^{i\tau_1 t}, \ldots, e^{i\tau_N t} \} \)
in \( S^{2N-1} \).
\( \mathcal{A}'_\omega z = (\omega_1 z_1, \ldots, \omega_N z_N) \)
Surjectivity of $\mathcal{C}^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N})$

Let $U \subset \mathcal{B}$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in \mathcal{C}^\infty(U', \mathfrak{g})$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$.

**Proof:** There is $F : \mathcal{N} \to S^{2N-1}$ such that

$$F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

If $p \in \pi^{-1}(U)$, then $h(p) \in \overline{O}_p$, so there is $\omega(p) \in G_0$ such that $F(h(p)) = \mathcal{A}'_{\omega(p)} F(p)$.

**Componentwise:** $F_\ell(h(p)) = \omega_\ell(p) F_\ell(p)$.

If all $F_\ell(p) \neq 0$ then $\omega(p)$ is unique:

All $\tau_j > 0$.

$\mathcal{A}_g = \text{action of } G_0 = \text{closure of } \{e^{i\tau_1 t}, \ldots, e^{i\tau_N t}\}$ in $S^{2N-1}$.

$\mathcal{A}_\omega z = (\omega_1 z_1, \ldots, \omega_N z_n)$
Surjectivity of $\mathcal{C}^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N})$

Let $U \subset \mathcal{B}$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in C^\infty(U', \mathfrak{g})$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$.

**Proof:** There is $F : \mathcal{N} \to S^{2N-1}$ such that

$$F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

If $p \in \pi^{-1}(U)$, then $h(p) \in \overline{O}_p$, so there is $\omega(p) \in G_0$ such that $F(h(p)) = \mathcal{A}'_{\omega(p)} F(p)$.

Componentwise: $F_\ell(h(p)) = \omega_\ell(p) F_\ell(p)$.

If all $F_\ell(p) \neq 0$ then $\omega(p)$ is unique: The $\omega_\ell$ are uniquely defined and smooth in an open dense set.

All $\tau_j > 0$.

$\mathcal{A}'_g = \text{action of } G_0 = \text{closure of } \{e^{i\tau_1 t}, \ldots, e^{i\tau_N t}\}$ in $S^{2N-1}$.

$\mathcal{A}'_{\omega} z = (\omega_1 z_1, \ldots, \omega_N z_N)$

No component of $F$ vanishes to infinite order at any point of $\mathcal{N}$ and span $F(\mathcal{N}) = \mathbb{C}^N$. 
Surjectivity of $\mathcal{C}^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N})$

Let $U \subset \mathcal{B}$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in \mathcal{C}^\infty(U', g)$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$.

**Proof:** There is $F : \mathcal{N} \rightarrow S^{2N-1}$ such that

$$F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^{N} \tau_j(z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

If $p \in \pi^{-1}(U)$, then $h(p) \in \overline{O_p}$, so there is $\omega(p) \in G_0$ such that $F(h(p)) = \mathcal{A}_{\omega(p)} F(p)$.

Componentwise: $F_\ell(h(p)) = \omega_\ell(p) F_\ell(p)$.

If all $F_\ell(p) \neq 0$ then $\omega(p)$ is unique: The $\omega_\ell$ are uniquely defined and smooth in an open dense set closed under $\alpha'_t$.

All $\tau_j > 0$.

$\mathcal{A}_g$ = action of $G_0$ = closure of $\{e^{i\tau_1 t}, \ldots, e^{i\tau_N t}\}$ in $S^{2N-1}$.

$\mathcal{A}^\prime_{\omega} z = (\omega_1 z_1, \ldots, \omega_N z_N)$

No component of $F$ vanishes to infinite order at any point of $\mathcal{N}$ and span $F(\mathcal{N}) = \mathbb{C}^N$.

$\alpha'_t$ = one-parameter group of diffeos generated by $\mathcal{T}$;
Surjectivity of $C^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N})$

Let $U \subset \mathcal{B}$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in C^\infty(U', g)$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$.

**Proof:** There is $F : \mathcal{N} \rightarrow S^{2N-1}$ such that

$$F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^{N} \tau_j(z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).$$

If $p \in \pi^{-1}(U)$, then $h(p) \in \overline{O}_p$, so there is $\omega(p) \in G_0$ such that $F(h(p)) = \mathcal{A}_\omega(p)F(p)$. Componentwise: $F_\ell(h(p)) = \omega_\ell(p)F_\ell(p)$.

If all $F_\ell(p) \neq 0$ then $\omega(p)$ is unique: The $\omega_\ell$ are uniquely defined and smooth in an open dense set closed under $\alpha'_t$. The condition $|\omega_\ell| = 1$ and (*) imply that the $\omega_\ell$ extend smoothly.
Surjectivity of $C^\infty(\mathcal{B}, g) \xrightarrow{\text{Exp}} \mathcal{J}^\infty(\mathcal{B}, \mathcal{N})$

Let $U \subset \mathcal{B}$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in C^\infty(U', g)$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$.

**Proof:** There is $F : \mathcal{N} \to S^{2N-1}$ such that

$$F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \overline{z}_j \partial \overline{z}_j).$$

If $p \in \pi^{-1}(U)$, then $h(p) \in \overline{O}_p$, so there is $\omega(p) \in G_0$ such that $F(h(p)) = \mathcal{A}_\omega(p) F(p).$

Componentwise: $F_\ell(h(p)) = \omega_\ell(p) F_\ell(p).$

If all $F_\ell(p) \neq 0$ then $\omega(p)$ is unique: The $\omega_\ell$ are uniquely defined and smooth in an open dense set closed under $\alpha'_t$. The condition $|\omega_\ell| = 1$ and (*) imply that the $\omega_\ell$ extend smoothly. $G_0$ is canonically isomorphic to $G$. 

All $\tau_j > 0$.

$\mathcal{A}_g$ = action of $G_0$ = closure of $\{e^{i\tau_1 t}, \ldots, e^{i\tau_N t}\}$ in $S^{2N-1}$.

$\mathcal{A}_\omega z = (\omega_1 z_1, \ldots, \omega_N z_N)$

No component of $F$ vanishes to infinite order at any point of $\mathcal{N}$ and span $F(\mathcal{N}) = \mathbb{C}^N$.

$\alpha'_t =$ one-parameter group of diffeos generated by $\mathcal{T}$;
Surjectivity of \( \mathcal{C}^\infty(\mathcal{B}, \mathfrak{g}) \overset{\text{Exp}}{\longrightarrow} \mathcal{I}^\infty(\mathcal{B}, \mathcal{N}) \)

Let \( U \subset \mathcal{B} \) be open, \( h \in \mathcal{I}^\infty(\pi^{-1}(U)), \, x_0 \in U \). There is a nbd \( U' \subset U \) of \( x_0 \) and \( f \in \mathcal{C}^\infty(U', \mathfrak{g}) \) s.t. \( h = \text{Exp}(f) \) in \( \pi^{-1}(U') \).

**Proof:** There is \( F : \mathcal{N} \to S^{2N-1} \) such that

\[
F_* \mathcal{T} = \mathcal{T}' = i \sum_{j=1}^{N} \tau_j (z_j \partial z_j - \bar{z}_j \partial \bar{z}_j).
\]

If \( p \in \pi^{-1}(U) \), then \( h(p) \in \overline{O}_p \), so there is \( \omega(p) \in G_0 \) such that \( F(h(p)) = \mathcal{A}'_{\omega(p)} F(p) \).

Component-wise: \( F_\ell(h(p)) = \omega_\ell(p) F_\ell(p) \).

If all \( F_\ell(p) \neq 0 \) then \( \omega(p) \) is unique: The \( \omega_\ell \) are uniquely defined and smooth in an open dense set closed under \( \alpha'_t \). The condition \( |\omega_\ell| = 1 \) and \((*)\) imply that the \( \omega_\ell \) extend smoothly. \( G_0 \) is canonically isomorphic to \( G \).

Conclusion...
Surjectivity of $\mathcal{C}^\infty(B, g) \xrightarrow{\operatorname{Exp}} \mathcal{I}^\infty(B, \mathcal{N})$

Let $U \subset B$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in C^\infty(U', g)$ s.t. $h = \operatorname{Exp}(f)$ in $\pi^{-1}(U')$.

Conclusion: There is $\omega : \pi^{-1}(U) \to G$ such that $h(\rho) = A_{\omega(\rho)}\rho$ with $\omega$ smooth and constant on orbits.

\[ \pi^{-1}(U) \xrightarrow{\omega} G \]
Surjectivity of $\mathcal{C}^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\text{Exp}} \mathcal{J}^\infty(\mathcal{B}, \mathcal{N})$

Let $U \subset \mathcal{B}$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in \mathcal{C}^\infty(U', \mathfrak{g})$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$.

Conclusion: There is $\omega : \pi^{-1}(U) \to G$ such that $h(p) = A_{\omega(p)}p$ with $\omega$ smooth and constant on orbits.

Next step: Find a lifting of $\omega$. 
Surjectivity of $\mathcal{C}^\infty(B, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(B, N)$

Let $U \subset B$ be open, $h \in \mathcal{I}^\infty(\pi^{-1}(U))$, $x_0 \in U$. There is a nbd $U' \subset U$ of $x_0$ and $f \in \mathcal{C}^\infty(U', g)$ s.t. $h = \text{Exp}(f)$ in $\pi^{-1}(U')$.

Conclusion: There is $\omega : \pi^{-1}(U) \to G$ such that $h(p) = A_{\omega(p)}p$ with $\omega$ smooth and constant on orbits.

Next step: Find a lifting of $\omega$. This is accomplished by funding a tubular neighborhood of $\pi^{-1}(x_0)$. This is done using an invariant metric so the neighborhood is also of the form $\pi^{-1}(U')$ for some $U'$ and observing that $\pi^{-1}(U')$ contracts to $\pi^{-1}(x_0)$. 

\[ \begin{array}{ccc} g & \xrightarrow{\text{Exp}} & \mathcal{I}^\infty(B, N) \\ \downarrow{f} & & \downarrow{\text{exp}} \\ \pi^{-1}(U') & \xrightarrow{\omega} & G \end{array} \]
Surjectivity of \( \mathcal{C}^\infty(B, g) \xrightarrow{\text{Exp}} \mathcal{I}^\infty(B, \mathcal{N}) \)

Let \( U \subset B \) be open, \( h \in \mathcal{I}^\infty(\pi^{-1}(U)) \), \( x_0 \in U \). There is a nbd \( U' \subset U \) of \( x_0 \) and \( f \in \mathcal{C}^\infty(U', g) \) s.t. \( h = \text{Exp}(f) \) in \( \pi^{-1}(U') \).

Conclusion: There is \( \omega : \pi^{-1}(U) \rightarrow G \) such that \( h(p) = \mathcal{A}_\omega(p) p \) with \( \omega \) smooth and constant on orbits.

Next step: Find a lifting of \( \omega \).
This is accomplished by funding a tubular neighborhood of \( \pi^{-1}(x_0) \). This is done using an invariant metric so the neighborhood is also of the form \( \pi^{-1}(U') \) for some \( U' \) and observing that
\[
\pi^{-1}(U') \text{ contracts to } \pi^{-1}(x_0).
\]
\( \mathcal{C}(\mathcal{B}, g) \) is a fine sheaf

Let \( \mu \) denote the normalized Haar measure of \( G \).

For \( U \subset \mathcal{B} \) open and \( f \in C^\infty(\pi^{-1}(U)) \) define \( Av f : \pi^{-1}(U) \to \mathbb{C} \) by

\[
Av f(p) = \int_G f(\mathcal{A}_g p) d\mu(g)
\]

Then \( Av f \) is smooth. If \( f \in C^\infty_c(\pi^{-1}(U)) \), then \( Av f \in C^\infty_c(\pi^{-1}(U)) \).
$\mathcal{C}(B, g)$ is a fine sheaf

Let $\mu$ denote the normalized Haar measure of $G$.

For $U \subset B$ open and $f \in C^\infty(\pi^{-1}(U))$ define $\text{Av} f : \pi^{-1}(U) \to \mathbb{C}$ by

$$\text{Av} f(p) = \int_G f(A_g p) d\mu(g)$$

Then $\text{Av} f$ is smooth. If $f \in C_c^\infty(\pi^{-1}(U))$, then $\text{Av} f \in C_c^\infty(\pi^{-1}(U))$.

Double fibration:

$$\rho \text{ is the canonical projection}$$

$$G \times \mathcal{N}$$

$$\mathcal{N}$$
$\mathcal{C}(B, g)$ is a fine sheaf

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Double fibration:

$\rho$ is the canonical projection

$\mathcal{A} : (g, p) \mapsto \mathcal{A}_g p$ is a locally trivial fibration. The fiber over $p \in \mathcal{N}$ is

$$\{(g, \mathcal{A}_g^{-1} p) : g \in G\}$$
$$\mathcal{C}(\mathcal{B}, g)$$ is a fine sheaf

Let $\mu$ denote the normalized Haar measure of $G$.

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Then $\text{Av} f$ is smooth. If $f \in C^\infty_c(\pi^{-1}(U))$, then $\text{Av} f \in C^\infty_c(\pi^{-1}(U))$.

Double fibration:

\[ G \times \mathcal{N} \xrightarrow{\rho} \mathcal{N} \xrightarrow{\mathcal{A}} \mathcal{N} \]

$\rho$ is the canonical projection

$\mathcal{A} : (g, p) \mapsto \mathcal{A}_g p$ is a locally trivial fibration. The fiber over $p \in \mathcal{N}$ is

\[ \{(g, \mathcal{A}_g^{-1} p) : g \in G\} \]

$\text{Av} f = \rho_*(\mathcal{A}^* f \mu)$ gives $f \in C^\infty(\pi^{-1}(U)) \implies \text{Av} f \in C^\infty(\pi^{-1}(U))$
C(B, g) is a fine sheaf

Let $\mu$ denote the normalized Haar measure of $G$.

For $U \subset B$ open and $f \in C^\infty(\pi^{-1}(U))$ define $\text{Av} f : \pi^{-1}(U) \to \mathbb{C}$ by

$$\text{Av} f(p) = \int_G f(\mathcal{A}_g p) d\mu(g)$$

Then $\text{Av} f$ is smooth. If $f \in C^\infty_c(\pi^{-1}(U))$, then $\text{Av} f \in C^\infty_c(\pi^{-1}(U))$.

Partitions of unity:

Let $\{U_a\}$ be an open cover of $B$. If $\{\chi_\gamma\}_{\gamma \in \Gamma}$ a smooth partition of unity subordinate to $\{\pi^{-1}(U_a)\}$, then $\{\text{Av} \chi_\gamma\}_{\gamma \in \Gamma}$ is also a partition of unity subordinate to $\{\pi^{-1}(U_a)\}$, with $\text{Av} \chi_\gamma \in C^\infty(B)$.

This gives that $C(B, g)$ is a fine sheaf.
\( \mathcal{C}(\mathcal{B}, \mathfrak{g}) \) is a fine sheaf

Let \( \mu \) denote the normalized Haar measure of \( G \).

For \( U \subset \mathcal{B} \) open and \( f \in C^\infty(\pi^{-1}(U)) \) define \( \text{Av} f : \pi^{-1}(U) \to \mathbb{C} \) by

\[
\text{Av} f(p) = \int_{\mathcal{G}} f(\mathfrak{A}_g p) d\mu(g)
\]

Then \( \text{Av} f \) is smooth. If \( f \in C^\infty_c(\pi^{-1}(U)) \), then \( \text{Av} f \in C^\infty_c(\pi^{-1}(U)) \).

Partitions of unity:

Let \( \{U_a\} \) be an open cover of \( \mathcal{B} \). If \( \{\chi_\gamma\}_{\gamma \in \Gamma} \) a smooth partition of unity subordinate to \( \{\pi^{-1}(U_a)\} \), then \( \{\text{Av} \chi_\gamma\}_{\gamma \in \Gamma} \) is also a partition of unity subordinate to \( \{\pi^{-1}(U_a)\} \), with \( \text{Av} \chi_\gamma \in C^\infty(\mathcal{B}) \).

This gives that \( \mathcal{C}(\mathcal{B}, \mathfrak{g}) \) is a fine sheaf.

\[
\cdots \to \check{H}^1(\mathcal{B}, \mathcal{C}(\mathcal{B}, \mathfrak{g})) \to \check{H}^1(\mathcal{B}, \mathcal{I}(\mathcal{B}, \mathcal{N})) \to \check{H}^2(\mathcal{B}, \mathcal{L}) \to \check{H}^2(\mathcal{B}, \mathcal{C}(\mathcal{B}, \mathfrak{g})) \to \cdots
\]
\( \mathcal{C}(B, g) \) is a fine sheaf

Let \( \mu \) denote the normalized Haar measure of \( G \).

For \( U \subset B \) open and \( f \in C^\infty(\pi^{-1}(U)) \) define \( \text{Av} f : \pi^{-1}(U) \to \mathbb{C} \) by

\[
\text{Av} f(p) = \int_G f(\mathcal{A}_g p) d\mu(g)
\]

Then \( \text{Av} f \) is smooth. If \( f \in C^\infty_c(\pi^{-1}(U)) \), then \( \text{Av} f \in C^\infty_c(\pi^{-1}(U)) \).

Partitions of unity:

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This gives that \( \mathcal{C}(B, g) \) is a fine sheaf.

\[
\cdots \to \check{H}^1(B, \mathcal{C}(B, g)) \to \check{H}^1(B, \mathcal{I}(B, N)) \to \check{H}^2(B, \mathcal{L}) \to \check{H}^2(B, \mathcal{C}(B, g)) \to \cdots
\]
\( \mathcal{C}(\mathcal{B}, g) \) is a fine sheaf

Let \( \mu \) denote the normalized Haar measure of \( G \).

For \( U \subset \mathcal{B} \) open and \( f \in C^\infty(\pi^{-1}(U)) \) define \( \text{Av} f : \pi^{-1}(U) \to \mathbb{C} \) by

\[
\text{Av} f(p) = \int_{G} f(\mathcal{A}_g p) d\mu(g)
\]

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Partitions of unity:

Let \( \{U_a\} \) be an open cover of \( \mathcal{B} \). If \( \{\chi_\gamma\}_{\gamma \in \Gamma} \) a smooth partition of unity subordinate to \( \{\pi^{-1}(U_a)\} \), then \( \{\text{Av} \chi_\gamma\}_{\gamma \in \Gamma} \) is also a partition of unity subordinate to \( \{\pi^{-1}(U_a)\} \), with \( \text{Av} \chi_\gamma \in C^\infty(\mathcal{B}) \).

This gives that \( \mathcal{C}(\mathcal{B}, g) \) is a fine sheaf.

\[
\cdots \to \check{H}^1(\mathcal{B}, \mathcal{C}(\mathcal{B}, g)) \to \check{H}^1(\mathcal{B}, \mathcal{I}(\mathcal{B}, \mathcal{N})) \cong \check{H}^2(\mathcal{B}, \mathcal{I}) \to \check{H}^2(\mathcal{B}, \mathcal{C}(\mathcal{B}, g)) \to \cdots
\]

\( \approx \mathbb{Z}^d \) for some \( d \)
Back to boundary structures

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$ such that:

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ is an elliptic structure with $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_{\mathbb{C}} \mathcal{T}$;
- There is a $\overline{\mathbb{D}}$-closed element $\beta \in C^\infty(\mathcal{N}, \overline{\mathcal{V}}^*)$ such that $\mathbf{i}_T \beta = -i$. 


Back to boundary structures

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- There is a $\overline{\mathbb{D}}$-closed element $\beta \in C^\infty(\mathcal{N}, \mathcal{V}^*)$ such that $i_\mathcal{T} \beta = -i$.

$(\mathcal{N}, \mathcal{T}, \mathcal{V}), (\mathcal{N}', \mathcal{T}', \mathcal{V}') \in \mathcal{F}_{\text{ell}}$ are globally ell-equivalent if there is an equivariant diffeomorphism $h : \mathcal{N}' \rightarrow \mathcal{N}$ such that $h_* \mathcal{V}' = \overline{\mathcal{V}}$. 
Back to boundary structures

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$ such that:

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$(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}), (\mathcal{N}', \mathcal{T}', \overline{\mathcal{V}}') \in \mathcal{F}_{\text{ell}}$ are globally $\text{ell}$-equivalent if there is an equivariant diffeomorphism $h : \mathcal{N}' \rightarrow \mathcal{N}$ such that $h_* \overline{\mathcal{V}}' = \overline{\mathcal{V}}$.

They are locally $\text{ell}$-equivalent if there are open covers $\{U_a\}_{a \in A}$ of $\mathcal{N}$ and $\{U'_a\}_{a \in A}$ of $\mathcal{N}'$ by $\mathcal{T}$, resp. $\mathcal{T}'$-invariant open sets and equivariant diffeomorphisms $h_a : U'_a \rightarrow U_a$ for each $a \in A$ such that $h_a h_b^{-1}$ satisfies

$$h_a h_b^{-1}(p) \in \overline{\mathcal{O}}_p \quad \text{for all } p$$

and preserves $\overline{\mathcal{V}}$. 
Fix $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}$. Let $U \subset \mathcal{B}$ be open.
Fix $(\mathcal{N}, \mathcal{T}, \mathcal{V}) \in \mathcal{F}_{\text{ell}}$. Let $U \subset B$ be open.

- $\mathcal{I}^\mathcal{V}(U)$ is the subgroup of $\mathcal{I}^\infty(U)$ whose elements preserve $\mathcal{V}$. The associated sheaf is $\mathcal{I}^\mathcal{V}(B, \mathcal{N})$.

\[ h \in \mathcal{I}^\infty(U) \text{ preserves } \mathcal{V} \text{ if } h_*(\mathcal{V}|_U) \subset \mathcal{V}|_{\pi^{-1}(U)}. \]
Fix \((\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}\). Let \(U \subset \mathcal{B}\) be open.

- \(\mathcal{I}^{\overline{\mathcal{V}}}(U)\) is the subgroup of \(\mathcal{I}^{\infty}(U)\) whose elements preserve \(\overline{\mathcal{V}}\). The associated sheaf is \(\mathcal{I}^{\overline{\mathcal{V}}}(\mathcal{B}, \mathcal{N})\).

\[ h \in \mathcal{I}^{\infty}(U) \text{ preserves } \overline{\mathcal{V}} \text{ if } h_*(\overline{\mathcal{V}}|_U) \subset \overline{\mathcal{V}}|_{\pi^{-1}(U)}. \]

- \(\mathcal{C}^{\overline{\mathcal{V}}}(U, \mathfrak{g})\) is the subspace of \(\mathcal{C}^{\infty}(U, \mathfrak{g})\) whose image by \(\text{Exp}\) is \(\mathcal{I}^{\overline{\mathcal{V}}}(U)\). The associated sheaf is denoted \(\mathcal{C}^{\overline{\mathcal{V}}}(\mathcal{B}, \mathfrak{g})\).
Fix \((\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_\text{ell}\). Let \(U \subset \mathcal{B}\) be open.

- \(\mathcal{I}^{\overline{\mathcal{V}}}(U)\) is the subgroup of \(\mathcal{I}^\infty(U)\) whose elements preserve \(\overline{\mathcal{V}}\). The associated sheaf is \(\mathcal{I}^{\overline{\mathcal{V}}}(\mathcal{B}, \mathcal{N})\).

  \(h \in \mathcal{I}^\infty(U)\) preserves \(\overline{\mathcal{V}}\) if \(h_*(\overline{\mathcal{V}}|_U) \subset \overline{\mathcal{V}}|_{\pi^{-1}(U)}\).

- \(\mathcal{C}^{\overline{\mathcal{V}}}(U, g)\) is the subspace of \(C^\infty(U, g)\) whose image by \(\text{Exp}\) is \(\mathcal{I}^{\overline{\mathcal{V}}}(U)\). The associated sheaf is denoted \(\mathcal{C}^{\overline{\mathcal{V}}}(\mathcal{B}, g)\).

The sequence

\[
0 \to \mathcal{I} \to \mathcal{C}^{\overline{\mathcal{V}}}(\mathcal{B}, g) \to \mathcal{I}^{\overline{\mathcal{V}}}(\mathcal{B}, \mathcal{N}) \to 0
\]

is exact.
Fix \((\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}\). Let \(U \subset \mathcal{B}\) be open.

- \(\mathcal{I}^\overline{V}(U)\) is the subgroup of \(\mathcal{I}^\infty(U)\) whose elements preserve \(\overline{\mathcal{V}}\). The associated sheaf is \(\mathcal{S}^\overline{V}(\mathcal{B}, \mathcal{N})\).

- \(\mathcal{C}^\overline{V}(U, \mathfrak{g})\) is the subspace of \(\mathcal{C}^\infty(U, \mathfrak{g})\) whose image by \(\text{Exp}\) is \(\mathcal{I}^\overline{V}(U)\). The associated sheaf is denoted \(\mathcal{C}^\overline{V}(\mathcal{B}, \mathfrak{g})\).

The sequence

\[ 0 \rightarrow \mathcal{Z} \rightarrow \mathcal{C}^{\overline{V}}(\mathcal{B}, \mathfrak{g}) \rightarrow \mathcal{I}^{\overline{V}}(\mathcal{B}, \mathcal{N}) \rightarrow 0 \]

is exact.

**Theorem:**

*Let \((\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}\). There is a natural one-to-one correspondence between the elements of \(H^1(\mathcal{B}, \mathcal{S}^{\overline{V}}(\mathcal{N}))\) and the global ell-equivalence classes of elements of \(\mathcal{F}_{\text{ell}}\) which are locally ell-equivalent to \(\mathcal{N}\).*
Cast

\( p' \in \overline{O}_p \)

but \( p \notin \overline{O}_{p'} \)

\[ \cos^2 \theta \partial_\theta + 5 \partial_\psi \]