Classification and embedding theorems for manifolds with $\mathbb{R}$-action

Gerardo Mendoza

Temple University

Beirut, November 2011
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

1. Embedding Theorems

Let $B$ be a compact manifold, let $E \to B$ be a complex line bundle. Then there is $N$ and an embedding $\phi: B \to \mathbb{CP}^N$ such that $E \cong \phi^*\Gamma$.

Recall: $\mathbb{CP}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$.

$\Gamma \to \mathbb{CP}^N$ is the line bundle whose fiber at $p \in \mathbb{CP}^N$ is the vector space $p\cdot \Gamma$.

2. If $B$ is a compact complex manifold and $E \to B$ is a positive line bundle, then there is a holomorphic embedding $\phi: B \to \mathbb{CP}^N$. This is Kodaira's embedding theorem.

Positive means that $E \to B$ has a holomorphic connection whose curvature $\sum_{\mu,\nu} \Omega_{\mu\nu} dz^\mu \wedge d\bar{z}^\nu$ is such that $i\pi \sum_{\mu,\nu} \Omega_{\mu\nu} dz^\mu \otimes d\bar{z}^\nu$ is positive definite.
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $\mathcal{B}$ be a compact manifold, let $E \rightarrow \mathcal{B}$ be a complex line bundle. Then there is $N$ and an embedding

$$ \phi : \mathcal{B} \rightarrow \mathbb{CP}^N $$

such that $E \cong \phi^* \Gamma$.

Recall: $\mathbb{CP}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$.

$\Gamma_{\mathbb{CP}^N}$ is the line bundle whose fiber at $p \in \mathbb{CP}^N$ is the vector space $p \cdot \Gamma_{\mathbb{CP}^N}$.

2. If $\mathcal{B}$ is a compact complex manifold and $E \rightarrow \mathcal{B}$ is a positive line bundle, then there is a holomorphic embedding

$$ \phi : \mathcal{B} \rightarrow \mathbb{CP}^N $$

This is Kodaira's embedding theorem.

Positive means that $E \rightarrow \mathcal{B}$ has a holomorphic connection whose curvature $\sum_{\mu,\nu} \Omega_{\mu\nu} dz_{\mu} \wedge d\bar{z}_{\nu}$ is such that $i \frac{\pi}{2} \sum_{\mu,\nu} \Omega_{\mu\nu} dz_{\mu} \otimes d\bar{z}_{\nu}$ is positive definite.
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $\mathcal{B}$ be a compact manifold, let $E \rightarrow \mathcal{B}$ be a complex line bundle. Then there is $N$ and an embedding

   $\phi : \mathcal{B} \rightarrow \mathbb{C}P^N$

   such that $E \cong \phi^* \Gamma$.

Recall: $\mathbb{C}P^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$.

Recall: $\Gamma \rightarrow \mathbb{C}P^N$ is the line bundle whose fiber at $p \in \mathbb{C}P^N$ is the vector space $p \cdot \Gamma_{\mathbb{C}P^N}$.
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $B$ be a compact manifold, let $E \to B$ be a complex line bundle.
   Then there is $N$ and an embedding
   \[ \phi : B \to \mathbb{CP}^N \]
   such that $E \approx \phi^* \Gamma$.

Recall: $\mathbb{CP}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$.
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $\mathcal{B}$ be a compact manifold, let $E \to \mathcal{B}$ be a complex line bundle. Then there is $N$ and an embedding

$$\phi : \mathcal{B} \to \mathbb{CP}^N$$

such that $E \cong \phi^* \Gamma$. 

Recall: $\mathbb{CP}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$. $\Gamma \to \mathbb{CP}^N$ is the line bundle whose fiber at $p \in \mathbb{CP}^N$ is the vector space $p$. 

Positive means that $E \to \mathcal{B}$ has a holomorphic connection whose curvature $\sum_{\mu, \nu} \Omega_{\mu \nu} dz^\mu \wedge d\bar{z}^\nu$ is such that $i^2 \pi \sum_{\mu, \nu} \Omega_{\mu \nu} dz^\mu \otimes d\bar{z}^\nu$ is positive definite.
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $\mathcal{B}$ be a compact manifold, let $E \to \mathcal{B}$ be a complex line bundle. Then there is $N$ and an embedding $\phi: \mathcal{B} \to \mathbb{CP}^N$ such that $E \cong \phi^\ast \Gamma$.

   Recall: $\mathbb{CP}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$. $\Gamma \to \mathbb{CP}^N$ is the line bundle whose fiber at $p \in \mathbb{CP}^N$ is the vector space $p$. 

2. If $\mathcal{B}$ is a compact complex manifold and $E \to \mathcal{B}$ is a positive line bundle, then there is a holomorphic embedding $\phi: \mathcal{B} \to \mathbb{CP}^N$. This is Kodaira's embedding theorem. 

Recall: positive means that $E \to \mathcal{B}$ has a holomorphic connection whose curvature $\sum_{\mu,\nu} \Omega_{\mu\nu} dz^\mu \wedge dz^\nu$ is such that $i^2 \pi \sum_{\mu,\nu} \Omega_{\mu\nu} dz^\mu \otimes dz^\nu$ is positive definite.
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $\mathcal{B}$ be a compact manifold, let $E \to \mathcal{B}$ be a complex line bundle. Then there is $N$ and an embedding

   $$ \phi : \mathcal{B} \to \mathbb{CP}^N $$

   such that $E \approx \phi^* \Gamma$.

   Recall: $\mathbb{CP}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$. $\Gamma \to \mathbb{CP}^N$ is the line bundle whose fiber at $p \in \mathbb{CP}^N$ is the vector space $p$.

   $$ \mathcal{B} \times \Gamma \quad \Gamma \quad \mathcal{B} \to \mathbb{CP}^N $$
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $B$ be a compact manifold, let $E \rightarrow B$ be a complex line bundle. Then there is $N$ and an embedding

$$\phi : B \rightarrow \mathbb{CP}^N$$

such that $E \approx \phi^* \Gamma$.

Recall: $\mathbb{CP}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$. $\Gamma \rightarrow \mathbb{CP}^N$ is the line bundle whose fiber at $p \in \mathbb{CP}^N$ is the vector space $p$.

$B \times \Gamma \supset \phi^* \Gamma$

$\Gamma$

$\phi$

$B \rightarrow \mathbb{CP}^N$
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $\mathcal{B}$ be a compact manifold, let $E \to \mathcal{B}$ be a complex line bundle. Then there is an embedding

$$\phi : \mathcal{B} \to \mathbb{CP}^N$$

such that $E \approx \phi^* \Gamma$. 

Recall: $\mathbb{CP}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$. $\Gamma \to \mathbb{CP}^N$ is the line bundle whose fiber at $p \in \mathbb{CP}^N$ is the vector space $p$. 

$$\phi^* \Gamma = \{(x, v) \in \mathcal{B} \times \Gamma : v \in \phi(x)\}$$

$$\mathcal{B} \times \Gamma \supset \phi^* \Gamma \rightarrow \Gamma$$

$$\mathcal{B} \rightarrow \mathbb{CP}^N$$

2. If $\mathcal{B}$ is a compact complex manifold and $E \to \mathcal{B}$ is a positive line bundle, then there is a holomorphic embedding $\phi : \mathcal{B} \to \mathbb{CP}^N$. This is Kodaira’s embedding theorem.
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $\mathcal{B}$ be a compact manifold, let $E \rightarrow \mathcal{B}$ be a complex line bundle.

   Then there is $N$ and an embedding
   \[ \phi : \mathcal{B} \rightarrow \mathbb{C}P^N \]
   such that $E \cong \phi^*\Gamma$.

2. If $\mathcal{B}$ is a compact complex manifold and $E \rightarrow \mathcal{B}$ is a positive line bundle, then there is a holomorphic embedding
   \[ \phi : \mathcal{B} \rightarrow \mathbb{C}P^N. \]
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $B$ be a compact manifold, let $E \to B$ be a complex line bundle.

Then there is $N$ and an embedding

$$\phi : B \to \mathbb{CP}^N$$

such that $E \cong \phi^*\Gamma$.

2. If $B$ is a compact complex manifold and $E \to B$ is a positive line bundle, then there is a holomorphic embedding

$$\phi : B \to \mathbb{CP}^N.$$  

This is Kodaira’s embedding theorem.
I will discuss generalizations of the following four theorems concerning line bundles over compact manifolds:

A. Embeddings Theorems

1. Let $B$ be a compact manifold, let $E \to B$ be a complex line bundle.

   Then there is $N$ and an embedding
   
   $$\phi : B \to \mathbb{CP}^N$$

   such that $E \cong \phi^* \Gamma$.

2. If $B$ is a compact complex manifold and $E \to B$ is a positive line bundle, then there is a holomorphic embedding

   $$\phi : B \to \mathbb{CP}^N.$$

   This is Kodaira’s embedding theorem.

Recall: $\mathbb{CP}^N$ is the manifold whose points are the one-dimensional subspaces of $\mathbb{C}^{N+1}$. $\Gamma \to \mathbb{CP}^N$ is the line bundle whose fiber at $p \in \mathbb{CP}^N$ is the vector space $p$. $\phi^* \Gamma = \{(x, v) \in B \times \Gamma : v \in \phi(x)\}$.

Positive means that $E \to B$ has a holomorphic connection whose curvature $\sum_{\mu, \nu} \Omega_{\mu \nu} dz^\mu \wedge d\bar{z}^\nu$ is such that

$$\frac{i}{2\pi} \sum_{\mu, \nu} \Omega_{\mu \nu} dz^\mu \otimes d\bar{z}^\nu$$

is positive definite.
B. Classification Theorems

3. Let $\mathcal{B}$ be a compact manifold. There is a natural one-to-one correspondence between elements of $\check{H}^2(\mathcal{B}, \mathbb{Z})$ and the family of isomorphism classes of line bundles over $\mathcal{B}$. 
B. Classification Theorems

3. Let $\mathcal{B}$ be a compact manifold. There is a natural one-to-one correspondence between elements of $\check{H}^2(\mathcal{B}, \mathbb{Z})$ and the family of isomorphism classes of line bundles over $\mathcal{B}$, by way of the first Chern class.
B. Classification Theorems

3. Let $\mathcal{B}$ be a compact manifold. There is a natural one-to-one correspondence between elements of $\check{H}^2(\mathcal{B}, \mathbb{Z})$ and the family of isomorphism classes of line bundles over $\mathcal{B}$.

4. Let $\mathcal{B}$ be a compact complex manifold, let $\mathcal{M}^* \to \mathcal{B}$ be the sheaf of germs of meromorphic functions. There is a one-to-one correspondence between of $H^1(\mathcal{B}, \mathcal{M}^*)$ and the set of isomorphism classes of holomorphic line bundles over $\mathcal{B}$.
Outline

In each case I will first reinterpret the classical theorem,
Outline

In each case I will first reinterpret the classical theorem,

1. Let $B$ be a compact manifold, let $E \to B$ be a complex line bundle. There is $N$ and an embedding $\phi : B \to \mathbb{CP}^N$ such that $E \cong \phi^*\Gamma$.

2. If $B$ is a compact complex manifold and $E \to B$ is a positive line bundle, then there is a holomorphic embedding $\phi : B \to \mathbb{CP}^N$.

The proofs of the generalizations of 1. and 2. use an idea of S. Bochner.


The proof of 2. also uses ideas of L. Boutet de Monvel.


3. Let $B$ be a compact manifold. There is a natural one-to-one correspondence between elements of $H^2(B, \mathbb{Z})$ and the family of isomorphism classes of line bundles over $B$.

4. Let $B$ be a compact complex manifold, let $M \to B$ be the sheaf of germs of meromorphic functions. There is a one-to-one correspondence between of $H^1(B, M)$ and the set of isomorphism classes of holomorphic line bundles over $B$.

The proof of 3. follows closely the classical proof. The proof of 4. takes advantage of the description of holomorphic structures using connections.
Outline

In each case I will first reinterpret the classical theorem, then state the generalization.

1. Let $B$ be a compact manifold, let $E \to B$ be a complex line bundle. There is $N$ and an embedding $\phi : B \to \mathbb{C}P^N$ such that $E \approx \phi^*\Gamma$.

2. If $B$ is a compact complex manifold and $E \to B$ is a positive line bundle, then there is a holomorphic embedding $\phi : B \to \mathbb{C}P^N$.

The proofs of the generalizations of 1. and 2. use an idea of S. Bochner.


The proof of 2. also uses ideas of L. Boutet de Monvel.


3. Let $B$ be a compact manifold. There is a natural one-to-one correspondence between elements of $H^2(B, \mathbb{Z})$ and the family of isomorphism classes of line bundles over $B$.

4. Let $B$ be a compact complex manifold, let $M \to B$ be the sheaf of germs of meromorphic functions. There is a one-to-one correspondence between of $H^1(B, M)$ and the set of isomorphism classes of holomorphic line bundles over $B$.

The proof of 3. follows closely the classical proof. The proof of 4. takes advantage of the description of holomorphic structures using connections.
In each case I will first reinterpret the classical theorem, then state the generalization. After that I will sketch the proofs.

1. Let $B$ be a compact manifold, let $E \to B$ be a complex line bundle. There is $N$ and an embedding $\phi : B \to \mathbb{CP}^N$ such that $E \approx \phi^* \Gamma$.

2. If $B$ is a compact complex manifold and $E \to B$ is a positive line bundle, then there is a holomorphic embedding $\phi : B \to \mathbb{CP}^N$. 

The proofs of the generalizations of 1. and 2. use an idea of S. Bochner.


The proof of 2. also uses ideas of L. Boutet de Monvel.

Outline

In each case I will first reinterpret the classical theorem, then state the generalization. After that I will sketch the proofs.

The proofs of the generalizations of 1. and 2. use an idea of S. Bochner.

1. Let $\mathcal{B}$ be a compact manifold, let $E \to \mathcal{B}$ be a complex line bundle. There is $N$ and an embedding $\phi : \mathcal{B} \to \mathbb{C}P^N$ such that $E \cong \phi^* \Gamma$.

2. If $\mathcal{B}$ is a compact complex manifold and $E \to \mathcal{B}$ is a positive line bundle, then there is a holomorphic embedding $\phi : \mathcal{B} \to \mathbb{C}P^N$.

Outline

In each case I will first reinterpret the classical theorem, then state the generalization. After that I will sketch the proofs.

The proofs of the generalizations of 1. and 2. use an idea of S. Bochner.

The proof of 2. also uses ideas of L. Boutet de Monvel.

1. Let $\mathcal{B}$ be a compact manifold, let $E \to \mathcal{B}$ be a complex line bundle. There is $N$ and an embedding $\phi: \mathcal{B} \to \mathbb{CP}^N$ such that $E \approx \phi^* \Gamma$.

2. If $\mathcal{B}$ is a compact complex manifold and $E \to \mathcal{B}$ is a positive line bundle, then there is a holomorphic embedding $\phi: \mathcal{B} \to \mathbb{CP}^N$.


Outline

In each case I will first reinterpret the classical theorem, then state the generalization. After that I will sketch the proofs.

The proofs of the generalizations of 1. and 2. use an idea of S. Bochner.

The proof of 2. also uses ideas of L. Boutet de Monvel.

The proof of 3. follows closely the classical proof.

The proof of 4. takes advantage of the description of holomorphic structures using connections.

1. Let $B$ be a compact manifold, let $E \to B$ be a complex line bundle. There is $N$ and an embedding $\phi : B \to \mathbb{C}P^N$ such that $E \cong \phi^\ast \Gamma$.

2. If $B$ is a compact complex manifold and $E \to B$ is a positive line bundle, then there is a holomorphic embedding $\phi : B \to \mathbb{C}P^N$.

3. Let $B$ be a compact manifold. There is a natural one-to-one correspondence between elements of $H^2(B, \mathbb{Z})$ and the family of isomorphism classes of line bundles over $B$.

4. Let $B$ be a compact complex manifold, let $M \to B$ be the sheaf of germs of meromorphic functions. There is a one-to-one correspondence between of $H^1(B, M)$ and the set of isomorphism classes of holomorphic line bundles over $B$.


Brief overview of line bundles

Let $B$ be a manifold. A line bundle over $B$ is a manifold $E$ together with a smooth map $\varphi : E \to B$, an open cover $\mathcal{V} = \{ V_\alpha \}_{\alpha \in A}$ of $B$ and diffeomorphisms

Additionally, $\forall x \in B$:

$E_x = \varphi^{-1}(x)$ is a 1-dimensional vector space and $\varphi_\alpha|_{E_x}$ is linear for each $x \in V_\alpha$ and all $\alpha$.

Note: Let $\chi_\alpha \in C^\infty_c(V_\alpha)$. Then $\chi_\alpha \varphi_\alpha : E \to C$ is linear on $E_x$.

Embedding:

The family $\Psi = \{ \psi : E \to C : \psi$ is linear on each $E_x \}$ separates the sets $E_x : \psi|_{E_x} = \psi|_{E_x'} \forall \psi \in \Psi = \Rightarrow x = x'$

There are $\psi_1, ..., \psi_{N+1} \in \Psi$ such that $F : E \setminus 0 \to C^{N \setminus 0}$

$F(\eta) = (\psi_1(\eta), ..., \psi_{N+1}(\eta))$

is injective.

Then $B \ni x \mapsto F(E_x) \ni \mathbb{C}P^{N-1}$ is injective.

With additional well-chosen $\psi'$s in $\Psi$ it is an embedding.
Brief overview of line bundles

Let $\mathcal{B}$ be a manifold. A line bundle over $\mathcal{B}$ is a manifold $E$ together with a smooth map $\varphi : E \to \mathcal{B}$, an open cover $\mathcal{U} = \{V_\alpha\}_{\alpha \in A}$ of $\mathcal{B}$ and diffeomorphisms

$\varphi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C}$
Brief overview of line bundles

Let $B$ be a manifold. A line bundle over $B$ is a manifold $E$ together with a smooth map $\varphi : E \to B$, an open cover $\mathcal{U} = \{ V_\alpha \}_{\alpha \in A}$ of $B$ and diffeomorphisms

$$
\varphi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C}
$$

$\varphi$ $\varphi_\alpha$ $V_\alpha$
Brief overview of line bundles

Let $\mathcal{B}$ be a manifold. A line bundle over $\mathcal{B}$ is a manifold $E$ together with a smooth map $\varphi : E \to \mathcal{B}$, an open cover $\mathcal{U} = \{ V_\alpha \}_{\alpha \in \mathcal{A}}$ of $\mathcal{B}$ and diffeomorphisms

$$\varphi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C}$$

$$\varphi \downarrow \quad \circ \quad \varphi_\alpha$$

$$V_\alpha$$

$$\phi_\alpha(\eta) = (\varphi(\eta), \varphi_\alpha(\eta))$$
Let $B$ be a manifold. A line bundle over $B$ is a manifold $E$ together with a smooth map $\varphi : E \to B$, an open cover $\mathcal{U} = \{V_\alpha\}_{\alpha \in A}$ of $B$ and diffeomorphisms $\varphi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{C}$.

$$\varphi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C}$$

$\varphi : E \to B$ is the “trivial” line bundle.

$$\phi_\alpha(\eta) = (\varphi(\eta), \varphi_\alpha(\eta))$$
Brief overview of line bundles

Let $\mathcal{B}$ be a manifold. A line bundle over $\mathcal{B}$ is a manifold $E$ together with a smooth map $\phi : E \to \mathcal{B}$, an open cover $\mathcal{U} = \{V_\alpha\}_{\alpha \in A}$ of $\mathcal{B}$ and diffeomorphisms

$$\phi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C}$$

local trivializations

$$\phi_\alpha(\eta) = (\phi(\eta), \varphi_\alpha(\eta))$$

Additionally, $\forall x \in \mathcal{B}: E_p = \phi^{-1}(x)$ is a 1-dimensional vector space and

$$\varphi_\alpha|_{E_x} : E_x \to \mathbb{C}, \ E_x = \phi^{-1}(x)$$

is linear for each $x \in V_\alpha$ and all $\alpha$. 
Brief overview of line bundles

Let $B$ be a manifold. A line bundle over $B$ is a manifold $E$ together with a smooth map $\varphi : E \to B$, an open cover $\mathcal{U} = \{V_\alpha\}_{\alpha \in A}$ of $B$ and diffeomorphisms

\[
\varphi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C}
\]

local trivializations

\[
\phi_\alpha(\eta) = (\varphi(\eta), \varphi_\alpha(\eta))
\]

Additionally, $\forall x \in B$: $E_p = \varphi^{-1}(x)$ is a 1-dimensional vector space and

\[
\varphi_\alpha|_{E_x} : E_x \to \mathbb{C}, \ E_x = \varphi^{-1}(x)
\]

is linear for each $x \in V_\alpha$ and all $\alpha$.

Note: Let $\chi_\alpha \in C_c^\infty(V_\alpha)$. Then

$\chi_\alpha \varphi_\alpha : E \to \mathbb{C}$ is linear on $E_x$. 
**Overview of line bundles**

Let $\mathcal{B}$ be a manifold. A line bundle over $\mathcal{B}$ is a manifold $\mathcal{E}$ together with a smooth map $\varphi : \mathcal{E} \to \mathcal{B}$, an open cover $\mathcal{U} = \{ V_\alpha \}_{\alpha \in A}$ of $\mathcal{B}$ and diffeomorphisms $\varphi^{-1}(V_\alpha) \cong V_\alpha \times \mathbb{C}$. 

**Embedding:**

\[
\begin{array}{ccc}
\varphi^{-1}(V_\alpha) & \xrightarrow{\phi_\alpha} & V_\alpha \times \mathbb{C} \\
\downarrow \varphi & & \downarrow \varphi_\alpha \\
V_\alpha & & \mathcal{B} \times \mathbb{C}
\end{array}
\]

$\varphi_\alpha(\eta) = (\varphi(\eta), \varphi_\alpha(\eta))$

Additionally, $\forall x \in \mathcal{B}: E_p = \varphi^{-1}(x)$ is a 1-dimensional vector space and

$\varphi_\alpha|_{E_x} : E_x \to \mathbb{C}$, $E_x = \varphi^{-1}(x)$

is linear for each $x \in V_\alpha$ and all $\alpha$.

**Note:** Let $\chi_\alpha \in C^\infty_c(V_\alpha)$. Then $\chi_\alpha \varphi_\alpha : \mathcal{E} \to \mathbb{C}$ is linear on $E_x$. 
Brief overview of line bundles

Let $B$ be a manifold. A line bundle over $B$ is a manifold $E$ together with a smooth map $\varphi : E \to B$, an open cover $\mathcal{U} = \{V_\alpha\}_{\alpha \in A}$ of $B$ and diffeomorphisms

\[ \varphi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C} \]

\[ \varphi \]

\[ \varphi_\alpha \]

\[ B \times \mathbb{C} \text{ is the "trivial" line bundle} \]

local trivializations

Embedding: The family $\psi = \{\psi : E \to \mathbb{C} : \psi \text{ is linear on each } E_x\}$ separates the sets $E_x$:

$\psi|_{E_x} = \psi|_{E_{x'}} \forall \psi \in \psi \implies x = x'$

Additionally, $\forall x \in B: E_p = \varphi^{-1}(x)$ is a 1-dimensional vector space and

$\varphi_\alpha|_{E_x} : E_x \to \mathbb{C}, \ E_x = \varphi^{-1}(x)$

is linear for each $x \in V_\alpha$ and all $\alpha$.

Note: Let $\chi_\alpha \in C_c^\infty(V_\alpha)$. Then

$\chi_\alpha \varphi_\alpha : E \to \mathbb{C}$ is linear on $E_x$. 
**Brief overview of line bundles**

Let $\mathcal{B}$ be a manifold. A line bundle over $\mathcal{B}$ is a manifold $E$ together with a smooth map $\varphi : E \to \mathcal{B}$, an open cover $\mathcal{U} = \{ V_\alpha \}_{\alpha \in A}$ of $\mathcal{B}$ and diffeomorphisms

$$\varphi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C}$$

local trivializations

$\varphi$ $\xrightarrow{\varphi_\alpha} V_\alpha$

$\varphi^{-1}(V_\alpha)$

$\varphi_\alpha$

$\varphi_\alpha(\eta) = (\varphi(\eta), \varphi_\alpha(\eta))$

Additionally, $\forall x \in \mathcal{B}$: $E_p = \varphi^{-1}(x)$ is a 1-dimensional vector space and

$$\varphi_\alpha|_{E_x} : E_x \to \mathbb{C}, \ E_x = \varphi^{-1}(x)$$

is linear for each $x \in V_\alpha$ and all $\alpha$.

**Embedding**: The family

$$\psi = \{ \psi : E \to \mathbb{C} : \psi \text{ is linear on each } E_x \}$$

separates the sets $E_x$:

$$\psi|_{E_x} = \psi|_{E_x}, \forall \psi \in \psi \implies x = x'$$

There are $\psi_1, \ldots, \psi_{N+1} \in \psi$ such that

$$F : E \setminus 0 \to \mathbb{C}^N \setminus 0,$$

$$F(\eta) = (\psi_1(\eta), \ldots, \psi_{N+1}(\eta))$$

is injective.
Brief overview of line bundles

Let $B$ be a manifold. A line bundle over $B$ is a manifold $E$ together with a smooth map $\varphi : E \to B$, an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ of $B$ and diffeomorphisms $\varphi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C}$.

**local trivializations**

$\varphi^{-1}(V_\alpha) \xrightarrow{\phi_\alpha} V_\alpha \times \mathbb{C}$

$\varphi : E \to B$

$\varphi_\alpha : V_\alpha \to B$

$\phi_\alpha(\eta) = (\varphi(\eta), \varphi_\alpha(\eta))$

Additionally, $\forall x \in B : E_p = \varphi^{-1}(x)$ is a 1-dimensional vector space and

$\varphi_\alpha|_{E_x} : E_x \to \mathbb{C}, \ E_x = \varphi^{-1}(x)$ is linear for each $x \in V_\alpha$ and all $\alpha$.

**Embedding**: The family $\psi = \{\psi : E \to \mathbb{C} : \psi \text{ is linear on each } E_x\}$ separates the sets $E_x$:

$\psi|_{E_x} = \psi|_{E_x'}, \forall \psi \in \psi \implies x = x'$

There are $\psi_1, \ldots, \psi_{N+1} \in \psi$ such that

$F : E \setminus 0 \to \mathbb{C}^N \setminus 0$,

$F(\eta) = (\psi_1(\eta), \ldots, \psi_{N+1}(\eta))$

is injective. Then

$B \ni x \mapsto F(E_x) \ni \mathbb{C} \mathbb{P}^{N-1}$

is injective.
Brief overview of line bundles

Let $B$ be a manifold. A line bundle over $B$ is a manifold $E$ together with a smooth map $\varphi : E \to B$, an open cover $\mathcal{U} = \{ V_\alpha \}_{\alpha \in A}$ of $B$ and diffeomorphisms

$$\varphi^{-1}(V_\alpha) \xrightarrow{\varphi_\alpha} V_\alpha \times \mathbb{C}$$

local trivializations

Embedding: The family

$$\psi = \{ \psi : E \to \mathbb{C} : \psi \text{ is linear on each } E_x \}$$

separates the sets $E_x$:

$$\psi|_{E_x} = \psi|_{E_x'}, \forall \psi \in \Psi \implies x = x'$$

There are $\psi_1, \ldots, \psi_{N+1} \in \Psi$ such that

$$F : E \setminus 0 \to \mathbb{C}^N \setminus 0,$$

$$F(\eta) = (\psi_1(\eta), \ldots, \psi_{N+1}(\eta))$$

is injective. Then

$$B \ni x \mapsto F(E_x) \ni \mathbb{CP}^{N-1}$$

is injective. With additional well-chosen $\psi'$s in $\Psi$ it is an embedding.
Circle bundles

The circle bundle of the line bundle $E \to B$ is the set of unit length of $E$ with respect to some Hermitian metric.
Circle bundles

The circle bundle of the line bundle $E \to B$ is the set of unit length of $E$ with respect to some Hermitian metric. There is an $S^1$-action on $SE$:

$$S^1 \times SE \ni (e^{it}, p) \mapsto e^{it} p \in SE$$
Circle bundles

The circle bundle of the line bundle $E \to B$ is the set of unit length of $E$ with respect to some Hermitian metric. There is an $S^1$-action on $SE$:

$$S^1 \times SE \ni (e^{it}, p) \mapsto e^{it} p \in SE$$

View this as an $\mathbb{R}$-action:

$$\mathbb{R} \times SE \ni (t, p) \mapsto e^{it} p \in SE.$$
Circle bundles

The circle bundle of the line bundle $E \to B$ is the set of unit length of $E$ with respect to some Hermitian metric. There is an $S^1$-action on $SE$:

$$S^1 \times SE \ni (e^{it}, p) \mapsto e^{it}p \in SE$$

View this as an $\mathbb{R}$-action:

$$\mathbb{R} \times SE \ni (t, p) \mapsto e^{it}p \in SE.$$

Let $\mathcal{T}$ be its infinitesimal generator:

$$\mathcal{T}_p = \dot{\gamma}_p(0).$$
Circle bundles

The circle bundle of the line bundle $E \to B$ is the set of unit length of $E$ with respect to some Hermitian metric. There is an $S^1$-action on $SE$:

$$S^1 \times SE \ni (e^{it}, p) \mapsto e^{it} p \in SE$$

View this as an $\mathbb{R}$-action:

$$\mathbb{R} \times SE \ni (t, p) \mapsto e^{it} p \in SE.$$ 

Let $\mathcal{T}$ be its infinitesimal generator:

$$\mathcal{T}_p = \dot{\gamma}_p(0).$$

$\mathcal{T}$ is a smooth real nowhere zero vector field on $SE$. Its projection on $B$ is $\pi_* \mathcal{T} = 0$. since $\pi(\gamma_p(t)) = \pi(p)$. 

(Temple University)
Circle bundles

The circle bundle of the line bundle $E \to B$ is the set of unit length of $E$ with respect to some Hermitian metric. There is an $S^1$-action on $SE$:

$$S^1 \times SE \ni (e^{it}, p) \mapsto e^{it} p \in SE$$

View this as an $\mathbb{R}$-action:

$$\mathbb{R} \times SE \ni (t, p) \mapsto e^{it} p \in SE.$$  

Let $\mathcal{T}$ be its infinitesimal generator:

$$\mathcal{T}_p = \dot{\gamma}_p(0).$$

$\mathcal{T}$ is a smooth real nowhere zero vector field on $SE$. Its projection on $B$ is $\pi_* \mathcal{T} = 0$. Since $\pi(\gamma_p(t)) = \pi(p)$

When $E = B \times S^1 \subset B \times \mathbb{C}$,

$$\mathcal{T} = i(z \partial_z - \bar{z} \partial_{\bar{z}}) = -y \partial_x + x \partial_y$$
Circle bundles

The circle bundle of the line bundle $E \to B$ is the set of unit length of $E$ with respect to some Hermitian metric. There is an $S^1$-action on $SE$:

$$S^1 \times SE \ni (e^{it}, p) \mapsto e^{it} p \in SE$$

View this as an $\mathbb{R}$-action:

$$\mathbb{R} \times SE \ni (t, p) \mapsto e^{it} p \in SE.$$ 

Let $\mathcal{T}$ be its infinitesimal generator:

$$\mathcal{T}_p = \dot{\gamma}_p(0).$$

$\mathcal{T}$ is a smooth real nowhere zero vector field on $SE$. Its projection on $B$ is $\pi_* \mathcal{T} = 0$.

When $E = B \times S^1 \subset B \times \mathbb{C}$,

$$\mathcal{T} = i(z \partial_z - \overline{z} \partial_{\overline{z}}) = -y \partial_x + x \partial_y$$

$$\partial_z = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

$$\partial_{\overline{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

$z = x + iy,$
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

$\mathcal{T}|_{\mathcal{N}} = 0$

The effect of (3) is to make $p \sim p' \iff p' \in O_p$ a relation of equivalence.

Here is an example where $\sim$ is not a relation of equivalence: (No symmetry)

$q \neq p \Rightarrow p' \in O_p$ but $p / \in O_p'$
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that:

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

$\mathcal{N}$ is like $SE$,
$\mathcal{T}$ gives an $\mathbb{R}^1$-action $LT_g = 0$.

The effect of (3) is to make $p \sim p' \iff p' \in \text{closure of orbit of } \mathcal{T}$ through $p$ a relation of equivalence.

Here is an example where $\sim$ is not a relation of equivalence: (No symmetry)
$q \not\sim p \sim q \not\sim p' \in \text{closure of orbit of } \mathcal{T}$. 

$\cos 2\theta \frac{\partial}{\partial \theta} + 5 \frac{\partial}{\partial \psi}$

(Temple University)

Embedding theorems

Beirut, November 2011 7 / 18
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;

2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;

3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

$\mathcal{L}_T g = 0$

$\mathcal{N}$ is like $SE$

$\mathcal{T}$ gives an $\mathbb{R}^1$-action

$\text{closure of orbit}$

Here is an example where $\sim$ is not a relation of equivalence: (No symmetry)

$q\overset{p}{\sim}q\overset{p}{\sim}p'\overset{q}{\sim}p$ but $p/\in O_{p'}$
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, T)$ such that

1) $\mathcal{N}$ is compact;
2) $T$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $T$-invariant Riemannian metric on $\mathcal{N}$.

The effect of (3) is to make $p \sim p' \iff p' \in \overline{O}_p$ a relation of equivalence.

$L_T g = 0$
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The effect of (3) is to make

$$p \sim p' \iff p' \in \overline{O}_p$$

a relation of equivalence.

Here is an example where $\sim$ is not a relation of equivalence: (No symmetry)
Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The effect of (3) is to make

$$p \sim p' \iff p' \in \overline{O_p}$$

a relation of equivalence.

Here is an example where $\sim$ is not a relation of equivalence: (No symmetry)
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The set of closures of orbits, $\mathcal{B}_\mathcal{N}$, is a Hausdorff space.
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The set of closures of orbits, $\mathcal{B}_\mathcal{N}$, is a Hausdorff space

$$\mathcal{B}_\mathcal{N} = \mathcal{N}/\sim,$$

$$p \sim p' \iff p' \in \overline{O}_p$$
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The set of closures of orbits, $\mathcal{B}_\mathcal{N}$, is a Hausdorff space

$$\pi_\mathcal{N} : \mathcal{N} \to \mathcal{B}_\mathcal{N}$$

is like the circle bundle of a Hermitian line bundle with $\mathcal{T}$ as the infinitesimal generator of the action

$$(e^{it}, p) \mapsto e^{it} \cdot p = e^{it} p.$$

$\mathcal{B}_\mathcal{N} = \mathcal{N} / \sim, \quad p \sim p' \iff p' \in \overline{O}_p$
Generalization

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that

1) $\mathcal{N}$ is compact;
2) $\mathcal{T}$ is a nowhere zero real vector field on $\mathcal{N}$;
3) there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The set of closures of orbits, $\mathcal{B}_\mathcal{N}$, is a Hausdorff space

$$\pi_\mathcal{N} : \mathcal{N} \to \mathcal{B}_\mathcal{N}$$

is like the circle bundle of a Hermitian line bundle with $\mathcal{T}$ as the infinitesimal generator of the action

$$(e^{it}, p) \mapsto e^{it} \cdot p = e^{it}p.$$ 

I’ll write $\alpha_t$ for the action of a general $\mathcal{T}$. In general the orbits of $\mathcal{T}$ need not be periodic (need not be compact).
Embeddings

Suppose $\varphi : E \rightarrow B$ is a complex line bundle. There is an embedding $\phi : B \rightarrow \mathbb{CP}^N$ such that $E$ is isomorphic to $\phi^*\Gamma$: 

$m$
Embeddings

Suppose \( \varphi : E \rightarrow B \) is a complex line bundle. There is an embedding \( \phi : B \rightarrow \mathbb{CP}^N \) such that \( E \) is isomorphic to \( \phi^* \Gamma \):

\[
\Phi : E \rightarrow \{(x, v) \in B \times \mathbb{C}^N : v \in \phi(x)\}
\]

\( F : \{\eta \in E : |\eta| = 1\} \rightarrow S^2 \mathbb{N}^+ \subset \mathbb{C}^{N+1} \setminus \{0\} \) is an embedding and \( F^* T = i \sum_j (z_j \partial z_j - z_j \partial z_j) \).
Embeddings

Suppose \( \varphi : E \to B \) is a complex line bundle. There is an embedding \( \phi : B \to \mathbb{C}P^N \) such that \( E \) is isomorphic to \( \phi^* \Gamma \):

\[
\Phi : E \to \{(x, v) \in B \times \mathbb{C}^N : v \in \phi(x)\}
\]

\[
\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N
\]
Suppose \( \varphi : E \to B \) is a complex line bundle. There is an embedding \( \phi : B \to \mathbb{CP}^N \) such that \( E \) is isomorphic to \( \phi^* \Gamma \):

\[
\Phi : E \to \{(x, v) \in B \times \mathbb{C}^N : v \in \phi(x)\}
\]

\[
\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N
\]
Embeddings

Suppose $\varphi : E \rightarrow B$ is a complex line bundle. There is an embedding $\phi : B \rightarrow \mathbb{CP}^N$ such that $E$ is isomorphic to $\phi^* \Gamma$:

$$\Phi : E \rightarrow \{(x, \nu) \in B \times \mathbb{C}^N : \nu \in \phi(x)\}$$

$$\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N, F(\eta) \in \phi(\varphi(\eta))$$
Embeddings

Suppose $\phi: E \to B$ is a complex line bundle. There is an embedding $\phi: B \to \mathbb{CP}^N$ such that $E$ is isomorphic to $\phi^* \Gamma$:

$$\Phi : E \to \{(x, v) \in B \times \mathbb{C}^N : v \in \phi(x)\}$$

$$\Phi(\eta) = (\phi(\eta), F(\eta)) \in B \times \mathbb{C}^N, \ F(\eta) \in \phi(\phi(\eta))$$

$F : E \to \mathbb{C}^{N+1}$ is injective

$F|_{E_x}$ is linear injective for each $x$
Embeddings

Suppose \( \phi : E \to B \) is a complex line bundle. There is an embedding \( \phi : B \to \mathbb{CP}^N \) such that \( E \) is isomorphic to \( \phi^* \Gamma \):

\[
\Phi : E \to \{(x, \nu) \in B \times \mathbb{C}^N : \nu \in \phi(x)\}
\]

\[
\Phi(\eta) = (\phi(\eta), F(\eta)) \in B \times \mathbb{C}^N , \quad F(\eta) \in \phi(\phi(\eta))
\]

Suppose \( E \) has a Hermitian metric such that \(|\eta| = 1 \implies |F(\eta)| = 1|:

\[
F : E \to \mathbb{C}^{N+1} \text{ is injective}
\]

\[
F|_{E_x} \text{ is linear injective for each } x
\]
Embeddings

Suppose \( \varphi : E \to B \) is a complex line bundle. There is an embedding \( \phi : B \to \mathbb{C}P^N \) such that \( E \) is isomorphic to \( \phi^* \Gamma \):

\[
\Phi : E \to \{(x, v) \in B \times \mathbb{C}^N : v \in \phi(x)\}
\]

\[
\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N, \ F(\eta) \in \phi(\varphi(\eta))
\]

Suppose \( E \) has a Hermitian metric such that \(|\eta| = 1 \implies |F(\eta)| = 1\):

\[
F : \{\eta \in E : |\eta| = 1\} \to S^{2N+1} \subset \mathbb{C}^{N+1} \setminus 0
\]
Embeddings

Suppose \( \varphi : E \to B \) is a complex line bundle. There is an embedding \( \phi : B \to \mathbb{C}P^N \) such that \( E \) is isomorphic to \( \phi^* \Gamma \):

\[
\Phi : E \to \{(x, v) \in B \times \mathbb{C}^N : v \in \phi(x)\}
\]

\[
\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N, \quad F(\eta) \in \phi(\varphi(\eta))
\]

Suppose \( E \) has a Hermitian metric such that \( |\eta| = 1 \implies |F(\eta)| = 1 \):

\[
F : \{\eta \in E : |\eta| = 1\} \to S^{2N+1} \subset \mathbb{C}^{N+1} \setminus 0 = SE = \text{circle bundle of } E
\]

\( F : E \to \mathbb{C}^{N+1} \) is injective

\( F|_{E_x} \) is linear injective for each \( x \)
Embeddings

Suppose $\varphi : E \to B$ is a complex line bundle. There is an embedding $\phi : B \to \mathbb{C}P^N$ such that $E$ is isomorphic to $\phi^*\Gamma$:

$$
\Phi : E \to \{(x, v) \in B \times \mathbb{C}^N : v \in \phi(x)\}
$$

$$
\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N, \quad F(\eta) \in \phi(\varphi(\eta))
$$

Suppose $E$ has a Hermitian metric such that $|\eta| = 1 \implies |F(\eta)| = 1$:

$$
F : \{\eta \in E : |\eta| = 1\} \to S^{2N+1} \subset \mathbb{C}^{N+1} \setminus 0
$$

Since $F|_{E_x}$ is linear,

$$
F(e^{it}\eta) = e^{it}F(\eta)
$$

Let $T$ be the infinitesimal generator of the action $(t, \eta) \mapsto e^{it}\eta$.  

$F : E \to \mathbb{C}^{N+1}$ is injective

$F|_{E_x}$ is linear injective for each $x$
**Embeddings**

Suppose $\varphi : E \to B$ is a complex line bundle. There is an embedding $\phi : B \to \mathbb{C}P^N$ such that $E$ is isomorphic to $\phi^*\Gamma$:

$$
\Phi : E \to \{(x, v) \in B \times \mathbb{C}^N : v \in \phi(x)\}
$$

$$
\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N, F(\eta) \in \phi(\varphi(\eta))
$$

Suppose $E$ has a Hermitian metric such that $|\eta| = 1 \implies |F(\eta)| = 1$:

$$
F : \{\eta \in E : |\eta| = 1\} \to S^{2N+1} \subset \mathbb{C}^{N+1} \setminus \{0\}
$$

Since $F|_{E_x}$ is linear,

$$
F(e^{it}\eta) = e^{it} F(\eta)
$$

Let $\mathcal{T}$ be the infinitesimal generator of the action $(t, \eta) \mapsto e^{it}\eta$. This is a vector field on $SE$.

\[ F : E \to \mathbb{C}^{N+1} \text{ is injective} \]

\[ F|_{E_x} \text{ is linear injective for each } x \]
Embeddings

Suppose $\varphi : E \to B$ is a complex line bundle. There is an embedding $\phi : B \to \mathbb{CP}^N$ such that $E$ is isomorphic to $\phi^*\Gamma$:

$$\Phi : E \to \{ (x, v) \in B \times \mathbb{C}^N : v \in \phi(x) \}$$

$$\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N, \quad F(\eta) \in \phi(\varphi(\eta))$$

Suppose $E$ has a Hermitian metric such that $|\eta| = 1 \implies |F(\eta)| = 1$:

$$F : \{ \eta \in E : |\eta| = 1 \} \to S^{2N+1} \subset \mathbb{C}^{N+1}\setminus 0$$

Since $F|_{E_x}$ is linear,

$$F(e^{it}\eta) = e^{it}F(\eta)$$

Let $\mathcal{T}$ be the infinitesimal generator of the action $(t, \eta) \mapsto e^{it}\eta$. This is a vector field on $SE$. The infinitesimal generator of the “same” action on $S^{2N+1}$ is $\mathcal{T}' = i \sum_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j})$. 

$F : E \to \mathbb{C}^{N+1}$ is injective

$F|_{E_x}$ is linear injective for each $x$
Embeddings

Suppose $\varphi : E \to B$ is a complex line bundle. There is an embedding $\phi : B \to \mathbb{CP}^N$ such that $E$ is isomorphic to $\phi^*\Gamma$:

$$\Phi : E \to \{ (x, v) \in B \times \mathbb{C}^N : v \in \phi(x) \}$$

$$\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N, F(\eta) \in \phi(\varphi(\eta))$$

Suppose $E$ has a Hermitian metric such that $|\eta| = 1 \implies |F(\eta)| = 1$:

$$F : \{ \eta \in E : |\eta| = 1 \} \to S^{2N+1} \subset \mathbb{C}^{N+1}\setminus 0$$

Since $F|_{E_x}$ is linear,

$$F(e^{it}\eta) = e^{it}F(\eta)$$

Let $T$ be the infinitesimal generator of the action $(t, \eta) \mapsto e^{it}\eta$. This is a vector field on $SE$. The infinitesimal generator of the “same” action on $S^{2N+1}$ is $T' = i \sum_{j}(z^j \partial z^j - \bar z^j \partial \bar z^j)$.

$$F : SE \to S^{2N+1} \text{ is an embedding}$$
Embeddings

Suppose $\varphi : E \to \mathcal{B}$ is a complex line bundle. There is an embedding $\phi : \mathcal{B} \to \mathbb{CP}^N$ such that $E$ is isomorphic to $\phi^*\Gamma$:

$$
\Phi : E \to \{ (x, v) \in \mathcal{B} \times \mathbb{C}^N : v \in \phi(x) \}
$$

$$
\Phi(\eta) = (\varphi(\eta), F(\eta)) \in \mathcal{B} \times \mathbb{C}^N, \ F(\eta) \in \phi(\varphi(\eta))
$$

Suppose $E$ has a Hermitian metric such that $|\eta| = 1 \implies |F(\eta)| = 1$:

$$
F : \{ \eta \in E : |\eta| = 1 \} \to S^{2N+1} \subset \mathbb{C}^{N+1} \setminus \{0\}
$$

Since $F|_{E_x}$ is linear,

$$
F(e^{it}\eta) = e^{it}F(\eta)
$$

Let $\mathcal{T}$ be the infinitesimal generator of the action $(t, \eta) \mapsto e^{it}\eta$. This is a vector field on $SE$. The infinitesimal generator of the “same” action on $S^{2N+1}$ is $\mathcal{T}' = i \sum_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j})$.

$$
F : SE \to S^{2N+1} \text{ is an embedding and } F_*\mathcal{T} = i \sum_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}).
$$
Embeddings

Suppose \( \varphi : E \to B \) is a complex line bundle. There is an embedding \( \phi : B \to \mathbb{CP}^N \) such that \( E \) is isomorphic to \( \phi^\ast \Gamma \):

\[
\Phi : E \to \{ (x, \nu) \in B \times \mathbb{C}^N : \nu \in \phi(x) \} \\
\Phi(\eta) = (\varphi(\eta), F(\eta)) \in B \times \mathbb{C}^N, \ F(\eta) \in \phi(\varphi(\eta))
\]

Suppose \( E \) has a Hermitian metric such that \( |\eta| = 1 \implies |F(\eta)| = 1 \):

\[
F : \{ \eta \in E : |\eta| = 1 \} \to S^{2N+1} \subset \mathbb{C}^{N+1} \setminus 0
\]

Since \( F|_{E_x} \) is linear,

\[
F(e^{it} \eta) = e^{it} F(\eta)
\]

Let \( \mathcal{T} \) be the infinitesimal generator of the action \( (t, \eta) \mapsto e^{it} \eta \). This is a vector field on \( SE \). The infinitesimal generator of the “same” action on \( S^{2N+1} \) is \( \mathcal{T}' = i \sum_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}) \).

\[
F : SE \to S^{2N+1} \text{ is an embedding and } F_\ast \mathcal{T} = i \sum_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}).
\]
Embeddings

Let \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\).

1) \(\mathcal{N}\) is compact;
2) \(\mathcal{T}\) is a nowhere zero real vector field on \(\mathcal{N}\);
3) there is a \(\mathcal{T}\)-invariant Riemannian metric on \(\mathcal{N}\).
Embeddings

Let \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\). There is an embedding

\[ F : \mathcal{N} \rightarrow S^{2N+1} \]

for some \(N\) such that

\[ F_* \mathcal{T} = i \sum_j \tau_j(z^j \partial z^j - \bar{z}^j \partial \bar{z}^j) \]

for some positive \(\tau_j \in \mathbb{R}\).

1) \(\mathcal{N}\) is compact;
2) \(\mathcal{T}\) is a nowhere zero real vector field on \(\mathcal{N}\);
3) there is a \(\mathcal{T}\)-invariant Riemannian metric on \(\mathcal{N}\).
Embeddings

Let \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\). There is an embedding

\[ F : \mathcal{N} \to S^{2N+1} \]

for some \(N\) such that

\[ F_* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{\bar{z}^j} - \bar{z}^j \partial_{z^j}) \]

for some positive \(\tau_j \in \mathbb{R}\).

Write \(\mathcal{T}'\) for the vector field on the right. It is

1) \(\mathcal{N}\) is compact;
2) \(\mathcal{T}\) is a nowhere zero real vector field on \(\mathcal{N}\);
3) there is a \(\mathcal{T}\)-invariant Riemannian metric on \(\mathcal{N}\).
Embeddings

Let \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\). There is an embedding

\[ F : \mathcal{N} \to S^{2N+1} \]

for some \(N\) such that

\[ F_\ast \mathcal{T} = i \sum_j \tau_j (z^j \partial z^j - \bar{z}^j \partial \bar{z}^j) \]

for some positive \(\tau_j \in \mathbb{R}\).

Write \(\mathcal{T}'\) for the vector field on the right. It is tangent to \(S^{2N+1}\) and preserves the standard metric of \(S^{2N+1}\).

1) \(\mathcal{N}\) is compact;
2) \(\mathcal{T}\) is a nowhere zero real vector field on \(\mathcal{N}\);
3) there is a \(\mathcal{T}\)-invariant Riemannian metric on \(\mathcal{N}\).
Embeddings

Let \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\). There is an embedding

\[ F : \mathcal{N} \rightarrow S^{2N+1} \]

for some \(N\) such that

\[ F^* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}) \]

for some positive \(\tau_j \in \mathbb{R}\).

Write \(\mathcal{T}'\) for the vector field on the right. It is tangent to \(S^{2N+1}\) and preserves the standard metric of \(S^{2N+1}\). So the pair

\((S^{2N+1}, \mathcal{T}')\)

belongs to the class \(\mathcal{F}\).
Embeddings

Let \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\). There is an embedding

\[
F : \mathcal{N} \to S^{2N+1}
\]

for some \(N\) such that

\[
F_* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j})
\]

for some positive \(\tau_j \in \mathbb{R}\).

Write \(\mathcal{T}'\) for the vector field on the right. It is tangent to \(S^{2N+1}\) and preserves the standard metric of \(S^{2N+1}\). So the pair

\((S^{2N+1}, \mathcal{T}')\)

belongs to the class \(\mathcal{F}\). Let

\[
\omega = (\omega^1, \ldots, \omega^{N+1}) \in S^{2N+1}.
\]

Then

\[
\alpha'_t \omega = (e^{i\tau_1 t} \omega^1, \ldots, e^{i\tau_{N+1} t} \omega^{N+1}).
\]
Embeddings

Let \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\). There is an embedding

\[ F : \mathcal{N} \to S^{2N+1} \]

for some \(N\) such that

\[ F_\ast \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}) \]

for some positive \(\tau_j \in \mathbb{R}\).

Write \(\mathcal{T}'\) for the vector field on the right. It is tangent to \(S^{2N+1}\) and preserves the standard metric of \(S^{2N+1}\). So the pair

\((S^{2N+1}, \mathcal{T}')\)

belongs to the class \(\mathcal{F}\). Let

\[ \omega = (\omega^1, \ldots, \omega^{N+1}) \in S^{2N+1}. \]

Then

\[ a'_t \omega = (e^{i\tau_1 t} \omega^1, \ldots, e^{i\tau_{N+1} t} \omega^{N+1}). \]

The closure of the orbit of \(\omega\) is a torus of dimension

\[ \dim \operatorname{span}_\mathbb{Q} \{ \tau_j : \omega^j \neq 0 \} \]
Outline of proof

Embedding of \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) in \(\mathbb{C}^N\).
Outline of proof

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute.
Outline of proof

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute.

Embedding of \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) in \(\mathbb{C}^N \setminus \{0\}\).

\(\mathcal{E}_\lambda = \ker(\Delta - \lambda)\) is invariant under \(\mathcal{T}\), 

\(-i\mathcal{T}\) is symmetric, so \(\text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\).
Outline of proof

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[
E_{\tau, \lambda} = \{ \phi \in C^\infty(\mathcal{N}) : \Delta \phi = \lambda \phi, \quad -i \mathcal{T} \phi = \tau \phi \},
\]

\[
\text{spec}(-i \mathcal{T}, \Delta) = \{ (\tau, \lambda) : E_{\tau, \lambda} \neq 0 \}.
\]
Outline of proof

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[\mathcal{E}_\lambda = \ker(\Delta - \lambda)\] is invariant under \(\mathcal{T}\),

\[\mathcal{E}_\lambda = \mathcal{E}_\lambda,\] is symmetric, so \(\text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\)

Embedding of \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) in \(\mathbb{C}^N \setminus 0\).

\[\mathcal{E}_\lambda = \ker(\Delta - \lambda)\] is invariant under \(\mathcal{T}\),

\[\mathcal{E}_\lambda = \mathcal{E}_\lambda,\] is symmetric, so \(\text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\)

Pick orthonormal bases

\[\{\phi_{\tau,\lambda,j} \in \mathcal{E}_{\tau,\lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau,\lambda}\}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta)\]
Outline of proof

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[
\mathcal{E}_\lambda = \ker(\Delta - \lambda)\text{ is invariant under } \mathcal{T}, \\
-\mathcal{T}\text{ is symmetric, so spec}(-\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}
\]

\[
\mathcal{E}_{\tau, \lambda} = \{\phi \in C^\infty(\mathcal{N}) : \Delta \phi = \lambda \phi, \quad -\mathcal{I} \phi = \tau \phi\},
\]

\[
\text{spec}(-\mathcal{I}, \Delta) = \{(\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0\}.
\]

Pick orthonormal bases

\[
\{\phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \text{dim } \mathcal{E}_{\tau, \lambda}\}, \quad (\tau, \lambda) \in \text{spec}(-\mathcal{I}, \Delta)
\]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C} T_{p_0}^* \mathcal{N} = \text{span}\{d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-\mathcal{I}, \Delta), \quad j = 1, \ldots, \text{dim } \mathcal{E}_{\tau, \lambda}\};\)
Outline of proof

Embedding of \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) in \(\mathbb{C}^N \setminus \{0\}\).

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[
\mathcal{E}_\lambda = \ker(\Delta - \lambda) \text{ is invariant under } \mathcal{T},
\]

\(-i\mathcal{T}\) is symmetric, so \(\text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\).

Let

\[
\mathcal{E}_{\tau,\lambda} = \{ \phi \in C^\infty(\mathcal{N}) : \Delta \phi = \lambda \phi, \ -i\mathcal{T} \phi = \tau \phi \},
\]

\[
\text{spec}(-i\mathcal{T}, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau,\lambda} \neq 0 \}.
\]

Pick orthonormal bases

\[
\{ \phi_{\tau,\lambda,j} \in \mathcal{E}_{\tau,\lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau,\lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta)
\]

(1) for all \(p_0 \in \mathcal{N}, \mathbb{C}T_{p_0}^* \mathcal{N} = \text{span}\{d\phi_{\tau,\lambda,j}(p_0) : (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), ~ j = 1, \ldots, N_{\tau,\lambda}\};
\]

From (1) with continuity and compactness get an immersion

\[
F = (\phi_{\tau_1,\lambda_1,j_1}, \ldots, \phi_{\tau_N,\lambda_N,j_N}) : SE \to \mathbb{C}^N
\]
Outline of proof

Pick \((\mathcal{N}, \mathcal{T}) \in \mathscr{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[
\mathcal{E}_\lambda = \ker(\Delta - \lambda)\text{ is invariant under }\mathcal{T},
\]

\(-i\mathcal{T}\) is symmetric, so

\[
\text{spec}\left(-i\mathcal{T} |_{\mathcal{E}_\lambda}\right) \subset \mathbb{R}
\]

Pick orthonormal bases

\[
\mathcal{E}_\tau, \lambda = \{ \phi \in C^\infty(\mathcal{N}) : \Delta \phi = \lambda \phi, \ -i\mathcal{T} \phi = \tau \phi \},
\]

\[
\text{spec}(-i\mathcal{T}, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0 \}.
\]

Pick orthonormal bases

\[
\{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta)
\]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C} T_{p_0}^* \mathcal{N} = \text{span}\{ d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), \ j = 1, \ldots, N_{\tau, \lambda} \} \);

(2) the functions \(\phi_{\tau, \lambda, j}, (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), \ j = 1, \ldots, N_{\tau, \lambda}\), separate points of \(\mathcal{N}\).

From (1) with continuity and compactness get an immersion

\[
F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \to \mathbb{C}^N
\]
Outline of proof

Pick \((N, T) \in \mathcal{F}\) and a \(T\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(T\) commute. Let

\[ \mathcal{E}_\lambda = \ker(\Delta - \lambda) \text{ is invariant under } T, \]

\[ -iT \text{ is symmetric, so } \text{spec}(-iT|_{\mathcal{E}_\lambda}) \subset \mathbb{R} \]

Pick \(E_{\tau, \lambda} = \{ \phi \in C^\infty(N) : \Delta \phi = \lambda \phi, -iT \phi = \tau \phi \}, \)

\[ \text{spec}(-iT, \Delta) = \{ (\tau, \lambda) : E_{\tau, \lambda} \neq 0 \}. \]

Pick orthonormal bases

\[ \{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \text{dim } \mathcal{E}_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-iT, \Delta) \]

(1) for all \(p_0 \in N\), \(\mathbb{C} T_{p_0}^*N = \text{span}\{ d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-iT, \Delta), j = 1, \ldots, N_{\tau, \lambda} \}; \)

(2) the functions \(\phi_{\tau, \lambda, j}, (\tau, \lambda) \in \text{spec}(-iT, \Delta), j = 1, \ldots, N_{\tau, \lambda}, \)

separate points of \(N\).

From (1) with continuity and compactness get an immersion

\[ F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \to \mathbb{C}^N \]

If \(\tau_\ell < 0\), replace \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) by \(\overline{\phi_{\tau_\ell, \lambda_\ell, j_\ell}}\).
Outline of proof

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let

\[\mathcal{E}_\lambda = \ker(\Delta - \lambda)\]

is invariant under \(\mathcal{T}\), \(-i\mathcal{T}\) is symmetric, so \(\text{spec}(-i\mathcal{T}|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\).

Pick orthonormal bases

\[\{\phi_{\tau,\lambda,j} \in \mathcal{E}_{\tau,\lambda} : j = 1, \ldots, \text{dim} \mathcal{E}_{\tau,\lambda}\}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta)\]

(1) for all \(p_0 \in \mathcal{N}\), \(\mathbb{C}T^*_p \mathcal{N} = \text{span}\{d\phi_{\tau,\lambda,j}(p_0) : (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), j = 1, \ldots, N_{\tau,\lambda}\};\)

(2) the functions \(\phi_{\tau,\lambda,j}, (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), j = 1, \ldots, N_{\tau,\lambda},\)

separate points of \(\mathcal{N}\).

From (1) with continuity and compactness get an immersion

\[F = (\phi_{\tau_1,\lambda_1,j_1}, \ldots, \phi_{\tau_N,\lambda_N,j_N}) : SE \rightarrow \mathbb{C}^N \setminus 0\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell,\lambda_\ell,j_\ell}\) separate points.
Outline of proof

Pick \((N, T) \in \mathcal{F}\) and a \(T\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(T\) commute. Let

\[
\mathcal{E}_\lambda = \{ \phi \in C^\infty(N) : \Delta \phi = \lambda \phi, \ -i T \phi = \tau \phi \},
\]

\[
\text{spec}(-i T, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0 \}.
\]

Pick orthonormal bases

\[
\{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-i T, \Delta)
\]

(1) for all \(p_0 \in N\), \(\mathbb{C} T_{p_0}^* N = \text{span}\{ d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-i T, \Delta), j = 1, \ldots, N_{\tau, \lambda} \};
\]

(2) the functions \(\phi_{\tau, \lambda, j}, (\tau, \lambda) \in \text{spec}(-i T, \Delta), j = 1, \ldots, N_{\tau, \lambda},\)

separate points of \(N\).

From (1) with continuity and compactness get an immersion

\[
F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \to \mathbb{C}^N \setminus 0
\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) separate points.
Outline of proof

Pick \((N, T) \in \mathcal{F}\) and a \(T\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(T\) commute. Let

\[
\mathcal{E}_{\tau, \lambda} = \{ \phi \in C^\infty(N) : \Delta \phi = \lambda \phi, \quad -iT \phi = \tau \phi \},
\]

\[
\text{spec}(-iT, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0 \}.
\]

Pick orthonormal bases

\[
\{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-iT, \Delta)
\]

(1) for all \(p_0 \in N\), \(C^*T_{p_0}N = \text{span}\{ d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-iT, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda} \};\)

(2) the functions \(\phi_{\tau, \lambda, j} \), \((\tau, \lambda) \in \text{spec}(-iT, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda},\)

separate points of \(N\).

From (1) with continuity and compactness get an immersion

\[
F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \to C^N \setminus 0
\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) separate points.
Outline of proof

Pick \((N, T) \in \mathcal{F}\) and a \(T\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(T\) commute.

\[
\mathcal{E}_{\tau, \lambda} = \{ \phi \in C^\infty(N) : \Delta \phi = \lambda \phi, -i T \phi = \tau \phi \},
\]
\[
\text{spec}(-i T, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0 \}.
\]

Pick orthonormal bases \(\{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \text{dim} \mathcal{E}_{\tau, \lambda} \}, (\tau, \lambda) \in \text{spec}(-i T, \Delta)\)

(1) for all \(p_0 \in N\), \(\mathbb{C} T_{p_0}^* N = \text{span}\{ d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-i T, \Delta), j = 1, \ldots, N_{\tau, \lambda} \} \);

(2) the functions \(\phi_{\tau, \lambda, j}, (\tau, \lambda) \in \text{spec}(-i T, \Delta), j = 1, \ldots, N_{\tau, \lambda}, \) separate points of \(N\).

From (1) with continuity and compactness get an immersion

\[
F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \to \mathbb{C}^N \setminus 0
\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) separate points.

\(\mathcal{E}_\lambda = \ker(\Delta - \lambda)\) is invariant under \(T\), 
\(-i T\) is symmetric, so \(\text{spec}(-i T|_{\mathcal{E}_\lambda}) \subset \mathbb{R}\)

\(\alpha_t = \) one parameter group of diffeos generated by \(T\)

\(-i T \phi = \tau \phi \implies \phi(\alpha_t p) = e^{i \tau t} \phi(p)\)

If \(\tau_\ell < 0\), replace \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) by \(\overline{\phi_{\tau_\ell, \lambda_\ell, j_\ell}}\).
Outline of proof

Pick \((\mathcal{N}, \mathcal{T}) \in \mathcal{F}\) and a \(\mathcal{T}\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(\mathcal{T}\) commute. Let
\[
\mathcal{E}_{\tau, \lambda} = \{ \phi \in C^\infty(\mathcal{N}) : \Delta \phi = \lambda \phi, \quad -i\mathcal{T} \phi = \tau \phi \},
\]
\[
\text{spec}(-i\mathcal{T}, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0 \}.
\]
Pick orthonormal bases
\[
\{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta)
\]
\[(1)\] for all \(p_0 \in \mathcal{N}, \mathbb{C} \mathcal{T}^*_{p_0} \mathcal{N} = \text{span}\{ d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda} \};
\]
\[(2)\] the functions \(\phi_{\tau, \lambda, j}, (\tau, \lambda) \in \text{spec}(-i\mathcal{T}, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda},\)

separate points of \(\mathcal{N}\).

From (1) with continuity and compactness get an immersion
\[
F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \to \mathbb{C}^N \setminus 0
\]
Use property (2) to ensure that the functions \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) separate points.
Outline of proof

Pick \((N, T) \in F\) and a \(T\)-invariant metric, let \(\Delta\) be the Laplacian. Note that \(\Delta\) and \(T\) commute. Let
\[
\mathcal{E}_{\tau, \lambda} = \{ \phi \in C^\infty(N) : \Delta \phi = \lambda \phi, \quad -iT \phi = \tau \phi \},
\]
\[
\text{spec}(-iT, \Delta) = \{ (\tau, \lambda) : \mathcal{E}_{\tau, \lambda} \neq 0 \}.
\]

Pick orthonormal bases
\[
\{ \phi_{\tau, \lambda, j} \in \mathcal{E}_{\tau, \lambda} : j = 1, \ldots, \dim \mathcal{E}_{\tau, \lambda} \}, \quad (\tau, \lambda) \in \text{spec}(-iT, \Delta)
\]

(1) for all \(p_0 \in N\), \(\mathbb{C} T_{p_0}^* N = \text{span}\{ d\phi_{\tau, \lambda, j}(p_0) : (\tau, \lambda) \in \text{spec}(-iT, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda} \} \);

(2) the functions \(\phi_{\tau, \lambda, j}, \quad (\tau, \lambda) \in \text{spec}(-iT, \Delta), \quad j = 1, \ldots, N_{\tau, \lambda}, \)

separate points of \(N\).

From (1) with continuity and compactness get an immersion
\[
F = (\phi_{\tau_1, \lambda_1, j_1}, \ldots, \phi_{\tau_N, \lambda_N, j_N}) : SE \to \mathbb{C}^N \setminus 0
\]

Use property (2) to ensure that the functions \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}\) separate points.

Since \(\phi_{\tau_\ell, \lambda_\ell, j_\ell}(a_t p) = e^{i\tau_\ell t} \phi_{\tau_\ell, \lambda_\ell, j_\ell}(p), \quad F_* T = i \sum \tau_\ell (z^\ell \partial_{\overline{z}^\ell} - \overline{z}^\ell \partial_{z^\ell})\)
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold;
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold;

$$T^{0,1}B$$
Line bundles over complex manifolds

Suppose now that \( \mathcal{B} \) is a compact complex manifold; 

\[
\begin{align*}
\mathcal{T}^{0,1}_\mathcal{B} & \quad \text{span}_{\mathbb{C}} \{ \partial z_1, \ldots, \partial z_n \}, \\
\frac{\partial}{\partial z^j} & = \frac{1}{2} \left( \frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j} \right),
\end{align*}
\]

\( z^1, \ldots, z^n \) local complex coordinates, 
\( z^j = x^j + i y^j \).
Line bundles over complex manifolds

Suppose now that $\mathcal{B}$ is a compact complex manifold; $\pi^1 : E \rightarrow \mathcal{B}$ is still only a complex line bundle.

\[ T^{0,1}\mathcal{B} \]

Embedding theorems
Line bundles over complex manifolds

Suppose now that $\mathcal{B}$ is a compact complex manifold; $\pi^1: E \rightarrow \mathcal{B}$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$\nabla = \{ v \in \mathbb{C} TSE : \pi_* v \in T^{0,1} \mathcal{B} \}.$

This is an “elliptic structure” on $SE$

\[ \frac{\partial}{\partial z^j} = \frac{1}{2} \left( \frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j} \right). \]
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold; $\pi^1 : E \to B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$$\bar{\mathcal{V}} = \{ \nu \in \mathbb{C}TSE : \pi_* \nu \in T^{0,1}B \}.$$ 

This is an “elliptic structure” on $SE$.
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold; $\pi^1 : E \to B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let $\overline{\mathcal{V}} = \{v \in \mathbb{C} TSE : \pi_* v \in T^{0,1} B \}$. This is an “elliptic structure” on $SE$ such that $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span } T$. \[ \overline{\mathcal{V}} \text{ is involutive and } \mathcal{V} + \overline{\mathcal{V}} = \mathbb{C} TSE \]
Line bundles over complex manifolds

Suppose now that $\mathcal{B}$ is a compact complex manifold; $\pi^1: E \to \mathcal{B}$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$$\mathcal{V} = \{ \nu \in \mathbb{C} TSE : \pi_* \nu \in T^{0,1}\mathcal{B} \}.$$ 

This is an “elliptic structure” on $SE$ such that $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span} \mathcal{T}$. 

$\overline{\mathcal{V}}$ is involutive and $\mathcal{V} + \overline{\mathcal{V}} = \mathbb{C} TSE$

$SE =$ circle bundle

$\mathcal{T}$ is the infinitesimal generator of the $S^1$ action on $SE$
Line bundles over complex manifolds

Suppose now that \( B \) is a compact complex manifold; \( \pi^1: E \to B \) is still only a complex line bundle. Fix a Hermitian connection \( h \). Let
\[
\overline{\mathcal{V}} = \{ \nu \in \mathbb{C}TSE : \pi_* \nu \in T^{0,1}B \}.
\]
This is an “elliptic structure” on \( SE \) such that
\[
\mathcal{V} \cap \overline{\mathcal{V}} = \text{span} \; \mathcal{T}.
\]
Let
\[
\nabla : C^\infty(B; E) \to C^\infty(B; E \otimes \mathbb{C}T^*B)
\]
be a Hermitian connection. Its horizontal bundle \( \mathcal{H} \) is tangent to \( SE \).
Line bundles over complex manifolds

Suppose now that $\mathcal{B}$ is a compact complex manifold; $\pi^1 : E \rightarrow \mathcal{B}$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$$\nabla = \{ \nabla \in \mathbb{C}TSE : \pi_* \nabla \in T^{0,1}\mathcal{B} \}.$$ 

This is an “elliptic structure” on $SE$ such that $\nabla \cap \overline{\nabla} = \text{span} \mathcal{T}$. Let

$$\nabla : C^\infty(\mathcal{B}; E) \rightarrow C^\infty(\mathcal{B}; E \otimes \mathbb{C}T^*\mathcal{B})$$

be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

$$\mathcal{H} = \{ \nabla \in TSE : \langle \frac{1}{2i} (\zeta d\zeta - \zeta d\bar{\zeta}) - i \pi^* \omega, \nabla \rangle = 0 \}$$

$\eta$ local frame over $\mathcal{V}$, $|\eta| = 1$, $\nabla \eta = \eta \otimes \omega$, $\zeta \eta$ arbitrary element of $E|_\mathcal{V}$. 
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold; $\pi^1 : E \to B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let $\nabla : C^\infty(B; E) \to C^\infty(B; E \otimes \mathbb{C}T^*B)$ be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$. Let

$$\mathcal{H} = \{ \nu \in TSE : \langle \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, \nu \rangle = 0 \}$$

$$\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.$$
Line bundles over complex manifolds \( \bar{\mathcal{V}} \) is involutive and \( \mathcal{V} + \bar{\mathcal{V}} = \mathbb{C}TSE \)

Suppose now that \( \mathcal{B} \) is a compact complex manifold; \( \pi^1 : E \to \mathcal{B} \) is still only a complex line bundle. Fix a Hermitian connection \( h \). Let

\[
\bar{\mathcal{V}} = \{ \nu \in \mathbb{C}TSE : \pi_* \nu \in T^{0,1} \mathcal{B} \}.
\]

This is an “elliptic structure” on \( SE \) such that \( \mathcal{V} \cap \bar{\mathcal{V}} = \text{span} \, \mathcal{T} \).

Let

\[
\nabla : C^\infty(\mathcal{B}; E) \to C^\infty(\mathcal{B}; E \otimes \mathbb{C} T^* \mathcal{B})
\]

be a Hermitian connection. Its horizontal bundle \( \mathcal{H} \) is tangent to \( SE \).

Let

\[
\mathcal{H} = \{ \nu \in TSE : \langle \frac{1}{2i} (\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^* \omega, \nu \rangle = 0 \}
\]

\( \theta = 1 \)-form vanishing on \( \mathcal{H} \) such that \( \langle \theta, \mathcal{T} \rangle = 1 \).
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold; $\pi^1: E \to B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$\mathcal{V} = \{ \nu \in \mathbb{C}TSE : \pi_*\nu \in T^0,1B \}.$

This is an “elliptic structure” on $SE$ such that $\mathcal{V} \cap \mathcal{V} = \text{span } \mathcal{T}.$

Let

$\nabla : C^\infty(B; E) \to C^\infty(B; E \otimes \mathbb{C}T*B)$

be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

Let

$\mathcal{H} = \{ \nu \in TS_\theta = \{ \frac{1}{2i}(\zeta d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, \nu \} = 0 \}$

$\theta = 1$-form vanishing on $\mathcal{H}$ such that $\langle \theta, \mathcal{T} \rangle = 1.$

Let $\beta = -i\theta|_{\mathcal{V}}.$
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold; $\pi : E \to B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let $\mathcal{V} = \{ \nu \in \mathbb{C}T^*_SE : \pi_*\nu \in T^{0,1}B \}$. This is an “elliptic structure” on $SE$ such that $\nu \cap \overline{\nu} = \text{span} \ T$. Let

$$\nabla : C^\infty(B; E) \to C^\infty(B; E \otimes \mathbb{C}T^*B)$$

be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

Let $\mathcal{H} = \{ \nu \in T\nu \theta = \langle \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, \nu \rangle = 0 \}$

$$\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, T \rangle = 1.$$ Let $\beta = -i\theta|_{\mathcal{V}}$. 

$\overline{\nu}$ is involutive and $\nu + \overline{\nu} = \mathcal{C}T^*SE$
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold; $\pi^1 : E \to B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$$\bar{\nabla} = \{ \nu \in \mathbb{C}T^1 \otimes \pi_* \nu \in T^{0,1}B \}.$$ 

This is an “elliptic structure” on $SE$ such that $\nu \cap \bar{\nabla} = \text{span} \, \mathcal{T}$. Let

$$\nabla : C^\infty(B; E) \to C^\infty(B; E \otimes \mathbb{C}T^*B)$$

be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

Let

$$\mathcal{H} = \{ \nu \in TS^\mathfrak{h} \theta = \frac{1}{2i} (\zeta d\zeta - \zeta d\bar{\zeta}) - i \pi^* \omega, \nu \} = 0 \}$$

$\theta = 1$-form vanishing on $\mathcal{H}$ such that $\langle \theta, \mathcal{T} \rangle = 1$.

Let $\beta = -i \theta|_{\bar{\nu}}$. Then $\bar{\mathcal{K}}_\beta = \ker \beta$ is involutive iff $\overline{\nabla} \beta = 0$: $\nabla \beta$ is involutive and $\nabla + \overline{\nabla} = C T^1 \mathcal{T}.$
Line bundles over complex manifolds

Suppose now that $\mathcal{B}$ is a compact complex manifold; $\pi^1: E \rightarrow \mathcal{B}$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$$\mathcal{V} = \{ v \in \mathbb{C}TSE : \pi_* v \in T^{0,1}\mathcal{B} \}.$$ 

This is an “elliptic structure” on $SE$ such that $\mathcal{V} \cap \mathcal{V} = \text{span} \, \mathcal{T}$.

Let $\nabla : C^\infty(\mathcal{B}; E) \rightarrow C^\infty(\mathcal{B}; E \otimes \mathbb{C}T^*\mathcal{B})$ be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

Let $$\mathcal{H} = \{ v \in TSE \theta = \frac{1}{2i} (\bar{\zeta} d \zeta - \zeta d \bar{\zeta}) - i \pi^* \omega, v \} = 0$$

$\theta = 1$-form vanishing on $\mathcal{H}$ such that $\langle \theta, \mathcal{T} \rangle = 1$.

Let $\beta = -i \theta|_{\mathcal{V}}$. Then $\mathcal{K}_\beta = \ker \beta$ is involutive iff $\overline{\mathcal{D}} \beta = 0$:

$$2(\overline{\mathcal{D}} \beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle$$ 

$X, Y \in \mathcal{K}_\beta$
Line bundles over complex manifolds

Suppose now that \( B \) is a compact complex manifold; \( \pi^1 : E \to B \) is still only a complex line bundle. Fix a Hermitian connection \( h \). Let

\[
\nabla = \{ \nu \in \mathbb{C} TSE : \pi_* \nu \in T^{0,1}B \}.
\]

This is an “elliptic structure” on \( SE \) such that

\[
\nabla \cap \nabla = \text{span} \, \mathcal{T}.
\]

Let

\[
\nabla : C^\infty(B; E) \to C^\infty(B; E \otimes \mathbb{C} T^*B)
\]

be a Hermitian connection. Its horizontal bundle \( \mathcal{H} \) is tangent to \( SE \).

Let

\[
\mathcal{H} = \{ \nu \in \mathcal{T}SE \theta = \langle \frac{1}{2i}(\zeta d\zeta - d\zeta) - i\pi^* \omega, \nu \rangle = 0 \}
\]

\[
\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.
\]

Let \( \beta = -i\theta|_\nabla \). Then \( \overline{K}_\beta = \ker \beta \) is involutive iff \( \overline{D} \beta = 0 \):

\[
2(\overline{D} \beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -\langle \beta, [X, Y] \rangle
\]

\( X, Y \in \overline{K}_\beta \)
Line bundles over complex manifolds

Suppose now that \( B \) is a compact complex manifold; \( \pi^1 : E \to B \) is still only a complex line bundle. Fix a Hermitian connection \( h \). Let

\[
\nabla = \{ v \in \mathbb{C} TSE : \pi_* v \in T^{0,1} B \}.
\]

This is an “elliptic structure” on \( SE \) such that \( V \cap \nabla = \text{span} \mathcal{T} \).

Let

\[
\nabla : C^\infty(\mathcal{B}; E) \to C^\infty(\mathcal{B}; E \otimes \mathbb{C} T^* \mathcal{B})
\]

be a Hermitian connection. Its horizontal bundle \( \mathcal{H} \) is tangent to \( SE \).

Let

\[
\mathcal{H} = \{ v \in TSE \theta = \langle \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^* \omega, v \rangle = 0 \}
\]

\( \theta = 1\)-form vanishing on \( \mathcal{H} \) such that \( \langle \theta, \mathcal{T} \rangle = 1 \).

Let \( \beta = -i\theta|_V \). Then \( \mathcal{K}_\beta = \ker \beta \) is involutive iff \( \overline{\partial} \beta = 0 \):

\[
2(\overline{\partial} \beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -\langle \beta, [X, Y] \rangle
\]

\[
= -i(X\langle \theta, Y \rangle - Y\langle \theta, X \rangle - \langle \theta, [X, Y] \rangle)
\]

\( X, Y \in \mathcal{K}_\beta \).
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold; $\pi^1: E \rightarrow B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let
\[
\overline{V} = \{ \nu \in \mathbb{C} T^*B : \pi_* \nu \in T^0,1B \}.
\]
This is an “elliptic structure” on $SE$ such that $V \cap \overline{V} = \text{span } T$.

Let $\nabla: C^\infty(B; E) \rightarrow C^\infty(B; E \otimes \mathbb{C} T^*B)$ be a Hermitian connection. Its horizontal bundle $H$ is tangent to $SE$.

Let $\mathcal{H} = \{ \nu \in TS\theta = \langle \frac{1}{2i} (\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, \nu \rangle = 0 \}$ \quad $\text{local frame over } V$, $|\eta| = 1$, $\nabla \eta = \eta \otimes \omega$, $\zeta \eta$ arbitrary element of $E|_V$.

\[\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, T \rangle = 1.\]

Let $\beta = -i\theta|_\mathcal{V}$. Then $\mathcal{K}_\beta = \text{ker } \beta$ is involutive iff $\overline{D}\beta = 0$:
\[
2(\overline{D}\beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -i\text{d}\theta(X, Y)
\]
\[
= -i(X\langle \theta, Y \rangle - Y\langle \theta, X \rangle - \langle \theta, [X, Y] \rangle)
\]
$x, y \in \mathcal{K}_\beta$
Line bundles over complex manifolds

Suppose now that $\mathcal{B}$ is a compact complex manifold; $\pi^1 : E \to \mathcal{B}$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$$\overline{\mathcal{V}} = \{ \nu \in \mathbb{C}TSE : \pi_* \nu \in T^{0,1}\mathcal{B} \}.$$ 

This is an “elliptic structure” on $SE$ such that

$$\mathcal{V} \cap \overline{\mathcal{V}} = \text{span} \ T.$$ 

Let

$$\mathcal{H} = \{ \nu \in TS : \theta = \frac{1}{2i}(\xi d\xi - \xi d\overline{\xi}) - i\pi^* \omega, \nu \} = 0$$

$$\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, T \rangle = 1.$$ 

Let $\beta = -i\theta|_{\overline{\mathcal{V}}}$. Then $\mathcal{K}_\beta = \ker \beta$ is involutive iff $\overline{\mathcal{D}} \beta = 0$:

$$2(\overline{\mathcal{D}} \beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -i\theta(X, Y)$$

$$= -(\pi^* d\omega)(X, Y) \langle \theta, X \rangle - \langle \theta, [X, Y] \rangle$$

$X, Y \in \mathcal{K}_\beta$
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold; $\pi^1 : E \to B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$$\nabla = \{ \nu \in \mathbb{C}TSB : \pi_*\nu \in T^{0,1}B \}.$$ 

This is an “elliptic structure” on $SE$ such that $\nabla \cap \overline{\nabla} = \text{span } \mathcal{T}$. 

Let

$$\nabla : C^\infty(B ; E) \to C^\infty(B ; E \otimes \mathbb{C}T^*B)$$

be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$. 

Let

$$\mathcal{H} = \{ \nu \in TS \theta = \langle \frac{1}{2i} (\zeta d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, \nu \rangle = 0 \}$$ 

$\theta = 1$-form vanishing on $\mathcal{H}$ such that $\langle \theta, \mathcal{T} \rangle = 1$. 

Let $\beta = -i\theta|_\mathcal{V}$. Then $\overline{K}_\beta = \ker \beta$ is involutive iff $\overline{\nabla} \beta = 0$:

$$2(\overline{\nabla}\beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -id\theta(X, Y)$$ 

$$= -(\pi^*d\omega)(X, Y) \langle \zeta \rangle (\pi^*\partial\omega_0^1)(X, Y).$$

$X, Y \in \overline{K}_\beta$
Line bundles over complex manifolds

Suppose now that \( B \) is a compact complex manifold; \( \pi^1: E \to B \) is still only a complex line bundle. Fix a Hermitian connection \( h \). Let

\[
\overline{\mathcal{V}} = \{ \nu \in \mathbb{C}TSE : \pi_*\nu \in T^{0,1}B \}.
\]

This is an “elliptic structure” on \( SE \) such that

\[
\mathcal{V} \cap \overline{\mathcal{V}} = \text{span} \mathcal{T}.
\]

Let

\[
\nabla : C^\infty(B; E) \to C^\infty(B; E \otimes \mathbb{C}T^*B)
\]

be a Hermitian connection. Its horizontal bundle \( \mathcal{H} \) is tangent to \( SE \).

Let

\[
\mathcal{H} = \{ \nu \in TS\theta = \frac{1}{2i}(\bar{\zeta}d\zeta - \zeta d\bar{\zeta}) - i\pi^*\omega, \nu \} = 0
\]

\( \theta = 1\)-form vanishing on \( \mathcal{H} \) such that \( \langle \theta, \mathcal{T} \rangle = 1 \).

Let \( \beta = -i\theta|_{\overline{\mathcal{V}}} \). Then \( \overline{\mathcal{K}}_\beta = \ker \beta \) is involutive iff \( \overline{\mathcal{D}}\beta = 0 \):

\[
2(\overline{\mathcal{D}}\beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -id\theta(X, Y)
\]

\[
= -(\pi^*d\omega)(X, Y) - (X)(\pi^*\overline{\partial}\omega^{0,1})(X, Y).
\]

\( X, Y \in \overline{\mathcal{K}}_\beta \)

\( \beta \) is \( \overline{\mathcal{D}} \)-closed iff \( \Omega^{0,2} = \overline{\partial}\omega^{0,1} = 0 \)
Embedding theorems

Suppose now that $B$ is a compact complex manifold; $\pi^1: E \to B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$$\nabla = \{ \nabla \in \mathbb{C} TSE : \pi_* \nabla \in T^{0,1}B \}.$$ 

This is an “elliptic structure” on $SE$ such that $\nabla \cap \nabla = \text{span} \mathcal{T}$. Let

$$\cdots \to C^\infty(SE; \wedge^q \nabla^*) \xrightarrow{\overline{D}} C^\infty(SE; \wedge^{q+1} \nabla^*) \to \cdots$$

$$\nabla : C^\infty(B; E) \to C^\infty(B; E \otimes \mathbb{C} T^*B)$$

be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

Let

$$\mathcal{H} = \{ \nabla \in TSE \theta = \frac{1}{2i}(\zeta d\zeta - \zeta d\overline{\zeta}) - i \pi^* \omega, \nabla \} = 0 \}

\theta = 1\text{-form vanishing on } \mathcal{H} \text{ such that } \langle \theta, \mathcal{T} \rangle = 1.$$

Let $\beta = - i \theta|_\nabla$. Then $\overline{K}_\beta = \ker \beta$ is involutive iff $\overline{D} \beta = 0$:

$$2(\overline{D} \beta)(X, Y) = X \langle \beta, Y \rangle - Y \langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -i d\theta(X, Y)$$

$$= -(\pi^* d\omega)(X, Y) \langle \overline{\zeta} \rangle (\pi^* \overline{d}\omega^{0,1})(X, Y).$$

$X, Y \in \overline{K}_\beta$

$\beta$ is $\overline{D}$-closed iff $\Omega^{0,2} = \overline{\partial} \omega^{0,1} = 0$ (iff $\nabla$ is a holomorphic connection).
Line bundles over complex manifolds

Suppose now that $\mathcal{B}$ is a compact complex manifold; $\pi^1 : E \to \mathcal{B}$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let

$$\nabla = \{ \nu \in \mathbb{C}TSE : \pi_* \nu \in T^{0,1}\mathcal{B} \}.$$

This is an “elliptic structure” on $SE$ such that $\nu \cap \nabla = \text{span} \, \mathcal{T}$.

Let $\nabla : C^\infty(\mathcal{B}; E) \to C^\infty(\mathcal{B}; E \otimes \mathbb{C}T^*\mathcal{B})$

be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

Let

$$\mathcal{H} = \{ \nu \in TSE : \theta = \frac{1}{2i}(\zeta d\zeta - \zeta d\bar{\zeta}) - i\pi^* \omega, \nu \} = \text{Levi}_\theta(X, Y) = -id\theta(X, Y)$$

over $\mathcal{V}$, $\mathcal{H} = (\pi^* d\omega)(X, Y)$, $\zeta = \pi^* (\omega |_{\mathcal{V}})$

be a $1$-form vanishing on $\mathcal{H}$ such that $\langle \theta, T \rangle = 1$.

Let $\beta = -i\theta \big|_\nabla$. Then $\bar{\mathcal{K}}_\beta = \ker \beta$ is involutive iff $\overline{\partial} \beta = 0$:

$$2(\overline{\partial} \beta)(X, Y) = X \langle \beta, Y \rangle - Y \langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -i d\theta(X, Y)$$

$$= -((\pi^* d\omega)(X, Y)) \langle \zeta, (\pi^* \overline{\partial} \omega^{0,1})(X), Y \rangle.$$ 

$\beta$ is $\overline{\partial}$-closed iff $\Omega^{0,2} = \overline{\partial} \omega^{0,1} = 0$ (iff $\nabla$ is a holomorphic connection).
Line bundles over complex manifolds

Suppose now that $B$ is a compact complex manifold; $\pi^1 : E \to B$ is still only a complex line bundle. Fix a Hermitian connection $h$. Let $\mathcal{V} = \{ v \in \mathbb{C}TSE : \pi_* v \in T^{0,1}B \}$. This is an “elliptic structure” on $SE$ such that $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span } \mathcal{T}$.

Let $\nabla : C^\infty(B; E) \to C^\infty(B; E \otimes \mathbb{C}T^*B)$ be a Hermitian connection. Its horizontal bundle $\mathcal{H}$ is tangent to $SE$.

Let $\mathcal{H} = \{ v \in TSE \mid \theta = \frac{1}{2i} (\bar{\zeta} d\zeta - \zeta d\bar{\zeta}) - i \pi^* \omega, v \} = \text{Levi}_\theta(X, Y) = -id\theta(X, \bar{Y})$ over $\mathcal{V}$, \( \nabla \theta = (\pi^* d\omega)(X, \bar{Y}), \) \( \zeta = \pi^* \Omega(X, \bar{Y}) \) of the $S^1$ action on $SE$.

Let $\beta = -i \theta|_{\mathcal{V}}$. Then $\overline{\mathcal{K}}_\beta = \ker \beta$ is involutive iff $\overline{\nabla} \beta = 0$:

\[
2(\overline{\nabla} \beta)(X, Y) = X\langle \beta, Y \rangle - Y\langle \beta, X \rangle - \langle \beta, [X, Y] \rangle = -id\theta(X, Y) \\
= -(\pi^* d\omega)(X, \bar{Y}) \langle \zeta \rangle = \zeta (\pi^* \partial_{\omega}^0, 1)(X, Y). \\
X, Y \in \overline{\mathcal{K}}_\beta
\]

$\beta$ is $\overline{\nabla}$-closed iff $\Omega^{0,2} = \overline{\partial} \omega^{0,1} = 0$ (iff $\nabla$ is a holomorphic connection).
Suppose $E \to B$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$
Suppose $E \rightarrow B$ is positive. Define:

\[ E^m = E \otimes \cdots \otimes E \text{ (m times)}; \]

\[ \mathfrak{hol}(B; E^m) = \{ \eta \in C^\infty(B; E^m) : \bar{\partial} \eta = 0 \}, \quad N + 1 = \dim \mathfrak{hol}(B; E^m); \]
Suppose $E \to B$ is positive. Define:

\[ E^m = E \otimes \cdots \otimes E \quad (m \text{ times}); \]

\[ \mathcal{H}o\ell(B; E^m) = \{ \eta \in C^\infty(B; E^m) : \bar{\partial} \eta = 0 \}, \quad N + 1 = \dim \mathcal{H}o\ell(B; E^m); \]

\[ \text{ev}_x : \mathcal{H}o\ell(B; E^m) \to E^m_x, \quad \text{ev}_x(\eta) = \eta(x); \]
Suppose $E \to \mathcal{B}$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$

$$\mathcal{H}ol(\mathcal{B}; E^m) = \{ \eta \in C^\infty(\mathcal{B}; E^m) : \bar{\partial} \eta = 0 \}, \quad N + 1 = \dim \mathcal{H}ol(\mathcal{B}; E^m);$$

$$\text{ev}_x : \mathcal{H}ol(\mathcal{B}; E^m) \to E^m_x, \quad \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : \mathcal{B} \to \{ \text{subspaces of } \mathcal{H}ol(\mathcal{B}; E^m) \}, \quad \Phi(x) = \ker \text{ev}_x$$
Suppose $E \to B$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$

$\mathcal{H}ol(B; E^m) = \{\eta \in C^\infty(B; E^m) : \bar{\partial}\eta = 0\}$, $N + 1 = \dim \mathcal{H}ol(B; E^m)$;

$ev_x : \mathcal{H}ol(B; E^m) \to E^m_x$, $ev_x(\eta) = \eta(x)$;

$\Phi : B \to \{\text{subspaces of } \mathcal{H}ol(B; E^m)\}$, $\Phi(x) = \ker ev_x$

(Kodaira) If $m$ is large enough, then for all $x \in B$ there is

$\eta \in \mathcal{H}ol(B; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N$. Further,

the map $x \mapsto \Phi(x) \in \text{Gr}_N(\mathcal{H}ol(B; E^m)) \approx \mathbb{C}P^N$ is an embedding.
Suppose $E \to B$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \ times);$$

$$\mathfrak{Hol}(B; E^m) = \{ \eta \in C^\infty(B; E^m) : \overline{\partial} \eta = 0 \}, \ N + 1 = \dim \mathfrak{Hol}(B; E^m);$$

$$\text{ev}_x : \mathfrak{Hol}(B; E^m) \to E^m_x, \ \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : B \to \{ \text{subspaces of } \mathfrak{Hol}(B; E^m) \}, \ \Phi(x) = \ker \text{ev}_x$$

(Kodaira) If $m$ is large enough, then for all $x \in B$ there is $\eta \in \mathfrak{Hol}(B; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_N(\mathfrak{Hol}(B; E^m)) \approx \mathbb{CP}^N$ is an embedding.
Suppose $E \to B$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \text{ (m times)};$$

$$\mathcal{H}ol(B; E^m) = \{ \eta \in C^\infty(B; E^m) : \bar{\partial} \eta = 0 \}, \ N + 1 = \dim \mathcal{H}ol(B; E^m);$$

$$\text{ev}_x : \mathcal{H}ol(B; E^m) \to E^m_x, \ \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : B \to \{ \text{subspaces of } \mathcal{H}ol(B; E^m) \}, \ \Phi(x) = \ker \text{ev}_x$$

(Kodaira) If $m$ is large enough, then for all $x \in B$ there is $\eta \in \mathcal{H}ol(B; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_N(\mathcal{H}ol(B; E^m)) \approx \mathbb{C}P^N$ is an embedding.

If $p \in SE^*$ then $\varphi_m(p) = p \otimes \cdots \otimes p \in SE^{*m}$. 

because $E^m_x$ is one-dimensional
Suppose $E \to B$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$

$$\mathcal{H}ol(B; E^m) = \{ \eta \in C^\infty(B; E^m) : \overline{\partial} \eta = 0 \}, \ N + 1 = \dim \mathcal{H}ol(B; E^m);$$

$$\text{ev}_x : \mathcal{H}ol(B; E^m) \to E^m_x, \ \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : B \to \{ \text{subspaces of } \mathcal{H}ol(B; E^m) \}, \ \Phi(x) = \ker \text{ev}_x$$

(Kodaira) If $m$ is large enough, then for all $x \in B$ there is $\eta \in \mathcal{H}ol(B; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_N(\mathcal{H}ol(B; E^m)) \approx \mathbb{CP}^N$ is an embedding.

If $p \in SE^*$ then $\varphi_m(p) = p \otimes \cdots \otimes p \in SE^{*m}$. It makes sense to compose:

$$\eta \in \mathcal{H}ol(B; E^m), \ f_\eta(p) = \langle \eta, \varphi_m(p) \rangle$$
Suppose $E \to B$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \text{ (m times)};$$

$$\mathcal{H}\mathcal{O}\mathcal{L}(B; E^m) = \{ \eta \in C^\infty(B; E^m) : \overline{\partial} \eta = 0 \}, \quad N + 1 = \dim \mathcal{H}\mathcal{O}\mathcal{L}(B; E^m);$$

$$\text{ev}_x : \mathcal{H}\mathcal{O}\mathcal{L}(B; E^m) \to E^m_x, \quad \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : B \to \{ \text{subspaces of } \mathcal{H}\mathcal{O}\mathcal{L}(B; E^m) \}, \quad \Phi(x) = \ker \text{ev}_x.$$

(Kodaira) If $m$ is large enough, then for all $x \in B$ there is $\eta \in \mathcal{H}\mathcal{O}\mathcal{L}(B; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_N(\mathcal{H}\mathcal{O}\mathcal{L}(B; E^m)) \approx \mathbb{CP}^N$ is an embedding.

If $p \in SE^*$ then $\varphi_m(p) = \underbrace{p \otimes \cdots \otimes p}_{m \text{ times}} \in SE^{*m}$. It makes sense to compose:

$$\eta \in \mathcal{H}\mathcal{O}\mathcal{L}(B; E^m), \quad f_\eta(p) = \langle \eta, \varphi_m(p) \rangle$$

In Kodaira’s theorem, if $\eta_1, \ldots, \eta_N$ is a basis of $\mathcal{H}\mathcal{O}\mathcal{L}(B; E^m)$, then

$$SE^* \ni p \mapsto F(p) = (f_{\eta_1}(p), \ldots, f_{\eta_N}(p)) \in \mathcal{H}\mathcal{O}\mathcal{L}(B; E^m) \setminus 0$$

is m-to-1 and such that $F_* \mathcal{T} = i \sum_j m(z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j}).$
Suppose $E \to B$ is positive. Define:

$$E^m = E \otimes \cdots \otimes E \ (m \text{ times});$$

$$\mathcal{H}ol(B; E^m) = \{ \eta \in C^\infty(B; E^m) : \overline{\partial} \eta = 0 \}, \ N + 1 = \dim \mathcal{H}ol(B; E^m);$$

$$\text{ev}_x : \mathcal{H}ol(B; E^m) \to E^m_x, \ \text{ev}_x(\eta) = \eta(x);$$

$$\Phi : B \to \{ \text{subspaces of} \ \mathcal{H}ol(B; E^m) \}, \ \Phi(x) = \ker \text{ev}_x.$$

**Kodaira** If $m$ is large enough, then for all $x \in B$ there is $\eta \in \mathcal{H}ol(B; E^m)$ s.t. $\eta(p) \neq 0$. So $\dim \Phi(x) = N$. Further, the map $x \mapsto \Phi(x) \in \text{Gr}_N(\mathcal{H}ol(B; E^m)) \approx \mathbb{CP}^N$ is an embedding.

If $p \in SE^*$ then $\varphi_m(p) = \underbrace{p \otimes \cdots \otimes p}_{m \text{ times}} \in SE^*^m$. It makes sense to compose:

$$\eta \in \mathcal{H}ol(B; E^m), \ f_\eta(p) = \langle \eta, \varphi_m(p) \rangle \quad \Rightarrow \quad \varphi_m(e^{it}p) = e^{imt}\varphi_m(p) \quad \Rightarrow \quad f_\eta(e^{it}p) = e^{it}f_\eta(p).$$

In Kodaira's theorem, if $\eta_1, \ldots, \eta_N$ is a basis of $\mathcal{H}ol(B; E^m)$, then

$$\text{SE}^* \ni p \mapsto F(p) = (f_{\eta_1}(p), \ldots, f_{\eta_N}(p)) \in \mathcal{H}ol(B; E^m) \setminus 0$$

is $m$-to-1 and such that $F_\ast \mathcal{T} = i \sum_j m(z^j \partial_{\bar{z}^j} - \bar{z}^j \partial_{z^j})$. 
Generalization

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$ such that:

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ is an elliptic structure and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_{\mathbb{C}} \mathcal{T}$;
- there is $\beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$, $\overline{\mathcal{D}}\beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$. 

(Kernel)
Generalization

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \bar{\mathcal{V}})$ such that:

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\bar{\mathcal{V}} \subset \mathbb{C} \mathcal{T} \mathcal{N}$ is an elliptic structure and $\mathcal{V} \cap \bar{\mathcal{V}} = \text{span}_{\mathbb{C}} \mathcal{T}$;
- there is $\beta \in C^\infty(\mathcal{N}; \bar{\mathcal{V}}^*)$, $\bar{\partial} \beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$.  

$\overline{\mathcal{K}}_\beta = \ker \beta \subset \bar{\mathcal{V}}$ is a CR structure.
Generalization

Let \( \mathcal{F}_{\text{ell}} \) be the set of triples \((N, \mathcal{T}, \overline{V})\) such that:

1. \((N, \mathcal{T}) \in \mathcal{F} ;
2. \overline{V} \subset \mathbb{C}T_{\mathcal{N}} \) is an elliptic structure and \( V \cap \overline{V} = \text{span}_{\mathbb{C}} \mathcal{T} ;
3. \) there is \( \beta \in C^\infty(N; \overline{V}^*) \), \( \overline{D}\beta = 0 \), \( \langle \beta, \mathcal{T} \rangle = -i \).

If \( \beta, \beta' \in C^\infty(N; \overline{V}^*) \) are two sections as described, say

\[ \beta \sim \beta' \text{ iff there is } u \text{ real-valued such that } \beta' - \beta = \overline{D}u. \]

(\( \mathcal{K}_\beta = \ker \beta \subset \overline{V} \) is a CR structure)
**Generalization**

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$ such that:

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;

- $\overline{\mathcal{V}} \subset \mathbb{C} \mathcal{T} \mathcal{N}$ is an elliptic structure and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$;

- there is $\beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$, $\overline{\mathbb{D}} \beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$. \[\mathcal{K}_\beta = \ker \beta \subset \overline{\mathcal{V}}\] is a CR structure

If $\beta, \beta' \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$ are two sections as described, say $\overline{\mathbb{D}} u = du|_{\overline{\mathcal{V}}}$

$$\beta \sim \beta' \text{ iff there is } u \text{ real-valued such that } \beta' - \beta = \overline{\mathbb{D}} u.$$
Generalization

Let \( \mathcal{F}_{\text{ell}} \) be the set of triples \( (\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \) such that:

- \( (\mathcal{N}, \mathcal{T}) \in \mathcal{F} \);
- \( \overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N} \) is an elliptic structure and \( \mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_{\mathbb{C}} \mathcal{T} \);
- there is \( \beta \in \mathcal{C}^\infty(\mathcal{N}; \overline{\mathcal{V}}^*) \), \( \overline{D}\beta = 0 \), \( \langle \beta, \mathcal{T} \rangle = -i \).

If \( \beta, \beta' \in \mathcal{C}^\infty(\mathcal{N}; \overline{\mathcal{V}}^*) \) are two sections as described, say \( \beta \sim \beta' \) iff there is \( u \) real-valued such that \( \beta' - \beta = \overline{D}u \).

Let \( \beta \) be the class of \( \beta \).

Let \( (\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}} \) with \( \dim \mathcal{N} \geq 5 \). Fix \( \beta \). The following are equivalent:

- \( \exists \beta \in \beta \) such that \( \overline{\mathcal{K}}_\beta \) is definite.
- \( \exists \beta \in \beta \) and an equivariant CR embedding \( F : \mathcal{N}, \overline{\mathcal{K}}_\beta \to S^{2N+1} \) for some \( N \), with \( F^*\mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j}) \) and all \( \tau_j \) of the same sign.
Outline of proof

---

Embedding theorems

---

(Temple University)
Outline of proof

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, \mathcal{T}, \mathcal{V}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$. 
Outline of proof

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, T, \overline{V}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

$\mathcal{F}_{\text{ell}} =$ set of triples $(\mathcal{N}, T, \overline{V})$ s.t.:
- $(\mathcal{N}, T) \in \mathcal{F}$;
- $\overline{V} \subset \mathbb{C} T \mathcal{N}$ is an elliptic structure,
  $\mathcal{V} \cap \mathcal{V} = \text{span}_\mathbb{C} T$ and $\mathcal{V} \cap \mathcal{V} = \text{span}_\mathbb{C} T$;
- $\exists \beta \in C^\infty (\mathcal{N}; \overline{V}^*)$, $\overline{\partial} \beta = 0$, $\langle \beta, T \rangle = -i$.

$\overline{\mathcal{K}}_\beta = \ker \beta \subset \overline{V}$ is a CR structure

$\beta' \sim \beta$ iff there is $u$ real-valued such that $\beta' - \beta = \overline{\partial} u$.

---

(Temple University)  
Embedding theorems  
Beirut, November 2011  
15 / 18
Outline of proof

Suppose $\beta \in \beta$, let $\overline{\mathcal{K}}_\beta = \ker \beta$

From Cartan’s formula

$$\mathcal{L}_\mathcal{T} \beta = \overline{\mathcal{D}}(i_\mathcal{T} \beta) + i_\mathcal{T} \overline{\mathcal{D}} \beta = 0$$

deduce $\alpha_t : \overline{\mathcal{K}}_\beta \to \overline{\mathcal{K}}_\beta$.

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

$\mathcal{F}_{\text{ell}} = \text{set of triples } (\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \text{ s.t.}:
- (\mathcal{N}, \mathcal{T}) \in \mathcal{F};
- \overline{\mathcal{V}} \subset \mathbb{C} \mathcal{T} \mathcal{N}$ is an elliptic structure,
  $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$ and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$;
- $\exists \beta \in \mathcal{C}_\infty(\mathcal{N}; \overline{\mathcal{V}}^*), \overline{\mathcal{D}} \beta = 0, \langle \beta, \mathcal{T} \rangle = -i$.

$\overline{\mathcal{K}}_\beta = \ker \beta \subset \overline{\mathcal{V}}$ is a CR structure

$\beta' \sim \beta$ iff there is $u$ real-valued such that $\beta' - \beta = \overline{\mathcal{D}} u$. 
Outline of proof

Suppose $\beta \in \beta$, let $\overline{K}_\beta = \ker \beta$

From Cartan’s formula
\[ \mathcal{L}_\mathcal{T} \beta = \overline{\mathcal{D}}(i_\mathcal{T} \beta) + i_\mathcal{T} \overline{\mathcal{D}} \beta = 0 \]
deduce $\alpha_t : \overline{K}_\beta \to \overline{K}_\beta$.

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

$\mathcal{F}_{\text{ell}} = \text{set of triples } (\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}}) \text{ s.t. :}$
- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\overline{\mathcal{V}} \subset \mathbb{C} \mathcal{T}\mathcal{N}$ is an elliptic structure,
  $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$ and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$;
- $\exists \beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*), \overline{\mathcal{D}} \beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$.

$\overline{K}_\beta = \ker \beta \subset \overline{\mathcal{V}}$ is a CR structure

$\beta' \sim \beta$ iff there is $u$ real-valued such that $\beta' - \beta = \overline{\mathcal{D}} u$. 
Outline of proof

Suppose $\beta \in \mathcal{B}$, let $\overline{\mathcal{K}}_\beta = \ker \beta$

From Cartan’s formula

$$\mathcal{L}_T \beta = \overline{\mathcal{D}}(i_T \beta) + i_T \mathcal{D} \beta = 0$$

deduce $\alpha_t : \overline{\mathcal{K}}_\beta \to \overline{\mathcal{K}}_\beta$.

Let $\mathcal{H}_\beta$ be the subbundle of $TN$ s.t.

$$\mathbb{C}\mathcal{H}_\beta = \mathcal{K}_\beta \oplus \overline{\mathcal{K}}_\beta$$

Let $J : \mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{\mathcal{K}}_\beta = \{v + iJv : v \in \mathcal{H}_\beta\}$

Pick a $T$-invariant metric $g$. Then

$$\mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$$

is a $T$-invariant Hermitian metric on $\mathcal{H}_\beta$.

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, T, \overline{\mathcal{V}}) \in \mathcal{F}_{\text{ell}}$

with definite $\beta$.

$\mathcal{F}_{\text{ell}} = \text{set of triples } (\mathcal{N}, T, \overline{\mathcal{V}}) \text{ s.t.}:

- (\mathcal{N}, T) \in \mathcal{F};
- \overline{\mathcal{V}} \subset \mathbb{C}TN \text{ is an elliptic structure,}
  \mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} T \text{ and } \mathcal{V} \cap \mathcal{V} = \text{span}_\mathbb{C} T;
- \exists \beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*), \overline{\mathcal{D}} \beta = 0, \langle \beta, T \rangle = -i.$

$\overline{\mathcal{K}}_\beta = \ker \beta \subset \overline{\mathcal{V}}$ is a CR structure

$\beta' \sim \beta$ iff there is $u$ real-valued such that $\beta' - \beta = \overline{\mathcal{D}} u$. 

Redefine $g$ to be this on $\mathcal{H}_\beta$, and such that $T \perp \mathcal{H}_\beta$, $|T| = 1$

$\mathcal{H}_\beta + \text{span}_\mathbb{C} T = TN$. Now $\langle \theta, v \rangle = g(T, v)$.

With these data (Hermitian metric, Riemannian measure), let $\square_b = \text{Laplacian of } \partial_b$ complex in any degree.

$\square_b = \mathcal{L}_T \square_b = \square_b \mathcal{L}_T$. 

(Temple University)
Outline of proof

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, \mathcal{T}, \nabla) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$

$\mathcal{F}_{\text{ell}} = \text{set of triples } (\mathcal{N}, \mathcal{T}, \nabla) \text{ s.t.:}$

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\nabla \subset \mathbb{C} \mathcal{T} \mathcal{N}$ is an elliptic structure, $\mathcal{V} \cap \nabla = \text{span}_{\mathbb{C}} \mathcal{T}$ and $\mathcal{V} \cap \nabla = \text{span}_{\mathbb{C}} \mathcal{T}$;
- $\exists \beta \in \mathcal{C}^\infty (\mathcal{N}; \nabla^*)$, $\nabla \beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$.

$\overline{\mathcal{K}}_\beta = \ker \beta \subset \nabla$ is a CR structure

$\beta' \sim \beta$ iff there is $u$ real-valued such that $\beta' - \beta = \nabla u$.

Suppose $\beta \in \beta$, let $\overline{\mathcal{K}}_\beta = \ker \beta$

From Cartan’s formula

$\mathcal{L}_{\mathcal{T}} \beta = \nabla (i_{\mathcal{T}} \beta) + i_{\mathcal{T}} \nabla \beta = 0$

deduce $\alpha_t : \overline{\mathcal{K}}_\beta \to \overline{\mathcal{K}}_\beta$.

Let $\mathcal{H}_\beta$ be the subbundle of $\mathcal{T} \mathcal{N}$ s.t.

$\mathbb{C} \mathcal{H}_\beta = \mathcal{K}_\beta \oplus \overline{\mathcal{K}}_\beta \quad \mathcal{H}_\beta + \text{span} \mathcal{T} = \mathcal{T} \mathcal{N}$

Let $\mathcal{J} : \mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{\mathcal{K}}_\beta = \{ v + iJv : v \in \mathcal{H}_\beta \}$

Pick a $\mathcal{T}$-invariant metric $g$. Then

$\mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$

is a $\mathcal{T}$-invariant Hermitian metric on $\mathcal{H}_\beta$. Redefine $g$ to be this on $\mathcal{H}_\beta$, and such that $\mathcal{T} \perp \mathcal{H}_\beta$, $|\mathcal{T}| = 1$
Outline of proof

Suppose $\beta \in \beta$, let $\overline{K}_\beta = \ker \beta$

From Cartan’s formula

$$\mathcal{L}_T \beta = \overline{D}(i_T \beta) + i_T \overline{D} \beta = 0$$

deduce $a_t : \overline{K}_\beta \to \overline{K}_\beta$.

Let $\mathcal{H}_\beta$ be the subbundle of $T\mathcal{N}$ s.t.

$$\mathbb{C}\mathcal{H}_\beta = K_\beta \oplus \overline{K}_\beta \quad \mathcal{H}_\beta + \text{span} \mathcal{T} = T\mathcal{N}$$

Let $J : \mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{K}_\beta = \{ v + iJv : v \in \mathcal{H}_\beta \}$

Pick a $\mathcal{T}$-invariant metric $g$. Then

$$\mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$$

is a $\mathcal{T}$-invariant Hermitian metric on $\mathcal{H}_\beta$. Redefine $g$ to be this on $\mathcal{H}_\beta$, and such that $\mathcal{T} \perp \mathcal{H}_\beta$, $|\mathcal{T}| = 1$. Now $\langle \theta, v \rangle = g(\mathcal{T}, v)$. 

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, \mathcal{T}, \overline{V}) \in \mathcal{F}_\text{ell}$ with definite $\beta$.

$\mathcal{F}_\text{ell} = \text{set of triples } (\mathcal{N}, \mathcal{T}, \overline{V}) \text{ s.t.:}$

- $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$;
- $\overline{V} \subset \mathcal{C}T\mathcal{N}$ is an elliptic structure,
  $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$ and $\mathcal{V} \cap \overline{\mathcal{V}} = \text{span}_\mathbb{C} \mathcal{T}$;
- $\exists \beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$, $\overline{D} \beta = 0$, $\langle \beta, \mathcal{T} \rangle = -i$.

$\overline{K}_\beta = \ker \beta \subset \overline{V}$ is a CR structure

$\beta' \sim \beta$ iff there is $u$ real-valued such that $\beta' - \beta = \overline{D}u$. 

\[\text{(Temple University)}\]
Outline of proof

Suppose $\beta \in \beta$, let $\overline{K}_\beta = \ker \beta$

From Cartan’s formula

$$\mathcal{L}_\mathcal{T} \beta = \overline{\mathcal{D}}(i_T \beta) + i_T \overline{\mathcal{D}} \beta = 0$$

deduce $\alpha_t : \overline{K}_\beta \to \overline{K}_\beta$.

Let $\mathcal{H}_\beta$ be the subbundle of $TN$ s.t.

$$\mathbb{C} \mathcal{H}_\beta = \mathcal{K}_\beta \oplus \overline{\mathcal{K}}_\beta \quad \mathcal{H}_\beta + \text{span } \mathcal{T} = TN$$

Let $J : \mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{K}_\beta = \{ v + iJv : v \in \mathcal{H}_\beta \}$

Pick a $\mathcal{T}$-invariant metric $g$. Then

$$\mathcal{H}_\beta \times \mathcal{H}_\beta \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$$

is a $\mathcal{T}$-invariant Hermitian metric on $\mathcal{H}_\beta$. Redefine $g$ to be this on $\mathcal{H}_\beta$, and such that $\mathcal{T} \perp \mathcal{H}_\beta$, $|\mathcal{T}| = 1$. Now $\langle \theta, v \rangle = g(\mathcal{T}, v)$.

With these data (Hermitian metric, Riemannian measure), let

$$\Box_b = \text{Laplacian of } \overline{\partial}_b \text{ complex in any degree.}$$
Outline of proof

Suppose $\beta \in \beta$, let $\overline{\mathcal{K}}_{\beta} = \ker \beta$

From Cartan’s formula

$$\mathcal{L}_T \beta = \overline{D}(i_T \beta) + i_T \overline{D}\beta = 0$$

deduce $\alpha_t : \overline{\mathcal{K}}_{\beta} \to \overline{\mathcal{K}}_{\beta}$.

Let $\mathcal{H}_{\beta}$ be the subbundle of $T\mathcal{N}$ s.t.

$$\mathbb{C}\mathcal{H}_{\beta} = \mathcal{K}_{\beta} \oplus \overline{\mathcal{K}}_{\beta} \quad \mathcal{H}_{\beta} + \text{span} \, T = T\mathcal{N}$$

Let $J : \mathcal{H} \to \mathcal{H}$ be the complex structure such that $\overline{\mathcal{K}}_{\beta} = \{ v + iJv : v \in \mathcal{H}_{\beta} \} $

Pick a $T$-invariant metric $g$. Then

$$\mathcal{H}_{\beta} \times \mathcal{H}_{\beta} \ni (u, v) \mapsto \frac{1}{2}(g(u, v) + g(Ju, Jv)) \in \mathbb{R}$$

is a $T$-invariant Hermitian metric on $\mathcal{H}_{\beta}$. Redefine $g$ to be this on $\mathcal{H}_{\beta}$, and such that $T \perp \mathcal{H}_{\beta}$, $|T| = 1$. Now $\langle \theta, v \rangle = g(T, v)$.

With these data (Hermitian metric, Riemannian measure), let

$$\Box_b = \text{Laplacian of } \overline{\partial}_b \text{ complex in any degree.}$$

Note $\mathcal{L}_T \Box_b = \Box_b \mathcal{L}_T$. 

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, T, \overline{V}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.

$$\mathcal{F}_{\text{ell}} = \text{set of triples } (\mathcal{N}, T, \overline{V}) \text{ s.t.:}$$

- $(\mathcal{N}, T) \in \mathcal{F}$;
- $\overline{V} \subset \mathbb{C} T\mathcal{N}$ is an elliptic structure,
  $$\mathcal{V} \cap \overline{V} = \text{span}_\mathbb{C} T \text{ and } \mathcal{V} \cap \overline{V} = \text{span}_\mathbb{C} T;$$
- $\exists \beta \in C^\infty(\mathcal{N}; \overline{V}^*)$, $\overline{D}\beta = 0$, $\langle \beta, T \rangle = -i$.

$\mathcal{K}_{\beta} = \ker \beta \subset \overline{V}$ is a CR structure

$\beta' \sim \beta$ iff there is $u$ real-valued such that

$$\beta' - \beta = \overline{D}u.$$
Outline of proof

Let $\mathcal{H}_q^b = \ker \Box_b \subset L^2(\mathcal{N}; \wedge^q \mathcal{K}^*)$.

Let $\mathcal{D} = \{ \phi \in \mathcal{H}_q^b : L_T \phi \in L^2 \}$

$-i L_T : \mathcal{D} \subset \mathcal{H}_q^b \rightarrow \mathcal{H}_q^b$

is a selfadjoint operator with compact parametrix.

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, T, \bar{V}) \in \mathcal{F}_{ell}$ with definite $\beta$. 
Outline of proof

Let $\mathcal{H}^q_{\partial_b} = \ker \Box_b \subset L^2(\mathcal{N}; \wedge^q \mathcal{K}^*)$.

Let $D = \{ \phi \in \mathcal{H}^q_{\partial_b} : \mathcal{L}_T \phi \in L^2 \}$

\begin{equation}
(*) \quad -i\mathcal{L}_T : D \subset \mathcal{H}^q_{\partial_b} \to \mathcal{H}^q_{\partial_b}
\end{equation}

is a selfadjoint operator with compact parametrix. Consequently

\[ \text{spec}^q_0(-i\mathcal{L}_T) = \text{spectrum of } (*) \]

is a discrete subset of $\mathbb{R}$. 

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, T, \overline{V}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$. 

Embedding theorems

Beirut, November 2011 16 / 18
CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, \mathcal{T}, \mathcal{V}) \in \mathcal{F}_{ell}$ with definite $\beta$.

The proof exploits $[\Box_b, \mathcal{L}_T] = 0$ plus the fact that $\Box_b - \mathcal{L}_T^2$ is elliptic (and that $\mathcal{N}$ is compact).

Outline of proof

Let $\mathcal{H}^q_{\partial_b} = \ker \Box_b \subset L^2(\mathcal{N}; \Lambda^q \mathcal{K}^*)$.

Let $\mathcal{D} = \{ \phi \in \mathcal{H}^q_{\partial_b} : \mathcal{L}_T \phi \in L^2 \}$

$$(*) \quad -i\mathcal{L}_T : \mathcal{D} \subset \mathcal{H}^q_{\partial_b} \to \mathcal{H}^q_{\partial_b}$$

is a selfadjoint operator with compact parametrix. Consequently

$$\text{spec}^q_0(-i\mathcal{L}_T) = \text{spectrum of } (*)$$

is a discrete subset of $\mathbb{R}$. 

Theorem. Suppose that Levi $\theta$ is nondegenerate with $k$ positive and $n-k$ negative eigenvalues. Then

$$\dim \mathcal{N} = 2n+1$$

$\text{spec}^q_0(-i\mathcal{L}_T)$ is finite if $q \neq k, n-k$; $\text{spec}^n_{n-k}(\mathcal{L}_T)$ contains only finitely many positive elements, and $\text{spec}^n_{n-k}(\mathcal{L}_T)$ contains only finitely many negative elements.

The case $k = n$ (or $k = 0$) is like Kodaira's theorem on vanishing of cohomology.

If $k = n$ (or $k = 0$), then $\text{spec}^0_0(i\mathcal{L}_T)$ contains no negative (positive) elements.
Outline of proof

Let $\mathcal{H}^q_{\partial b} = \ker \Box_b \subset L^2(\mathcal{N}; \bigwedge^q \mathcal{K}^*)$.

Let $\mathcal{D} = \{ \phi \in \mathcal{H}^q_{\partial b} : \mathcal{L}_T \phi \in L^2 \}$

(*) $-i\mathcal{L}_T : \mathcal{D} \subset \mathcal{H}^q_{\partial b} \rightarrow \mathcal{H}^q_{\partial b}$

is a selfadjoint operator with compact parametrix. Consequently

$\text{spec}_0^q(-i\mathcal{L}_T) = \text{spectrum of (*)}$

is a discrete subset of $\mathbb{R}$.

**Theorem.** Suppose that Levi$_\theta$ is nondegenerate with $k$ positive and $n - k$ negative eigenvalues. Then

$\dim \mathcal{N} = 2n + 1$

$\text{spec}_0^q(-i\mathcal{L}_T)$ is finite if $q \neq k, \ n - k$;

$\text{spec}_0^k(-i\mathcal{L}_T)$ contains only finitely many positive elements, and

$\text{spec}_0^{n-k}(-i\mathcal{L}_T)$ contains only finitely many negative elements.
Outline of proof

Let \( \mathcal{H}^q_{\partial_b} = \ker \Box_b \subset L^2(\mathcal{N}; \bigwedge^q \bar{\mathcal{K}}^*) \).

Let \( D = \{ \phi \in \mathcal{H}^q_{\partial_b} : L_T \phi \in L^2 \} \)

\[ (*) \quad -iL_T : D \subset \mathcal{H}^q_{\partial_b} \rightarrow \mathcal{H}^q_{\partial_b} \]

is a selfadjoint operator with compact parametrix. Consequently

\[ \text{spec}^q_0(-iL_T) = \text{spectrum of \( (*) \) \) is a discrete subset of \( \mathbb{R} \).} \]

**Theorem.** Suppose that Levi\( _\theta \) is nondegenerate with \( k \) positive and \( n - k \) negative eigenvalues. Then

\[ \dim \mathcal{N} = 2n + 1 \]

\[ \text{spec}^q_0(-iL_T) \text{ is finite if } q \neq k, \ n - k; \]
\[ \text{spec}^k_0(-iL_T) \text{ contains only finitely many positive elements, and} \]
\[ \text{spec}^{n-k}_0(-iL_T) \text{ contains only finitely many negative elements.} \]

The case \( k = n \) (or \( k = 0 \)) is like Kodaira’s theorem on vanishing of cohomology.
Outline of proof

Let $\mathcal{H}^q_{\partial_b} = \ker \Box_b \subset L^2(\mathcal{N}; \bigwedge^q \mathcal{K}^*)$.

Let $D = \{ \phi \in \mathcal{H}^q_{\partial_b} : L_T \phi \in L^2 \}$

\[ (*) \quad -iL_T : D \subset \mathcal{H}^q_{\partial_b} \rightarrow \mathcal{H}^q_{\partial_b} \]

is a selfadjoint operator with compact parametrix. Consequently

\[ \text{spec}^q_0(-iL_T) = \text{spectrum of (*)} \]

is a discrete subset of $\mathbb{R}$.

**Theorem.** Suppose that Levi $\theta$ is nondegenerate with $k$ positive and $n - k$ negative eigenvalues. Then

\[ \dim \mathcal{N} = 2n + 1 \]

\[ \text{spec}^q_0(-iL_T) \text{ is finite if } q \neq k, \ n - k; \]

\[ \text{spec}^k_0(-iL_T) \text{ contains only finitely many positive elements, and} \]

\[ \text{spec}^{n-k}_0(-iL_T) \text{ contains only finitely many negative elements.} \]

The case $k = n$ (or $k = 0$) is like Kodaira’s theorem on vanishing of cohomology.

If $k = n$ (or 0), then $\text{spec}^0_0(iL_T)$ contains no negative (positive) elements.
Outline of proof

Let \( \mathcal{H}_q^{\partial b} = \ker \Box_b \subset L^2(\mathcal{N}; \bigwedge^q \overline{\mathcal{K}}^*) \).

Let \( D = \{ \phi \in \mathcal{H}_q^{\partial b} : \mathcal{L}_T \phi \in L^2 \} \)

\( (*) \quad -i\mathcal{L}_T : D \subset \mathcal{H}_q^{\partial b} \to \mathcal{H}_q^{\partial b} \)

is a selfadjoint operator with compact parametrix. Consequently

\[ \text{spec}_0^q(-i\mathcal{L}_T) = \text{spectrum of (*)} \]

is a discrete subset of \( \mathbb{R} \).

**Theorem.** Suppose that Levi_\( \theta \) is nondegenerate with \( k \) positive and \( n - k \) negative eigenvalues. Then

\[ \text{dim} \mathcal{N} = 2n + 1 \]

\[ \text{spec}_0^q(-i\mathcal{L}_T) \text{ is finite if } q \neq k, \ n - k; \]

\[ \text{spec}_0^k(-i\mathcal{L}_T) \text{ contains only finitely many positive elements, and} \]

\[ \text{spec}_0^{n-k}(-i\mathcal{L}_T) \text{ contains only finitely many negative elements.} \]

This is because \( \text{spec}_0^0(i\mathcal{L}_T) \) is an additive subgroup of \( \mathbb{R} \).

If \( k = n \) (or 0), then \( \text{spec}_0^0(i\mathcal{L}_T) \text{ contains no negative (positive) elements.} \)

The proof exploits \([\Box_b, \mathcal{L}_T] = 0\) plus the fact that \( \Box_b - \mathcal{L}_T^2 \) is elliptic (and that \( \mathcal{N} \) is compact).
Outline of proof

Let $\{\phi_\ell\}_{\ell=0}^\infty$ be an orthonormal basis of $H^0_{\partial b}$ consisting of eigenvectors of $-iL_T$, $\phi_\ell \in H^q_{\partial b, \tau_\ell}$.

Then there are $C, \mu > 0$ such that

$$\|\phi_\ell(p)\| \leq C(1 + |\tau_\ell|)^\mu$$

for all $p \in \mathcal{N}$, $\ell \in \mathbb{N}_0$. 

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, T, \overline{V}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$.
Outline of proof

Let $\{\phi_\ell\}_{\ell=0}^\infty$ be an orthonormal basis of $\mathcal{H}^0_{\partial b}$ consisting of eigenvectors of $-i\mathcal{L}_T$, $\phi_\ell \in \mathcal{H}^q_{\partial b, \tau_\ell}$. Then there are $C, \mu > 0$ such that

$$\|\phi_\ell(p)\| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \ell \in \mathbb{N}_0.$$  

This implies: The Fourier series of $u \in C^\infty(\mathcal{N}) \cap \mathcal{H}^0_{\partial b}$ converges in $C^\infty$. 

CR embedding in $\mathbb{C}^N$ of $(\mathcal{N}, T, \bar{V}) \in \mathcal{F}_{\text{ell}}$ with definite $\beta$. 

Consequence:
1. for all $p_0 \in \mathcal{N}$, span $\{d\phi_\ell(p_0) : \ell = 0, 1, 2, \ldots\}$ is the annihilator of $K_\beta$ in $C_T^*$;
2. the functions $\phi_\ell$, $\ell = 1, 2, \ldots$ separate points of $\mathcal{N}$.

The proofs of (1) and (2) use ideas of Boutet de Monvel. 

Theorem. Suppose $K_\beta$ is definite. There is an embedding $F : \mathcal{N} \to \mathbb{C}^N \setminus 0$ such that $F^* T = i \sum_j \tau_j (z_j \partial z_j - z_j \partial z_j)$ and all $\tau_j$ of the same sign.

$F$ is constructed using eigenfunctions. Getting one into $S^{2N+1}_2$ requires changing $\beta$ within the class $\beta$. This is like changing the Hermitian metric but not the holomorphic structure of a line bundle.
Outline of proof

Let \( \{ \phi_\ell \}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}_q^{0} \) consisting of eigenvectors of \(-iL_T\), \( \phi_\ell \in \mathcal{H}_q^{0, \partial_b, \tau_\ell} \).

Then there are \( C, \mu > 0 \) such that

\[
\| \phi_\ell(p) \| \leq C(1 + |\tau_\ell|)^\mu \quad \text{for all } p \in \mathcal{N}, \ \ell \in \mathbb{N}_0.
\]

Consequence:

1. for all \( p_0 \in \mathcal{N} \), \( \text{span}\{ d\phi_\ell(p_0) : \ell = 0, 1, \ldots \} \) is the annihilator of \( \overline{\mathcal{K}_\beta} \) in \( \mathbb{C} \mathcal{T}_{p_0}^{*} \mathcal{N} \);
2. the functions \( \phi_\ell \), \( \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

CR embedding in \( \mathbb{C}^N \) of \( (\mathcal{N}, T, \bar{V}) \in \mathcal{F}_{\text{ell}} \) with definite \( \beta \).
Outline of proof

Let \( \{ \phi_\ell \}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}_0^{\partial_b} \) consisting of eigenvectors of \(-iL_T\), \( \phi_\ell \in \mathcal{H}_b^{q,\tau_\ell} \).

Then there are \( C, \mu > 0 \) such that

\[
\| \phi_\ell(p) \| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \ \ell \in \mathbb{N}_0.
\]

This implies: The Fourier series of \( u \in C_\infty(\mathcal{N}) \cap \mathcal{H}_0^{\partial_b} \) converges in \( C_\infty \).

Consequence:

(1) for all \( p_0 \in \mathcal{N} \), \( \text{span}\{ d\phi_\ell(p_0) : \ell = 0, 1, \ldots \} \) is the annihilator of \( \overline{K}_\beta \) in \( \mathbb{C} T_{p_0}^* \mathcal{N} \);

(2) the functions \( \phi_\ell, \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

The proofs of (1) and (2) use ideas of Boutet de Monvel

CR embedding in \( \mathbb{C}^N \) of \( (\mathcal{N}, T, \overline{V}) \in \mathcal{F}_{\text{ell}} \) with definite \( \beta \).
Outline of proof

Let \( \{ \phi_\ell \}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}_0^0 \) consisting of eigenvectors of \(-iL_T\), \( \phi_\ell \in \mathcal{H}_q^0 \). Then there are \( C, \mu > 0 \) such that

\[
\| \phi_\ell(p) \| \leq C(1 + |\tau_\ell|)^\mu \quad \text{for all } p \in \mathcal{N}, \ \ell \in \mathbb{N}_0.
\]

Consequence:

(1) for all \( p_0 \in \mathcal{N} \), \( \text{span}\{d\phi_\ell(p_0) : \ell = 0, 1, \ldots\} \) is the annihilator of \( \mathcal{K}_\beta \) in \( \mathbb{C} T^*_{p_0} \mathcal{N} \);
(2) the functions \( \phi_\ell, \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

Theorem. Suppose \( \mathcal{K}_\beta \) is definite. There is an embedding

\[ F : \mathcal{N} \rightarrow \mathbb{C}^N \setminus 0 \quad \text{such that} \quad F^* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}) \]

and all \( \tau_j \) of the same sign.

\( F \) is constructed using eigenfunctions.
Outline of proof

Let \( \{ \phi_\ell \}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}_0^{\partial_b} \) consisting of eigenvectors of \(-iL_T\), \( \phi_\ell \in \mathcal{H}_q^{\partial_b, \tau_\ell} \).

Then there are \( C, \mu > 0 \) such that

\[
\| \phi_\ell(p) \| \leq C(1 + |\tau_\ell|)^\mu \quad \text{for all } p \in \mathcal{N}, \ \ell \in \mathbb{N}_0.
\]

This implies: The Fourier series of \( u \in C^\infty(\mathcal{N}) \cap \mathcal{H}_0^{\partial_b} \) converges in \( C^\infty \).

Consequence:

(1) for all \( p_0 \in \mathcal{N} \), \( \text{span}\{ d\phi_\ell(p_0) : \ell = 0, 1, \ldots \} \) is the annihilator of \( \overline{K}_\beta \) in \( \mathbb{C}T_{p_0}^* \mathcal{N} \);

(2) the functions \( \phi_\ell, \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

**Theorem.** Suppose \( \overline{K}_\beta \) is definite. There is an embedding

\[
F : \mathcal{N} \rightarrow \mathbb{C}^N \setminus 0
\]

such that

\[
F^* T = i \sum_j \tau_j (z^j \partial_{z^j} - \overline{z}^j \partial_{\overline{z}^j})
\]

and all \( \tau_j \) of the same sign.

\( F \) is constructed using eigenfunctions.

---

CR embedding in \( \mathbb{C}^N \) of \( (\mathcal{N}, T, \mathcal{V}) \in \mathcal{F}_{\text{ell}} \) with definite \( \beta \).
Outline of proof

Let \( \{\phi_\ell\}_{\ell=0}^\infty \) be an orthonormal basis of \( \mathcal{H}^0_{\partial_b} \) consisting of eigenvectors of \(-i\mathcal{L}_T\), \( \phi_\ell \in \mathcal{H}^q_{\partial_b,\tau_\ell} \).

Then there are \( C, \mu > 0 \) such that

\[
\|\phi_\ell(p)\| \leq C(1 + |\tau_\ell|)^\mu \text{ for all } p \in \mathcal{N}, \ \ell \in \mathbb{N}_0.
\]

Consequence:

1. for all \( p_0 \in \mathcal{N} \), \( \text{span}\{d\phi_\ell(p_0) : \ell = 0, 1, \ldots\} \) is the annihilator of \( \overline{\mathcal{K}_\beta} \) in \( \mathbb{C} T_{p_0}^* \mathcal{N} \);
2. the functions \( \phi_\ell, \ell = 1, 2, \ldots \) separate points of \( \mathcal{N} \).

**Theorem.** Suppose \( \overline{\mathcal{K}_\beta} \) is definite. There is an embedding

\[ F : \mathcal{N} \to \mathbb{C}^N \setminus 0 \text{ such that } \]

\[ F_* \mathcal{T} = i \sum_j \tau_j (z^j \partial_{\bar{z}^j} - \bar{z}^j \partial_{z^j}) \]

and all \( \tau_j \) of the same sign.

\( F \) is constructed using eigenfunctions.

CR embedding in \( \mathbb{C}^N \) of \( (\mathcal{N}, T, \mathcal{V}) \in \mathcal{F}_{\text{ell}} \) with definite \( \beta \).

This implies: The Fourier series of \( u \in C^\infty(\mathcal{N}) \cap \mathcal{H}^0_{\partial_b} \) converges in \( C^\infty \).

The proofs of (1) and (2) use ideas of Boutet de Monvel.

This is an embedding into \( \mathbb{C}^N \setminus 0 \).

Getting one into \( S^{2N+1} \) requires changing \( \beta \) within the class \( \beta \).

This is like changing the Hermitian metric but not the holomorphic structure of a line bundle.
End