RESOLVENTS OF ELLIPTIC CONE OPERATORS

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Abstract. We prove the existence of sectors of minimal growth for general closed extensions of elliptic cone operators under natural ellipticity conditions. This is achieved by the construction of a suitable parametrix and reduction to the boundary. Special attention is devoted to the clarification of the analytic structure of the resolvent.

1. Introduction

Motivated by Seeley's seminal work [21], and with the same intentions, the purpose of this paper is, first, to prove the existence of sectors of minimal growth for general closed extensions of elliptic cone differential operators under suitable ray conditions on the symbols of the operator; and second, to describe the structure of the resolvent as a pseudodifferential operator.

Previous relevant investigations in this direction assume that the coefficients are constant near the boundary, cf. [16], [17], or the technically convenient but rather restrictive dilation invariance of the domain, cf. [1], [11], [4], [12], [17]. Some of these works deal with special classes of operators such as Laplacians. In the general setting followed in this paper, the interactions of lower order terms in the Taylor expansion of the coefficients of the operators near the boundary lead to a domain structure beyond the minimal domain $\mathcal{D}_{\min}$ that brings up essential new difficulties not present in the constant coefficients case. Thus the investigation of the general case entails the development of new techniques.

Let M be a smooth compact n-manifold with boundary. Recall that a cone differential operator is a linear differential operator with smooth coefficients in the interior of M which locally near the boundary and in terms of coordinates $x, y_1, \dots, y_{n-1}$ with $x = 0$ on ∂M, is of the form

$$x^{-m} \sum_{k+|\alpha|\leq m} a_{k\alpha}(x,y) D_y^\alpha (xD_x)^k$$

with $a_{k\alpha}$ smooth up to the boundary and m a positive integer. Such an operator is called c-elliptic if it is elliptic in the interior in the usual sense, and near the boundary, if written as above, then

$$\sum_{k+|\alpha|=m} a_{k\alpha}(x,y)\eta^\alpha \xi^k$$

is an elliptic symbol up to $\{x = 0\}$. Fix some smooth defining function x for ∂M with $x > 0$ in the interior $\mathring{M}$ of M and denote by $x^{-m}\operatorname{Diff}^m_b(M;E)$ the space of

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cone operators of order at most m acting on sections of a Hermitian vector bundle $E \to M$.

Cone differential operators arise when introducing polar coordinates around a point, and for that reason they are of great interest in the study of operators on manifolds with conical singularities (cf. [9], [19]). In this context it is natural to base the L^2 theory of these operators, at least initially, on a c-density on M, which is a measure of the form $x^n\mathfrak{m}$ where $\mathfrak{m}$ is a smooth b-density, that is, $x\mathfrak{m}$ is a smooth everywhere positive density on M.

Let $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$, and write $L^2_c(M;E)$ for the space $L^2(M, x^n\mathfrak{m}; E)$. There are two canonical closed extensions one can specify for the unbounded operator

$$A : C_0^\infty(\mathring{M};E) \subset L^2_c(M;E) \to L^2_c(M;E), \tag{1.1}$$

namely the closure

$$A : \mathcal{D}_{\min} \subset L^2_c(M;E) \to L^2_c(M;E), \tag{1.2}$$

and

$$A : \mathcal{D}_{\max} \subset L^2_c(M;E) \to L^2_c(M;E), \tag{1.3}$$

with

$$\mathcal{D}_{\max} = \{u \in L^2_c(M;E) : Au \in L^2_c(M;E)\}.$$

Obviously, both $\mathcal{D}_{\min}$ and $\mathcal{D}_{\max}$ are complete in the graph norm,

$$\|u\|_A = \|u\|_{L^2_c} + \|Au\|_{L^2_c},$$

and $\mathcal{D}_{\min} \subset \mathcal{D}_{\max}$.

Suppose that A is c-elliptic. By a theorem of Lesch [11], $\mathcal{D}_{\min}$ has finite codimension in $\mathcal{D}_{\max}$, and all closed extensions of (1.1) are Fredholm and have domain $\mathcal{D}$ such that $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$. Moreover, if $A_{\mathcal{D}}$ denotes the closed extension with domain $\mathcal{D}$, then

$$\operatorname{ind}(A_{\mathcal{D}}) = \operatorname{ind}(A_{\mathcal{D}_{\min}}) + \dim(\mathcal{D}/\mathcal{D}_{\min}), \tag{1.4}$$

see Lesch, op. cit. and Gil-Mendoza [7]. Thus, if $\operatorname{ind}(A_{\mathcal{D}_{\min}})$ is already positive, then there is no extension of A with nonempty resolvent set. In fact, a necessary and sufficient condition for the existence of a closed extension $A_{\mathcal{D}}$ of (1.1) with nonempty resolvent set is that for some $\lambda \in \mathbb{C}$, $A_{\mathcal{D}_{\min}} - \lambda$ is injective and $A_{\mathcal{D}_{\max}} - \lambda$ is surjective, see [5].

Given a closed extension $A_{\mathcal{D}}$, we will prove in Section 6 (see Theorem 6.9) that under natural ellipticity conditions pertaining the symbol of A and the model operator $A_\wedge$, cf. (2.6), there exists a sector

$$\Lambda = \{z \in \mathbb{C} : z = re^{i\theta},\ r \geq 0,\ |\theta - \theta_0| \leq a\}$$

of minimal growth, i.e.,

$$A - \lambda : \mathcal{D} \to L^2_c(M;E)$$

is invertible for $\lambda \in \Lambda$ with $|\lambda|$ large, and

$$\|(A_{\mathcal{D}} - \lambda)^{-1}\|_{\mathscr{L}(L^2_c(M;E))} = O(|\lambda|^{-1}) \text{ as } |\lambda| \to \infty.$$

More precisely, we require that Λ is free of spectrum of the homogeneous principal c-symbol ${}^c\boldsymbol{\sigma}(A)$ of A on ${}^cT^*M\backslash\{0\}$, and that the model operator

$$A_\wedge - \lambda : \mathcal{D}_\wedge \to L^2_c(Y^\wedge; \pi_Y^* E|_Y)$$

is invertible for large $\lambda \in \Lambda$ with inverse bounded in the norm as $|\lambda| \to \infty$, where $Y^\wedge = \overline{\mathbb{R}}_+ \times Y$ is the stretched model cone with boundary $Y = \partial M$, and $\mathcal{D}_\wedge$ is a domain for $A_\wedge$ associated with $\mathcal{D}$ in a natural way (see [5]).

The proof of this result relies on the construction of a parameter-dependent parametrix

$$B(\lambda) : L^2_c(M;E) \to \mathcal{D}_{\min}(A), \tag{1.5}$$

which is a left-inverse for the operator $A_{\mathcal{D}_{\min}} - \lambda$ for large $|\lambda|$. Then, in order to deal with the finite dimensional contribution of the domain $\mathcal{D}$ beyond $\mathcal{D}_{\min}$, we follow the idea of reduction to the boundary motivated by the point of view that the choice of a domain corresponds to the choice of a boundary condition for the operator A.

More precisely, we add a suitable operator family $K(\lambda)$ to $A_{\mathcal{D}\min} - \lambda$ such that

$$\begin{pmatrix} A_{\mathcal{D}_{\min}} - \lambda & K(\lambda) \end{pmatrix} : \begin{array}{c} \mathcal{D}_{\min}(A) \\ \oplus \\ \mathbb{C}^{d''} \end{array} \to L^2_c(M;E) \tag{1.6}$$

is invertible for large $|\lambda|$, and consider (1.6) a "Dirichlet problem" for the operator $A - \lambda$. Following Schulze's viewpoint from the pseudodifferential edge-calculus [18, 19] we invert (1.6) in the context of operator matrices by adding generalized Green remainders to the parametrix $B(\lambda)$. We then multiply the inverse $\begin{pmatrix} B(\lambda) \\ T(\lambda) \end{pmatrix}$ from the left to the operator $A_{\mathcal{D}} - \lambda$, reducing the problem of inverting $A_{\mathcal{D}} - \lambda$ to the simpler problem of inverting the operator family

$$F(\lambda) = T(\lambda)(A - \lambda) : \mathcal{D}/\mathcal{D}_{\min} \to \mathbb{C}^{d''}. \tag{1.7}$$

The operator $F(\lambda)$ can be interpreted as the reduction to the boundary of $A - \lambda$ under the boundary condition $\mathcal{D}$ by (1.6), and it plays a similar role as, e.g., the Dirichlet-to-Neumann map in classical boundary value problems.

We prove that the resolvent can be written as

$$(A_{\mathcal{D}} - \lambda)^{-1} = B(\lambda) + (A_{\mathcal{D}} - \lambda)^{-1}\Pi(\lambda)$$

with $B(\lambda)$ from (1.5) and a finite dimensional smoothing pseudodifferential projection $\Pi(\lambda)$ onto a complement of the range of $A_{\mathcal{D}_{\min}} - \lambda$ in $L^2_c(M;E)$. The operators $B(\lambda)$ and $\Pi(\lambda)$ have complete asymptotic expansions as $|\lambda| \to \infty$ into homogeneous components in the interior and κ-homogeneous operator-valued components near the boundary, respectively, cf. Definition 2.9.

The structure of the paper is as follows: In Section 2 we recall basic facts about cone operators and their symbols. Section 3 is devoted to closed extensions in L^2 and in higher order Sobolev spaces. Section 4 concerns some relations between A and its symbols regarding the discreteness of the spectrum and the existence of sectors of minimal growth. In Section 5 we perform the construction of the parametrix (1.5) and establish the "Dirichlet problem" (1.6). Finally, in Section 6, we prove the results about the existence and norm estimates of the resolvent by investigating the operator (1.7).

2. Preliminaries

Let M be a smooth compact n-manifold with boundary and fix a defining function x for ∂M with $x > 0$ in $\mathring{M}$. Let $E \to M$ be a complex vector bundle and let

$\operatorname{Diff}^m(M;E)$ be the space of differential operators on $C^\infty(M;E)$ of order m. By $\operatorname{Diff}_b^m(M;E)$ we denote the subspace of totally characteristic operators of order m.

The elements of $x^{-m}\operatorname{Diff}_b^m(M;E)$, that is, differential operators of the form $A = x^{-m}P$ with $P \in \operatorname{Diff}_b^m(M;E)$, are the differential cone operators of order m.

According to [5] we associate with A an invariantly defined c-symbol

$$ {}^c\!\boldsymbol{\sigma}(A) \in C^\infty({}^cT^*M\backslash 0; \operatorname{End}({}^c\pi^*E)) $$

on the c-cotangent bundle ${}^cT^*M \to M$, where ${}^c\pi : {}^cT^*M \to M$ is the canonical projection map. Recall that ${}^cT^*M$ is the smooth vector bundle over M whose space of smooth sections is

$$ C^\infty_{\mathrm{cn}}(M;T^*M) = \{\eta \in C^\infty(M,T^*M) : \iota^*\eta = 0\}, $$

the space of 1-forms on M which are, over ∂M, sections of the conormal bundle of ∂M in M.

Let $\mathbf{x}^{-1} : {}^cT^*M \to {}^bT^*M$ be the natural isomorphism that is induced by the defining function x. Then the c-symbol of A and the b-symbol of x^mA are related as

$$ {}^c\!\boldsymbol{\sigma}(A)(\eta) = {}^b\!\boldsymbol{\sigma}(x^mA)(\mathbf{x}^{-1}(\eta)). $$

Definition 2.1. The operator $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ is called c-elliptic if

$$ {}^c\!\boldsymbol{\sigma}(A) \in C^\infty({}^cT^*M\backslash 0; \operatorname{End}({}^c\pi^*E)) $$

is an isomorphism. The family $\lambda \mapsto A - \lambda$ is called c-elliptic with parameter in a set $\Lambda \subset \mathbb{C}$ if

$$ {}^c\!\boldsymbol{\sigma}(A) - \lambda \in C^\infty(({}^cT^*M \times \Lambda)\backslash 0; \operatorname{End}({}^c\pi^*_\Lambda E)) $$

is an isomorphism. Here ${}^c\pi_\Lambda : ({}^cT^*M \times \Lambda)\backslash 0 \to M$ is the canonical map.

Let $E \to M$ be Hermitian, and $\mathfrak{m}$ be a positive b-density. Recall that the Hilbert space $L^2_b(M;E)$ is the L^2 space of sections of E with respect to the Hermitian form on E and the density $\mathfrak{m}$. Thus the inner product is

$$ (u,v)_{L^2_b} = \int (u,v)_E\,\mathfrak{m} \quad \text{if } u,\, v \in L^2_b(M;E). $$

For a nonnegative integer s the Sobolev space $H^s_b(M;E)$ is defined as

$$ H^s_b(M;E) = \{u \in L^2_b(M;E) : Pu \in L^2_b(M;E)\ \forall P \in \operatorname{Diff}^s_b(M;E)\}. $$

The spaces $H^s_b(M;E)$ for general $s \in \mathbb{R}$ are defined by interpolation and duality, and we set

$$ H^\infty_b(M;E) = \bigcap_s H^s_b(M;E), \quad H^{-\infty}_b(M;E) = \bigcup_s H^s_b(M;E). $$

The weighted spaces

$$ x^\mu H^s_b(M;E) = \{x^\mu u : u \in H^s_b(M;E)\} $$

are topologized so that $H^s_b(M;E) \ni u \mapsto x^\mu u \in x^\mu H^s_b(M;E)$ is an isomorphism. In the case of $s = 0$ one has

$$ x^\mu H^0_b(M;E) = x^\mu L^2_b(M;E) = L^2(M, x^{-2\mu}\mathfrak{m}; E), $$

and the Sobolev space based on $L^2(M, x^{-2\mu}\mathfrak{m}; E)$ and $\operatorname{Diff}^s_b(M;E)$ is isomorphic to $x^\mu H^s_b(M;E)$.

To define a Mellin transform consistent with the density $\mathfrak{m}$, pick a collar neighborhood $U_Y \cong Y \times [0,1)$ of the boundary $Y = \partial M$ in M, and a defining function $x : M \to \mathbb{R}$ such that

$$\mathfrak{m} = \frac{dx}{x} \otimes \pi_Y^* \mathfrak{m}_Y \text{ in } U_Y \tag{2.2}$$

for some smooth density $\mathfrak{m}_Y$ on Y, where $\pi_Y : Y \times [0,1) \to Y$ is the projection. Let ∂_x be the vector field tangent to the fibers of $U_Y \to Y$ such that $\partial_x x = 1$.

Fix $\omega \in C_0^\infty(-1,1)$ real valued, nonnegative and such that $\omega = 1$ in a neighborhood of 0. Also fix a Hermitian connection ∇ on E. The Mellin transform of an element $u \in C_0^\infty(\mathring{M}; E)$ is defined to be the entire function $\hat{u} : \mathbb{C} \to C^\infty(Y; E|_Y)$ such that for any $v \in C^\infty(Y; E|_Y)$

$$(x^{-i\sigma}\omega u, \pi_Y^* v)_{L_b^2(M;E)} = (\hat{u}(\sigma), v)_{L^2(Y;E|_Y)}$$

By $\pi_Y^* v$ we mean the section of E over U_Y obtained by parallel transport of v along the fibers of π_Y. The Mellin transform thus defined extends to the spaces $x^\mu H_b^s(M;E)$ so as to give holomorphic functions on $\{\sigma : \Im\sigma > -\mu\}$ with values in $H^s(Y; E|_Y)$. As is well known, the Mellin transform extends to the spaces $x^\mu L_b^2(M;E)$ in such a way that if $u \in x^\mu L_b^2(M;E)$ then $\hat{u}(\sigma)$ is holomorphic in $\{\Im\sigma > -\mu\}$ and belongs to $L^2(\{\Im\sigma = -\mu\} \times Y)$ with respect to $d\sigma \otimes \mathfrak{m}_Y$.

Let $A = x^{-m}P$ with $P \in \operatorname{Diff}_b^m(M;E)$, and let

$$\mathbb{C} \ni \sigma \mapsto \hat{P}(\sigma) \in \operatorname{Diff}^m(Y; E|_Y) \tag{2.3}$$

be the conormal symbol of P. Recall that $\hat{P}(\sigma)$ is elliptic for every $\sigma \in \mathbb{C}$ if A is c-elliptic. The boundary spectrum of A is

$$\operatorname{spec}_b(A) = \{\sigma \in \mathbb{C} : \hat{P}(\sigma) \text{ is not invertible}\},$$

which is discrete if A is c-elliptic, and the conormal symbol of A is defined to be that of the operator P.

Near Y one can write

$$P = \sum_{\ell=0}^{m} P'_\ell \circ (\nabla_{xD_x})^\ell$$

where the P'_ℓ are differential operators of order $m - \ell$ (defined on U_Y) such that for any smooth function $\phi(x)$ and section u of E over U_Y, $P'_\ell(\phi(x)u) = \phi(x)P'_\ell(u)$, in other words, of order zero in ∇_{xD_x}.

Definition 2.4. P is said to have coefficients independent of x near Y, or simply constant coefficients near the boundary, if

$$\nabla_{x\partial_x} P(u) = P(\nabla_{x\partial_x} u)$$

for any smooth section u of E supported in U_Y. Correspondingly, A is said to have coefficients independent of x near Y if this holds for P.

For $P \in \operatorname{Diff}_b^m(M;E)$ and any $N \in \mathbb{N}$ there are operators P_k, $\tilde{P}_N \in \operatorname{Diff}_b^m(M;E)$ such that

$$P = \sum_{k=0}^{N-1} P_k x^k + x^N \tilde{P}_N \tag{2.5}$$

where each P_k has coefficients independent of x near Y. If P_k has coefficients independent of x near Y, then so does its formal adjoint $P_k^\star$.

With $A = x^{-m}P$ we associate on the model cone $Y^\wedge = \overline{\mathbb{R}}_+ \times Y$ the operator

$$A_\wedge = x^{-m}P_0, \tag{2.6}$$

where $P_0 \in \operatorname{Diff}_b^m(Y^\wedge; E)$ is the constant term in the expansion (2.5) and has therefore coefficients independent of x.

For $\varrho > 0$ we consider the normalized dilation group action from sections of E to sections of E on $\mathring{Y}^\wedge$ defined by

$$(\kappa_\varrho u)(x, y) = \varrho^{m/2} u(\varrho x, y). \tag{2.7}$$

The normalizing factor $\varrho^{m/2}$ in the definition of κ_ϱ is added only because it makes

$$\kappa_\varrho : x^{-m/2}L_b^2(Y^\wedge; E) \to x^{-m/2}L_b^2(Y^\wedge; E)$$

an isometry, where the measure on L_b^2 refers to the b-density $\mathfrak{m} = \frac{dx}{x} \otimes \mathfrak{m}_Y$ on $Y^\wedge$.

Let $A^\star$ denote the formal adjoint of A acting on $x^{-m/2}L_b^2(M; E)$. Then

$$(A_\wedge)^\star = (A^\star)_\wedge.$$

The family $\lambda \mapsto A_\wedge - \lambda$ satisfies the homogeneity relation

$$A_\wedge - \varrho^m \lambda = \varrho^m \kappa_\varrho (A_\wedge - \lambda)\kappa_\varrho^{-1} \text{ for every } \varrho > 0. \tag{2.8}$$

Definition 2.9. A family of operators $A(\lambda)$ acting on a κ-invariant space of distributions on $Y^\wedge$ will be called κ-homogeneous of degree ν if

$$A(\varrho^m \lambda) = \varrho^\nu \kappa_\varrho A(\lambda) \kappa_\varrho^{-1}$$

for every $\varrho > 0$.

This notion of homogeneity is systematically used in Schulze's edge-calculus.

On $Y^\wedge$ it is convenient to introduce weighted Sobolev spaces with a particular structure at infinity consistent with the structure of the operators involved. Let $\omega \in C_0^\infty(\mathbb{R})$ be a nonnegative function with $\omega(r) = 1$ near $r = 0$. We follow Schulze (cf. [18]) and consider the space $H^s_{\mathrm{cone}}(Y^\wedge; E)$ consisting of distributions u such that given any coordinate patch Ω on Y diffeomorphic to an open subset of the sphere S^{n-1}, and given any function $\varphi \in C_0^\infty(\Omega)$, we have $(1-\omega)\varphi u \in H^s(\mathbb{R}^n; E)$ where $\mathbb{R}_+ \times S^{n-1}$ is identified with $\mathbb{R}^n \backslash \{0\}$ via polar coordinates.

For $s, \alpha \in \mathbb{R}$ we define $\mathcal{K}^{s,\alpha}(Y^\wedge; E)$ as the space of distributions u such that

$$\omega u \in x^\alpha H_b^s(Y^\wedge; E) \text{ and } (1-\omega)u \in x^{\frac{n-m}{2}} H^s_{\mathrm{cone}}(Y^\wedge; E)$$

for any cut-off function ω. Note that $H^0_{\mathrm{cone}}(Y^\wedge; E) = x^{-n/2}L_b^2(Y^\wedge; E)$.

It turns out that $C_0^\infty(\mathring{Y}^\wedge; E)$ is dense in $\mathcal{K}^{s,\alpha}(Y^\wedge; E)$, and

$$A_\wedge : \mathcal{K}^{s,\alpha}(Y^\wedge; E) \to \mathcal{K}^{s-m,\alpha-m}(Y^\wedge; E) \tag{2.10}$$

is bounded for every s and α. The group $\{\kappa_\varrho\}_{\varrho \in \mathbb{R}_+}$ is a strongly continuous group of isomorphisms on $\mathcal{K}^{s,\alpha}$ for every $s, \alpha \in \mathbb{R}$. As pointed out before, κ_ϱ defines an isometry on the space

$$\mathcal{K}^{0,-m/2}(Y^\wedge; E) = x^{-m/2}L_b^2(Y^\wedge; E),$$

which we will take as our reference Hilbert space on $Y^\wedge$.

3. Closed extensions

If $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$, then for any s and μ,

$$A : x^\mu H^s_b(M;E) \to x^{\mu-m} H^{s-m}_b(M;E)$$

is continuous. In order not to deal with the index μ, we normalize in such a way that if our original interests are in $x^\mu L^2_b(M;E)$, then we work with the operator $x^{-\mu-m/2}Ax^{\mu+m/2} \in x^{-m}\operatorname{Diff}^m_b(M;E)$ and base all the analysis on $x^{-m/2}L^2_b(M;E)$. The latter operator has the same c-symbol as A, so it is c-elliptic if and only if A is so, and it has the same spectral properties. This said, we assume that $\mu = -m/2$.

The closed extensions of elliptic cone operators on $x^{-m/2}L^2_b(M;E)$ have been studied by Lesch [11] and by two of the authors of the present work in [7], among others. It is important for our purposes to admit arbitrary regularity. In analogy with the $x^{-m/2}L^2_b$-case, two canonical closed extensions of the operator

$$A : C_0^\infty(\mathring{M};E) \subset x^{-m/2}H^s_b(M;E) \to x^{-m/2}H^s_b(M;E), \tag{3.1}$$

are singled out. Its closure

$$A : \mathcal{D}^s_{\min}(A) \subset x^{-m/2}H^s_b(M;E) \to x^{-m/2}H^s_b(M;E), \tag{3.2}$$

and

$$A : \mathcal{D}^s_{\max}(A) \subset x^{-m/2}H^s_b(M;E) \to x^{-m/2}H^s_b(M;E) \tag{3.3}$$

with

$$\mathcal{D}^s_{\max}(A) = \{u \in x^{-m/2}H^s_b(M;E) : Au \in x^{-m/2}H^s_b(M;E)\}.$$

Both $\mathcal{D}^s_{\min}(A)$ and $\mathcal{D}^s_{\max}(A)$ are complete in the graph norm

$$\|u\|_{A,s} = \|u\|_{x^{-m/2}H^s_b} + \|Au\|_{x^{-m/2}H^s_b}, \tag{3.4}$$

and $\mathcal{D}^s_{\min}(A) \subset \mathcal{D}^s_{\max}(A)$. Clearly, for any closed extension

$$A : \mathcal{D} \subset x^{-m/2}H^s_b(M;E) \to x^{-m/2}H^s_b(M;E)$$

of (3.1) we have $\mathcal{D}^s_{\min}(A) \subset \mathcal{D} \subset \mathcal{D}^s_{\max}(A)$, and $\mathcal{D}$ is closed (with respect to the graph norm of A). These facts do not involve c-ellipticity.

We will usually abbreviate $\mathcal{D}^s_{\min}(A)$ to $\mathcal{D}^s_{\min}$ and $\mathcal{D}^s_{\max}(A)$ to $\mathcal{D}^s_{\max}$ when the operator is clear from the context. The operator A with domain $\mathcal{D}$ will be denoted by $A_{\mathcal{D}}$.

The proof of the following proposition characterizing $\mathcal{D}^s_{\min}$, when A is c-elliptic and s is arbitrary, is a small variation of the characterization of $\mathcal{D}^0_{\min}$ as given in Gil-Mendoza [7].

Proposition 3.5. *Let $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$ be c-elliptic. Then*

(*i*) $\mathcal{D}^s_{\min} = \mathcal{D}^s_{\max} \cap \big(\bigcap_{\varepsilon>0} x^{m/2-\varepsilon}H^{s+m}_b(M;E)\big)$;

(*ii*) $\mathcal{D}^s_{\min} = x^{m/2}H^{s+m}_b(M;E)$ *if and only if* $\operatorname{spec}_b(A) \cap \{\Im\sigma = -m/2\} = \varnothing$.

The following theorem is also a straightforward generalization of the corresponding results for the case $s = 0$, cf. Lesch [11], Gil-Mendoza [7].

Theorem 3.6. *Let $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$ be c-elliptic.*

(*i*) *The closed extensions $A_{\mathcal{D}^s_{\min}}$ and $A_{\mathcal{D}^s_{\max}}$ of* (3.1) *are both Fredholm. Thus the space $\mathcal{D}^s_{\max}/\mathcal{D}^s_{\min}$ is finite dimensional.*

(*ii*) *There is a one to one correspondence between the domains $\mathcal{D}$ of the closed extensions of* (3.1) *and the subspaces of $\mathcal{D}^s_{\max}/\mathcal{D}^s_{\min}$.*

(*iii*) *For any sufficiently small* $\varepsilon > 0$, *the embeddings*

$$x^{m/2}H_b^{s+m}(M;E) \hookrightarrow \mathcal{D} \hookrightarrow x^{-m/2+\varepsilon}H_b^{s+m}(M;E)$$

are continuous.

(*iv*) *For any* $\mathcal{D}$ *with* $\mathcal{D}^s_{\min} \subset \mathcal{D} \subset \mathcal{D}^s_{\max}$, *the operator* $A : \mathcal{D} \to x^{-m/2}H_b^s(M;E)$ *is Fredholm with index*

$$\operatorname{ind} A_{\mathcal{D}} = \operatorname{ind} A_{\mathcal{D}^s_{\min}} + \dim \mathcal{D}/\mathcal{D}^s_{\min}. \tag{3.7}$$

Assuming that A is c-elliptic, the space $\mathcal{D}^s_{\max}/\mathcal{D}^s_{\min}$ can be identified with a (finite dimensional) subspace $\mathcal{E}^s_{\max} \subset \mathcal{D}^s_{\max}$ complementary to $\mathcal{D}^s_{\min}$. Thus, the domains of the various extensions of A based on $x^{-m/2}H_b^s(M;E)$ are of the form $\mathcal{D}^s_{\min} \oplus \mathcal{E}$ with $\mathcal{E} \subset \mathcal{E}^s_{\max}$. In fact, the complementary space can be chosen to be independent of s, a subspace $\mathcal{E}_{\max}$ of $x^{-m/2}H_b^\infty(M;E)$,

$$\mathcal{D}^s_{\max} = \mathcal{D}^s_{\min} \oplus \mathcal{E}_{\max} \quad \forall s \in \mathbb{R}.$$

A possible choice for $\mathcal{E}_{\max}$ is the orthogonal complement of $\mathcal{D}^0_{\min}(A)$ in $\mathcal{D}^0_{\max}(A)$ with respect to the inner product

$$(u,v)_A = (u,v)_{x^{-m/2}L^2_b} + (Au,Av)_{x^{-m/2}L^2_b}, \tag{3.8}$$

in other words, $\mathcal{E}_{\max} = \ker(A^\star A + I) \cap \mathcal{D}^0_{\max}$. Another way to describe the complementary space is by means of singular functions, see also Section 6.

Granted this, one can then speak of the "same" extension of A for different s; namely, if $\mathcal{E} \subset \mathcal{E}_{\max}$, let the extension of A based on $x^{-m/2}H_b^s(M;E)$ have domain

$$\mathcal{D}^s = \mathcal{D}^s_{\min} \oplus \mathcal{E}. \tag{3.9}$$

Then (3.7) reads

$$\operatorname{ind} A_{\mathcal{D}^s} = \operatorname{ind} A_{\mathcal{D}^s_{\min}} + \dim \mathcal{E}.$$

The index of $A_{\mathcal{D}^s_{\min}}$ is in fact independent of s. To see this, we first observe that the kernel of A in $x^{-m/2}H_b^{-\infty}(M;E)$ is contained in $x^{-m/2}H_b^\infty(M;E)$ and is therefore finite dimensional and contained in each space $x^{-m/2}H_b^s(M;E)$. Next, using the nonsingular sesquilinear pairing

$$x^{-m/2}H_b^s(M;E) \times x^{-m/2}H_b^{-s}(M;E) \ni (u,v) \mapsto (u,v)_{x^{-m/2}L^2_b} \in \mathbb{C},$$

we see that the annihilator of the range of $A_{\mathcal{D}^s_{\min}}$ is the kernel K^s of the formal adjoint $A^\star$ of A acting on $x^{-m/2}H_b^{-s}(M;E)$. Since $A^\star$ is also c-elliptic, its kernel in $x^{-m/2}H_b^{-\infty}(M;E)$ is a finite dimensional subspace of $x^{-m/2}H_b^\infty(M;E)$. Thus K^s is independent of s. Since the range of $A_{\mathcal{D}^s_{\min}}$ is closed, this range is the annihilator in $x^{-m/2}H^s(M;E)$ of K^s, so its codimension is independent of s. Hence $\operatorname{ind} A_{\mathcal{D}^s_{\min}}$ is independent of s. Thus:

Proposition 3.10. *Let* $\mathcal{E} \subset \mathcal{E}_{\max}$ *and define* $\mathcal{D}^s$ *as in* (3.9). *The index of*

$$A : \mathcal{D}^s \subset x^{-m/2}H_b^s(M;E) \to x^{-m/2}H_b^s(M;E) \tag{3.11}$$

is independent of s.

Let $P = x^m A$, an operator in $\operatorname{Diff}^m_b(M;E)$, and let $\lambda \in \mathbb{C}$. Since $A - \lambda = x^{-m}(P - \lambda x^m) \in x^{-m}\operatorname{Diff}^m_b(M;E)$, Proposition 4.1 of [7] gives that the minimal and maximal domains of $A - \lambda$ are those of A. Since $A - \lambda \in x^{-m}\operatorname{Diff}^m_b(M;E)$ is c-elliptic if A is c-elliptic, also the kernel of

$$A - \lambda : \mathcal{D}^s \subset x^{-m/2}H_b^s(M;E) \to x^{-m/2}H_b^s(M;E)$$

is independent of s if $\mathcal{D}^s$ is the domain in (3.9). Thus:

Proposition 3.12. *The spectrum of* (3.11) *is independent of* s.

Sometimes it is useful to approximate a c-elliptic operator $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ by operators having coefficients independent of x near the boundary Y of M, see Definition 2.4. A simple and efficient approximation of A can be obtained as follows.

Let U_Y be a collar neighborhood of Y. For small $\tau > 0$ let

$$\omega_\tau(x) = \omega(x/\tau)$$

where $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ is a cut-off function with $\omega = 1$ near 0. Given A let

$$A_\tau = \omega_\tau A_\wedge + (1-\omega_\tau)A. \tag{3.13}$$

For small enough $\tau > 0$ the operator A_τ is well defined, c-elliptic, and has the same conormal symbol and therefore the same boundary spectrum as A. Thus $\mathcal{D}_{\min}(A_\tau) = \mathcal{D}_{\min}(A)$. The following lemma was given in [6]. Related results can also be found in [11, Section 1.3].

Lemma 3.14. *As* $\tau \to 0$*,* $A_\tau \to A$ *in* $\mathscr{L}(\mathcal{D}_{\min}, x^{-m/2}L_b^2(M;E))$.

Proof. Since A is c-elliptic, there is a bounded parametrix $B : x^\gamma H_b^s \to x^{\gamma+m}H_b^{s+m}$ such that

$$R = I - BA : x^\gamma H_b^s \to x^\gamma H_b^\infty$$

is bounded for all s and γ. Write $A = x^{-m}P$ and expand $P = P_0 + x\tilde{P}_1$ as in (2.5). Then $x^{-m}P_0 = A_\wedge$, and with $\tilde{A} = x^{-m}\tilde{P}_1$ we get

$$A - A_\tau = x\omega_\tau\tilde{A} = x\omega_\tau\tilde{A}BA + x\omega_\tau\tilde{A}R = \tau\tilde{\omega}_\tau\tilde{A}BA + x\omega_\tau\tilde{A}R,$$

where $\tilde{\omega}_\tau(x) = (x/\tau)\omega(x/\tau)$. Now, $\tilde{A}B : x^{-m/2}L_b^2 \to x^{-m/2}L_b^2$ is bounded, so if $u \in \mathcal{D}_{\min}(A)$, then

$$\|\tau\tilde{\omega}_\tau\tilde{A}BAu\|_{x^{-m/2}L_b^2} \le c\,\tau\|Au\|_{x^{-m/2}L_b^2} \le c\,\tau\|u\|_A.$$

Let $0 < \alpha \ll 1$ and write $x\omega_\tau\tilde{A}R = \tau^{1-\alpha}(\frac{x}{\tau})^{1-\alpha}\omega_\tau\, x^\alpha\tilde{A}R$. The operator

$$x^\alpha\tilde{A}R : x^{m/2-\alpha}L_b^2 \to x^{-m/2}L_b^2$$

and the embedding $(\mathcal{D}_{\min}(A), \|\cdot\|_A) \hookrightarrow x^{m/2-\alpha}L_b^2$ are both continuous, so

$$\|x\omega_\tau\tilde{A}Ru\|_{x^{-m/2}L_b^2} \le \tilde{c}\,\tau^{1-\alpha}\|u\|_{x^{m/2-\alpha}L_b^2} \le c\,\tau^{1-\alpha}\|u\|_A.$$

Altogether,

$$\|(A-A_\tau)u\|_{x^{-m/2}L_b^2} \le C\,\tau^{1-\alpha}\|u\|_A \tag{3.15}$$

and thus $A_\tau \to A$ as $\tau \to 0$. $\square$

In a similar way it can be shown that, for the formal adjoints, we also have the convergence $A_\tau^\star \to A^\star$ as $\tau \to 0$.

An immediate consequence of this lemma is the following result originally given in [11, Section 1.3].

Corollary 3.16. *For A and A_τ as above, τ sufficiently small, we have*

$$\dim \mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A) = \dim \mathcal{D}_{\max}(A_\tau)/\mathcal{D}_{\min}(A_\tau).$$

Proof. We use the relative index formula (3.7) to obtain

$$\begin{aligned}\operatorname{ind} A_{\tau,\mathcal{D}_{\max}} &= \operatorname{ind} A_{\tau,\mathcal{D}_{\min}} + \dim \mathcal{D}_{\max}(A_\tau)/\mathcal{D}_{\min}(A_\tau),\\ \operatorname{ind} A_{\mathcal{D}_{\max}} &= \operatorname{ind} A_{\mathcal{D}_{\min}} + \dim \mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A).\end{aligned}$$

By construction, $\operatorname{ind} A_{\tau,\mathcal{D}_{\min}} = \operatorname{ind} A_{\mathcal{D}_{\min}}$ and similarly $\operatorname{ind} A^\star_{\tau,\mathcal{D}_{\min}} = \operatorname{ind} A^\star_{\mathcal{D}_{\min}}$ for τ sufficiently small. This implies $\operatorname{ind} A_{\tau,\mathcal{D}_{\max}} = \operatorname{ind} A_{\mathcal{D}_{\max}}$ since $A^\star_{\tau,\mathcal{D}_{\min}}$ and $A^\star_{\mathcal{D}_{\min}}$ are the Hilbert space adjoints of $A_{\tau,\mathcal{D}_{\max}}$ and $A_{\mathcal{D}_{\max}}$, respectively. In conclusion, the quotient spaces must have the same dimension. □

Similarly to the above, we consider extensions of the model operator

$$A_\wedge : C_0^\infty(\mathring{Y}^\wedge; E) \subset x^{-m/2} L_b^2(Y^\wedge; E) \to x^{-m/2} L_b^2(Y^\wedge; E).$$

Let $\mathcal{D}_{\wedge,\min} = \mathcal{D}_{\min}(A_\wedge)$ be the completion of $C_0^\infty(\mathring{Y}^\wedge; E)$ with respect to the norm induced by the inner product

$$(u, v)_{A_\wedge} = (u, v)_{x^{-m/2} L_b^2} + (A_\wedge u, A_\wedge v)_{x^{-m/2} L_b^2}, \tag{3.17}$$

and let

$$\mathcal{D}_{\wedge,\max} = \mathcal{D}_{\max}(A_\wedge) = \{u \in x^{-m/2} L_b^2(Y^\wedge; E) : A_\wedge u \in x^{-m/2} L_b^2(Y^\wedge; E)\}.$$

Then

$$A_\wedge : \mathcal{D}_{\wedge,\max} \subset x^{-m/2} L_b^2(Y^\wedge; E) \to x^{-m/2} L_b^2(Y^\wedge; E)$$

is closed and densely defined, and $\mathcal{D}_{\wedge,\min} \subset \mathcal{D}_{\wedge,\max}$. We have proved in [5] that

$$(1-\omega)\mathcal{D}_{\wedge,\max} = (1-\omega)\mathcal{D}_{\wedge,\min} = (1-\omega)\mathcal{K}^{m,m/2}(Y^\wedge; E)$$

for all cut-off functions $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ near zero, i.e., $\omega = 1$ in a neighborhood of zero and $\omega = 0$ near infinity.

Consequently, near infinity all domains $\mathcal{D}_{\wedge,\min} \subset \mathcal{D}_\wedge \subset \mathcal{D}_{\wedge,\max}$ of $A_\wedge$ coincide with $x^{\frac{n-m}{2}} H^m_{\mathrm{cone}}(Y^\wedge; E)$. On the other hand, near the boundary, the closed extensions of $A_\wedge$ are determined by its boundary spectrum which is the same as the boundary spectrum of A. For this reason, many of the results concerning the closed extensions of A find their analogs in the situation at hand. In fact, using an approximation A_τ as in (3.13) with τ small, one can easily describe the minimal and maximal extensions of $A_\wedge$ on $Y^\wedge$ in terms of those of A_τ on the manifold M. For instance, $u \in \mathcal{D}_{\max}(A_\wedge)$ if and only if $(1-\omega)u \in x^{\frac{n-m}{2}} H^m_{\mathrm{cone}}(Y^\wedge; E)$ and $\omega u \in \mathcal{D}_{\max}(A_\tau)$ for some cut-off function ω with small support and such that $\omega = 1$ near the boundary.

In particular, we have the embeddings

$$\mathcal{K}^{m,m/2}(Y^\wedge; E) \hookrightarrow \mathcal{D}_{\min}(A_\wedge) \hookrightarrow \mathcal{D}_{\max}(A_\wedge) \hookrightarrow \mathcal{K}^{m,-m/2+\varepsilon}(Y^\wedge; E).$$

for some small $\varepsilon > 0$.

Because of (2.8) (with $\lambda = 0$), both $\mathcal{D}_{\min}(A_\wedge)$ and $\mathcal{D}_{\max}(A_\wedge)$ are κ-invariant.

By the previous discussion, the following proposition is a direct consequence of Proposition 3.5 and Corollary 3.16.

Proposition 3.18. *Let $A \in x^{-m} \operatorname{Diff}_b^m(M; E)$ be c-elliptic. Then*

(i) $\mathcal{D}_{\wedge,\min} = \mathcal{D}_{\wedge,\max} \cap \big(\bigcap_{\varepsilon>0} \mathcal{K}^{m,m/2-\varepsilon}(Y^\wedge; E)\big)$;
(ii) $\mathcal{D}_{\wedge,\min} = \mathcal{K}^{m,m/2}(Y^\wedge; E)$ *if and only if* $\operatorname{spec}_b(A) \cap \{\Im\sigma = -m/2\} = \varnothing$;
(iii) $\dim \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} = \dim \mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$.

Finally, we define the background spectrum of $A_\wedge$ as

$$\begin{aligned}\text{bg-spec}\, A_\wedge = \{\lambda\in\mathbb{C} : A_{\wedge,\mathcal{D}_{\wedge,\min}} - \lambda \text{ is not injective, or}\\ A_{\wedge,\mathcal{D}_{\wedge,\max}} - \lambda \text{ is not surjective}\}.\end{aligned}$$

The complement $\text{bg-res}\, A_\wedge = \mathbb{C}\backslash \text{bg-spec}\, A_\wedge$ is the background resolvent set.

4. Ray conditions

The following theorem establishes the necessity of ray conditions on the symbols of A in order to have rays of minimal growth for A on some domain $\mathcal{D}$.

Theorem 4.1. *Let $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$ be c-elliptic. Suppose that there is a domain $\mathcal{D}$, a ray*

$$\Gamma = \{z\in\mathbb{C} : z = re^{i\theta_0} \text{ for } r>0\},$$

and a number $R>0$ such that $A-\lambda : \mathcal{D}\to x^{-m/2}L^2_b(M;E)$ is invertible for all $\lambda\in\Gamma$ with $|\lambda|>R$. Suppose further that for such λ, the resolvent

$$(A_{\mathcal{D}}-\lambda)^{-1} : x^{-m/2}L^2_b(M;E)\to\mathcal{D}$$

is uniformly bounded in λ. Then

$$\text{bg-spec}\, A_\wedge \cap \Gamma = \varnothing \quad \text{and} \quad \operatorname{spec}({}^c\boldsymbol{\sigma}(A))\cap\bar{\Gamma} = \varnothing \text{ on } {}^cT^*M\backslash 0. \tag{4.2}$$

Proof. The hypotheses imply that $A-\lambda : \mathcal{D}_{\min}(A)\to x^{-m/2}L^2_b(M;E)$ is injective for $\lambda\in\Gamma$ and that, in fact, if $u\in\mathcal{D}_{\min}(A)$, then

$$\|(A-\lambda)u\| \geq C\|u\|_A \tag{4.3}$$

for some constant $C>0$. Here $\|\cdot\|$ denotes the norm in $x^{-m/2}L^2_b$ and $\|\cdot\|_A$ is the graph norm. We first prove that

$$A_\wedge - \lambda : \mathcal{D}_{\min}(A_\wedge)\to x^{-m/2}L^2_b(Y^\wedge;E) \text{ is injective.}$$

Note that $\mathcal{D}_{\min}(A_\wedge)$ and $\mathcal{D}_{\max}(A_\wedge)$ are invariant under the dilation κ_ϱ. If $v\in \mathring{C}^\infty_0(Y^\wedge;E)$, then for $\varrho>0$ small, $\kappa^{-1}_\varrho v\in\mathcal{D}_{\min}(A_\wedge)$ is supported near Y, the boundary of $Y^\wedge$, and gives an element $\kappa^{-1}_\varrho v$ of $\mathcal{D}_{\min}(A)$. We have

$$\begin{aligned}\|(\varrho^m\kappa_\varrho A\kappa_\varrho^{-1}-\lambda)v\| &= \varrho^m\|\kappa_\varrho(A-\varrho^{-m}\lambda)\kappa_\varrho^{-1}v\|\\ &= \varrho^m\|(A-\varrho^{-m}\lambda)\kappa_\varrho^{-1}v\|\end{aligned}$$

because κ_ϱ is an isometry. Next, if $A-\lambda$ is injective, then obviously so is $A-\varrho^{-m}\lambda$ for $\varrho\le 1$, and by (4.3),

$$\varrho^m\|(A-\varrho^{-m}\lambda)\kappa_\varrho^{-1}v\| \geq C\varrho^m\|\kappa_\varrho^{-1}v\|_A.$$

But

$$\begin{aligned}\varrho^m\|\kappa_\varrho^{-1}v\|_A &= \varrho^m\|\kappa_\varrho^{-1}v\| + \varrho^m\|A\kappa_\varrho^{-1}v\|\\ &= \varrho^m\|v\| + \|\varrho^m\kappa_\varrho A\kappa_\varrho^{-1}v\|\end{aligned}$$

using again that κ_ϱ is an isometry. Thus

$$\|(\varrho^m\kappa_\varrho A\kappa_\varrho^{-1}-\lambda)v\| \geq C\big(\varrho^m\|v\| + \|\varrho^m\kappa_\varrho A\kappa_\varrho^{-1}v\|\big)$$

for some $C>0$ and all small ϱ. In view of the definition of $A_\wedge$, taking the limit as $\varrho\to 0$ we arrive at

$$\|(A_\wedge-\lambda)v\| \geq C\|A_\wedge v\| \tag{4.4}$$

for all $v \in C_0^\infty(\mathring{Y}^\wedge;E)$. Now, for $v \in \mathcal{D}_{\min}(A_\wedge)$ there is a sequence $\{v_k\} \subset C_0^\infty(\mathring{Y}^\wedge;E)$ such that $v_k \to v$ and $A_\wedge v_k \to A_\wedge v$ in $x^{-m/2}L_b^2$ as $k \to \infty$, so $(A_\wedge - \lambda)v_k \to (A_\wedge - \lambda)v$ in $x^{-m/2}L_b^2$. Thus, since (4.4) holds for the v_k, it holds for any $v \in \mathcal{D}_{\min}(A_\wedge)$.

The estimate (4.4) implies the injectivity of $A_\wedge - \lambda$ on $\mathcal{D}_{\min}(A_\wedge)$ for $\lambda \neq 0$. Indeed, if $(A_\wedge - \lambda)v = 0$, then $A_\wedge v = 0$, so $\lambda v = 0$. Thus $v = 0$ since $\lambda \neq 0$.

The surjectivity of $A_\wedge - \lambda : \mathcal{D}_{\max}(A_\wedge) \to x^{-m/2}L_b^2(Y^\wedge;E)$ follows from the injectivity of $A_\wedge^\star - \overline{\lambda} : \mathcal{D}_{\min}(A_\wedge^\star) \to x^{-m/2}L_b^2(Y^\wedge;E)$. The latter is a consequence of the injectivity of $(A^\star - \overline{\lambda})$ on $\mathcal{D}_{\min}(A^\star)$ for $\lambda \in \Gamma$ and the above argument. This proves the first assertion in (4.2).

We now prove the second assertion. Since A is c-elliptic, $A_\wedge$ is elliptic in the usual sense in the interior of $Y^\wedge$. So the usual elliptic *a priori* estimate holds in compact subsets of $\mathring{Y}^\wedge$. Thus there is a constant $C > 0$ such that

$$\|v\|_{\mathcal{K}^{m,m/2}} \le C\big(\|A_\wedge v\| + \|v\|\big)$$

for $v \in \mathcal{K}^{m,m/2}(Y^\wedge;E)$, $\operatorname{supp} v \subset \{1 \le x \le 2\} \times Y$. The inequality (4.4) now gives

$$\|v\|_{\mathcal{K}^{m,m/2}} \le C\big(\|(A_\wedge - \lambda)v\| + \|v\|\big) \tag{4.5}$$

for $v \in \mathcal{K}^{m,m/2}(Y^\wedge;E)$, $\operatorname{supp} v \subset \{1 \le x \le 2\} \times Y$, with some C independent of λ. By standard arguments (see e.g. Seeley [22]) this gives that $\boldsymbol{\sigma}(A_\wedge) - \lambda$ is invertible for $\lambda \in \Gamma$ when $1 \le x \le 2$. But

$$\boldsymbol{\sigma}(A_\wedge)(x,y;\xi,\eta) - \lambda = x^{-m}\big({}^c\boldsymbol{\sigma}(A_\wedge)(y;x\xi,\eta) - x^m\lambda\big).$$

In this formula we made use of the fact that the c-symbol of $A_\wedge$ is independent of x. Replacing $x\xi$ by ξ and $x^m\lambda$ by λ, and using that ${}^c\boldsymbol{\sigma}(A_\wedge) = {}^c\boldsymbol{\sigma}(A)|_Y$ we reach the conclusion that

$${}^c\boldsymbol{\sigma}(A) - \lambda$$

is invertible over Y, and therefore over a neighborhood of Y in M, when $\lambda \in \Gamma$. The hypothesis on A also implies estimates like (4.5) for A on compact subsets of the interior of M. Thus also $\boldsymbol{\sigma}(A) - \lambda$ is invertible over compact subsets of the interior of M when $\lambda \in \Gamma$. This gives the second statement in (4.2). □

The following is a partial converse of Theorem 4.1.

Theorem 4.6. *Let $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ be c-elliptic. If* (4.2) *holds, then there exists a domain $\mathcal{D}$ such that* $\operatorname{spec} A_{\mathcal{D}}$ *is discrete.*

Proof. We will use the parametrix from Section 5 to prove the statement. First of all, the compactness of M and the spectral condition on the symbol ${}^c\boldsymbol{\sigma}(A)$ imply that there exists some closed sector Λ with $\Gamma \subset \mathring{\Lambda}$ such that $\operatorname{spec}({}^c\boldsymbol{\sigma}(A)) \cap \Lambda = \varnothing$ on ${}^cT^*M\backslash 0$. Consequently, $A - \lambda$ is c-elliptic with parameter $\lambda \in \Lambda$, cf. Definition 2.1. We choose Λ in such a way that $\Lambda\backslash\{0\} \subset \operatorname{bg-res} A_\wedge$ also holds; this is possible because $\operatorname{bg-res} A_\wedge$ is a union of open sectors, see [5]. Then, for $\lambda \in \Lambda\backslash\{0\}$, we also have that $A_\wedge - \lambda : \mathcal{D}_{\min}(A_\wedge) \to x^{-m/2}L_b^2(Y^\wedge;E)$ is injective and therefore, by Theorem 5.29,

$$A - \lambda : \mathcal{D}_{\min}(A) \to x^{-m/2}L_b^2(M;E)$$

is injective for λ sufficiently large.

On the other hand, the surjectivity of $A_\wedge - \lambda : \mathcal{D}_{\max}(A_\wedge) \to x^{-m/2}L_b^2(Y^\wedge;E)$ implies the injectivity of $A_\wedge^\star - \bar{\lambda}$ on $\mathcal{D}_{\min}(A_\wedge^\star)$. Since $A^\star - \bar{\lambda}$ is also c-elliptic with parameter $\bar{\lambda}$ in the complex conjugate of Λ, we can use Theorem 5.29 with $A^\star$

instead of A to conclude that $A^\star - \bar{\lambda} : \mathcal{D}_{\min}(A^\star) \to x^{-m/2}L^2_b(M;E)$ is injective for $\bar{\lambda}$ sufficiently large. Thus, for such λ, we get the surjectivity of

$$A - \lambda : \mathcal{D}_{\max}(A) \to x^{-m/2}L^2_b(M;E).$$

In conclusion, for λ large, $A-\lambda$ is injective on $\mathcal{D}_{\min}$ and surjective on $\mathcal{D}_{\max}$, hence there exists a domain $\mathcal{D}$ such that

$$A_{\mathcal{D}} - \lambda : \mathcal{D} \to x^{-m/2}L^2_b(M;E)$$

is invertible. Thus $\operatorname{spec} A_{\mathcal{D}} \neq \mathbb{C}$, so it must be discrete. □

Observe that for $\lambda \in \Gamma$, $|\lambda| > R > 0$, the norm $\|(A_{\mathcal{D}} - \lambda)^{-1}\|_{\mathscr{L}(x^{-m/2}L^2_b(M;E),\mathcal{D})}$ is uniformly bounded if and only if

$$\|(A_{\mathcal{D}} - \lambda)^{-1}\|_{\mathscr{L}(x^{-m/2}L^2_b(M;E))} = O(|\lambda|^{-1}) \text{ as } |\lambda| \to \infty.$$

Stronger and more precise statements about resolvents of elliptic cone operators will be given in Section 6.

5. Parametrix construction

In this section we assume Λ to be a closed sector in $\mathbb{C}$ of the form

$$\Lambda = \{z \in \mathbb{C} : z = re^{i\theta} \text{ for } r \geq 0,\ \theta \in \mathbb{R},\ |\theta - \theta_0| \leq a\}$$

for some real θ_0 and $a > 0$. We assume that $A - \lambda$ is c-elliptic with parameter $\lambda \in \Lambda$ according to Definition 2.1, and that

$$A_\wedge - \lambda : \mathcal{D}_{\min}(A_\wedge) \to x^{-m/2}L^2_b(Y^\wedge;E) \text{ is injective if } \lambda \in \Lambda\backslash\{0\}. \tag{5.1}$$

Our goal is to construct a parameter-dependent parametrix of

$$A - \lambda : \mathcal{D}_{\min}(A) \to x^{-m/2}L^2_b(M;E) \tag{5.2}$$

by means of three crucial steps that we proceed to outline.

Step 1: The first step is concerned with the construction of a pseudodifferential parametrix $B_1(\lambda)$ of $A - \lambda : C_0^\infty(\overset{\circ}{M};E) \to C_0^\infty(\overset{\circ}{M};E)$ taking care of the degeneracy of the complete symbol of $A-\lambda$ near the boundary of M. The parametrix $B_1(\lambda)$ is constructed within a corresponding (sub)calculus of parameter-dependent pseudodifferential operators that are built upon degenerate symbols.

Step 2: In the second step the parametrix $B_1(\lambda)$ is refined to a parametrix

$$B_2(\lambda) : x^{-m/2}L^2_b(M;E) \to \mathcal{D}_{\min}(A)$$

which is continuous and pointwise a Fredholm inverse of $A-\lambda$. The remainders

$$B_2(\lambda)(A-\lambda) - 1 : \mathcal{D}_{\min}(A) \to \mathcal{D}_{\min}(A), \tag{5.3}$$

$$(A-\lambda)B_2(\lambda) - 1 : x^{-m/2}L^2_b(M;E) \to x^{-m/2}L^2_b(M;E) \tag{5.4}$$

are parameter-dependent smoothing pseudodifferential operators in

$$C_0^\infty(\overset{\circ}{M};E) \to C^\infty(\overset{\circ}{M};E)$$

since $B_2(\lambda)$ is a refinement of $B_1(\lambda)$, but the operator norms in the spaces (5.3) and (5.4) are not decreasing as $|\lambda| \to \infty$.

Step 3: While in the first two steps we only make use of the c-ellipticity with parameter, we now need the additional requirement that (5.1) holds. In view of the κ-homogeneity of $A_\wedge - \lambda$,

$$A_\wedge - \varrho^m\lambda = \varrho^m \kappa_\varrho (A_\wedge - \lambda)\kappa_\varrho^{-1} \text{ for } \lambda \neq 0,\ \varrho > 0,$$

we only need to require (5.1) for $|\lambda| = 1$. Recall that the minimal domain $\mathcal{D}_{\min}(A_\wedge)$ is invariant under the action of κ_ϱ.

Under the additional assumption (5.1) we will refine $B_2(\lambda)$ to obtain a parameter-dependent parametrix $B(\lambda)$ such that

$$B(\lambda)(A-\lambda) - 1 : \mathcal{D}_{\min}(A) \to \mathcal{D}_{\min}(A)$$

is compactly supported in $\lambda \in \Lambda$. In particular, for λ sufficiently large the operator family $A - \lambda : \mathcal{D}_{\min}(A) \to x^{-m/2}L^2_b(M;E)$ is injective, and the parametrix $B(\lambda)$ is a left-inverse. Moreover, for λ large, the smoothing remainder

$$\Pi(\lambda) = 1 - (A-\lambda)B(\lambda)$$

is a projection on $x^{-m/2}L^2_b(M;E)$ to a complement of the range of $A-\lambda$ on $\mathcal{D}_{\min}(A)$, i.e., $(A-\lambda)B(\lambda)$ is a projection onto $\mathrm{rg}(A_{\min} - \lambda)$.

For the final construction of $B(\lambda)$ we adopt Schulze's viewpoint from the pseudodifferential edge-calculus, see e.g. [19, 20], and add extra conditions of trace and potential type within a suitably defined class of Green remainders.

We now proceed to construct a suitable parametrix of $A - \lambda$ as outlined above. The first step is the parametrix construction in the interior of the manifold, assuming only that $A - \lambda$ is c-elliptic with parameter in a closed sector $\Lambda \subset \mathbb{C}$.

On M we fix a collar neighborhood diffeomorphic to $[0,1) \times Y$, $Y = \partial M$, and consider local coordinates of the form $[0,1)\times\Omega \subset \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}$ near the boundary, where $\Omega \subset \mathbb{R}^{n-1}$ corresponds to a chart on Y. Moreover, these coordinates are chosen in such a way that the push-forward of the vector bundle E is trivial on $[0,1)\times\Omega$ (e.g., choose Ω contractible).

In these coordinates the operator $A - \lambda$ takes the form

$$A - \lambda = x^{-m}\Big(\sum_{k+|\alpha|\le m} a_{k\alpha}(x,y) D_y^\alpha (xD_x)^k - x^m\lambda\Big), \tag{5.5}$$

where the $a_{k\alpha}$ are smooth matrix-valued coefficients on $[0,1)\times\Omega$. The c-ellipticity with parameter of the family $A - \lambda$ implies that, in the interior of M, it is elliptic with parameter in the usual sense, and in local coordinates near the boundary,

$$\sum_{k+|\alpha|=m} a_{k\alpha}(x,y)\eta^\alpha\xi^k - \lambda$$

is invertible for all $(\xi,\eta,\lambda) \in \big(\mathbb{R}\times\mathbb{R}^{n-1}\times\Lambda\big)\backslash\{0\}$ and $(x,y)\in[0,1)\times\Omega$.

From equation (5.5) we deduce that the complete symbol of $A-\lambda$ in $(0,1)\times\Omega$ is of the form $x^{-m}a(x,y,x\xi,\eta,x^m\lambda)$ for some parameter-dependent classical symbol $a(x,y,\xi,\eta,\lambda)$ of order m, and the c-ellipticity condition near the boundary is equivalent to the invertibility of the principal component $a_{(m)}(x,y,\xi,\eta,\lambda)$ of a. These observations give rise to the class of parameter-dependent pseudodifferential operators that we will consider below.

For the rest of this section we will work (without loss of generality) with scalar symbols; the general case of matrix-valued symbols is straightforward.

Sometimes we will denote the variables in $(0,1)\times\Omega$ by $z=(x,y)$ and $z'=(x',y')$, and the corresponding covariables in $\mathbb{R}^n$ by $\zeta=(\xi,\eta)\in\mathbb{R}\times\mathbb{R}^{n-1}$.

Definition 5.6. For $\mu\in\mathbb{R}$ let $\Psi^\mu(\Lambda)$ denote the space of all pseudodifferential operators

$$A(\lambda): C_0^\infty((0,1)\times\Omega)\to C^\infty((0,1)\times\Omega)$$

depending on the parameter $\lambda\in\Lambda$ of the form

$$A(\lambda)u(z)=\frac{1}{(2\pi)^n}\iint e^{i(z-z')\cdot\zeta}\,\tilde a(z,\zeta,\lambda)u(z')\,dz'\,d\zeta+C(\lambda)u(z) \tag{5.7}$$

for $z,z'\in(0,1)\times\Omega$, $\zeta\in\mathbb{R}^n$, where the family $C(\lambda)\in\Psi^{-\infty}(\Lambda)$ is a parameter-dependent smoothing operator of the form

$$C(\lambda)u(z)=\int k(z,z',\lambda)u(z')\,dz'$$

with rapidly decreasing integral kernel $k(z,z',\lambda)\in\mathscr{S}(\Lambda,C^\infty((0,1)\times\Omega\times(0,1)\times\Omega))$, and where the symbol $\tilde a(z,\zeta,\lambda)=\tilde a(x,y,\xi,\eta,\lambda)$ satisfies

$$\tilde a(x,y,\xi,\eta,\lambda)=x^{-\mu}a(x,y,x\xi,\eta,x^d\lambda)$$

with $a(x,y,\xi,\eta,\lambda)\in C^\infty([0,1)\times\Omega\times\mathbb{R}\times\mathbb{R}^{n-1}\times\Lambda)$ satisfying for all multi-indices α, β, and γ, the symbol estimates

$$|\partial^\alpha_{(x,y)}\partial^\beta_{(\xi,\eta)}\partial^\gamma_\lambda a(x,y,\xi,\eta,\lambda)|=O\big(\big(1+|\xi|+|\eta|+|\lambda|^{1/d}\big)^{\mu-|\beta|-d|\gamma|}\big)$$

as $|(\xi,\eta,\lambda)|\to\infty$, locally uniformly for $(x,y)\in[0,1)\times\Omega$. Here $d\in\mathbb{N}$ is a fixed parameter for the class $\Psi^\infty(\Lambda)$ which refers to the anisotropy; in the case of the operator $A-\lambda$ we have $d=m=\operatorname{ord}(A)$. Moreover, the symbol $a(x,y,\xi,\eta,\lambda)$ is assumed to be classical: It admits an asymptotic expansion

$$a\sim\sum_{j=0}^\infty\chi(\xi,\eta,\lambda)a_{(\mu-j)}(x,y,\xi,\eta,\lambda), \tag{5.8}$$

where $\chi\in C^\infty(\mathbb{R}\times\mathbb{R}^{n-1}\times\Lambda)$ is a function such that $\chi=0$ near the origin and $\chi=1$ for $|(\xi,\eta,\lambda)|$ large, and the components $a_{(\mu-j)}(x,y,\xi,\eta,\lambda)$ satisfy the homogeneity relation

$$a_{(\mu-j)}(x,y,\varrho\xi,\varrho\eta,\varrho^d\lambda)=\varrho^{\mu-j}a_{(\mu-j)}(x,y,\xi,\eta,\lambda)$$

for $\varrho>0$ and $(\xi,\eta,\lambda)\in(\mathbb{R}\times\mathbb{R}^{n-1}\times\Lambda)\backslash\{0\}$. The parameter-dependent principal symbol of $A(\lambda)$ is then given by $x^{-\mu}a_{(\mu)}(x,y,x\xi,\eta,x^d\lambda)$.

Note that the symbol $a(x,y,\xi,\eta,\lambda)$ is smooth in x up to $x=0$.

Proposition 5.9. *Let $A(\lambda)\in\Psi^{\mu_1}(\Lambda)$ and $B(\lambda)\in\Psi^{\mu_2}(\Lambda)$ with either $A(\lambda)$ or $B(\lambda)$ being properly supported, uniformly in $\lambda\in\Lambda$. Then the composition*

$$A(\lambda)B(\lambda): C_0^\infty((0,1)\times\Omega)\to C^\infty((0,1)\times\Omega)$$

belongs to $\Psi^{\mu_1+\mu_2}(\Lambda)$.

Proof. Let $\tilde a(x,y,\xi,\eta,\lambda)$ and $\tilde b(x,y,\xi,\eta,\lambda)$ be complete symbols associated with $A(\lambda)$ and $B(\lambda)$ according to (5.7). Then the corresponding complete symbol of the composition has the asymptotic expansion

$$\sum_{k+|\alpha|=0}^\infty\frac{1}{k!\alpha!}\partial_\xi^k\partial_\eta^\alpha\tilde a(x,y,\xi,\eta,\lambda)D_x^kD_y^\alpha\tilde b(x,y,\xi,\eta,\lambda). \tag{5.9a}$$

Now write

$$\tilde{a}(x,y,\xi,\eta,\lambda) = x^{-\mu_1} a(x,y,x\xi,\eta,x^d\lambda),$$
$$\tilde{b}(x,y,\xi,\eta,\lambda) = x^{-\mu_2} b(x,y,x\xi,\eta,x^d\lambda)$$

with a and b as in Definition 5.6. This gives

$$\partial_\xi^k \partial_\eta^\alpha \tilde{a}(x,y,\xi,\eta,\lambda) = x^{-\mu_1}\big(\partial_\xi^k \partial_\eta^\alpha a\big)(x,y,x\xi,\eta,x^d\lambda)x^k.$$

Since $(xD_x)D_y^\alpha \tilde{b}(x,y,\xi,\eta,\lambda)$ equals

$$x^{-\mu_2}\big((i\mu_2 + xD_x + \xi D_\xi + d\lambda_1 D_{\lambda_1} + d\lambda_2 D_{\lambda_2})D_y^\alpha b\big)(x,y,x\xi,\eta,x^d\lambda),$$

and since $x^k D_x^k = \sum\limits_{j=0}^{k} c_{kj}(xD_x)^j$ with some universal constants c_{kj}, we see that each term in the asymptotic expansion (**??**a) is of the form

$$\frac{1}{k!\alpha!}\partial_\xi^k \partial_\eta^\alpha \tilde{a}(x,y,\xi,\eta,\lambda) D_x^k D_y^\alpha \tilde{b}(x,y,\xi,\eta,\lambda) = x^{-(\mu_1+\mu_2)} p_{k,\alpha}(x,y,x\xi,\eta,x^d\lambda)$$

with a parameter-dependent symbol $p_{k,\alpha}$ of order $\mu_1+\mu_2-k-|\alpha|$ that satisfies the conditions of Definition 5.6. In conclusion, if p is such that

$$p(x,y,\xi,\eta,\lambda) \sim \sum_{k+|\alpha|=0}^{\infty} p_{k,\alpha}(x,y,\xi,\eta,\lambda),$$

then $x^{-(\mu_1+\mu_2)}p(x,y,x\xi,\eta,x^d\lambda)$ is a complete symbol of the composition $A(\lambda)B(\lambda)$ and the proposition follows. □

Definition 5.10. Let $A(\lambda) \in \Psi^\mu(\Lambda)$ have principal symbol $x^{-\mu} a_{(\mu)}(x,y,x\xi,\eta,x^d\lambda)$. The family $A(\lambda)$ is said to be c-elliptic with parameter $\lambda \in \Lambda$ if $a_{(\mu)}(x,y,\xi,\eta,\lambda)$ is invertible for all $(x,y) \in [0,1) \times \Omega$ and $(\xi,\eta,\lambda) \in (\mathbb{R} \times \mathbb{R}^{n-1} \times \Lambda)\backslash\{0\}$.

Proposition 5.11. *For $A(\lambda) \in \Psi^\mu(\Lambda)$ the following are equivalent:*

(*i*) *$A(\lambda)$ is c-elliptic with parameter $\lambda \in \Lambda$.*

(*ii*) *There exists a parametrix $Q(\lambda) \in \Psi^{-\mu}(\Lambda)$, properly supported (uniformly in λ), such that $A(\lambda)Q(\lambda)-1$ and $Q(\lambda)A(\lambda)-1$ both belong to $\Psi^{-\infty}(\Lambda)$.*

Proof. For the proof we need the auxiliary operator class $\Psi^{\mu,0}(\Lambda) = x^\mu \Psi^\mu(\Lambda)$. From the proof of Proposition 5.9 it is easy to see that composition gives rise to a map

$$\Psi^{\mu_1,0}(\Lambda) \times \Psi^{\mu_2,0}(\Lambda) \to \Psi^{\mu_1+\mu_2,0}(\Lambda)$$

provided that one of the factors is properly supported (uniformly in λ). Actually, it is not necessary to couple the weight factor and the order of the operators as it is done for the elements of $\Psi^\mu(\Lambda)$.

Let $A(\lambda) \in \Psi^\mu(\Lambda)$ be c-elliptic with parameter. Without loss of generality assume that $A(\lambda)$ is properly supported, uniformly in λ. Let $Q'(\lambda) \in \Psi^{-\mu}(\Lambda)$ be properly supported with complete symbol $x^\mu(\chi \cdot a_{(\mu)}^{-1})(x,y,x\xi,\eta,x^d\lambda)$, where χ is as in (5.8). Thus $A(\lambda)Q'(\lambda)-1$ and $Q'(\lambda)A(\lambda)-1$ both belong to $\Psi^{-1,0}(\Lambda)$ and are properly supported, uniformly in λ. For $k \in \mathbb{N}$ let $r_k(x,y,\xi,\eta,\lambda)$ be of order $-k$ such that $r_k(x,y,x\xi,\eta,x^d\lambda)$ is a complete symbol of $(Q'(\lambda)A(\lambda)-1)^k \in \Psi^{-k,0}(\Lambda)$. Let $r(x,y,\xi,\eta,\lambda)$ be of order -1 such that

$$r(x,y,\xi,\eta,\lambda) \sim \sum_{k=1}^{\infty}(-1)^k r_k(x,y,\xi,\eta,\lambda),$$

and let $R'(\lambda) \in \Psi^{-1,0}(\Lambda)$ be a properly supported operator having $r(x,y,x\xi,\eta,x^d\lambda)$ as complete symbol. Then

$$(1+R'(\lambda))Q'(\lambda)A(\lambda) - 1 \in \bigcap_{k\in\mathbb{N}} \Psi^{-k,0}(\Lambda) = \Psi^{-\infty}(\Lambda),$$

so $(1+R'(\lambda))Q'(\lambda) \in \Psi^{-\mu}(\Lambda)$ is a left parametrix of $A(\lambda)$. In the same way we obtain a right parametrix. The other direction of the proposition is immediate. □

We now pass to the collar neighborhood $[0,1)\times Y \subset M$: The restriction of the bundle E to $[0,1)\times Y$ is isomorphic to the pull-back of a bundle on Y. For simplicity, we also denote this bundle by E. The sections of E over $[0,1)\times Y$ are then represented as $C^\infty([0,1), C^\infty(Y;E))$. We consider families of pseudodifferential operators

$$A(\lambda) : C_0^\infty((0,1), C^\infty(Y;E)) \to C^\infty((0,1), C^\infty(Y;E))$$

on $(0,1)\times Y$ which depend anisotropically on the parameter $\lambda \in \Lambda$. With respect to the fixed splitting of variables these operators can be written as

$$A(\lambda)u(x) = \frac{1}{2\pi} \iint e^{i(x-x')\xi} \tilde{a}(x,\xi,\lambda)u(x')\,dx'\,d\xi + C(\lambda)u(x) \tag{5.12}$$

for $x, x' \in (0,1)$, $\xi \in \mathbb{R}$, where $C(\lambda) \in \Psi^{-\infty}(\Lambda)$ is a parameter-dependent smoothing operator

$$C(\lambda)u(x) = \int k(x,x',\lambda)u(x')\,dx'$$

with integral kernel $k(x,x',\lambda) \in \mathscr{S}(\Lambda, C^\infty((0,1)\times(0,1), L^{-\infty}(Y)))$. As in the local case, cf. Definition 5.6, we use here the notation $\Psi^{-\infty}(\Lambda)$ for the remainder class.

Moreover, the symbol $\tilde{a}(x,\xi,\lambda)$ is a smooth function of $x \in (0,1)$ taking values in the space $L^{\mu,(1,d)}(Y;\mathbb{R}\times\Lambda)$ of pseudodifferential operators of order $\mu \in \mathbb{R}$ on Y depending on the parameters $(\xi,\lambda) \in \mathbb{R}\times\Lambda$. Recall that a family of operators

$$B(\xi,\lambda) : C^\infty(Y;E) \to C^\infty(Y;E)$$

belongs to $L^{\mu,(1,d)}(Y;\mathbb{R}\times\Lambda)$ if, in a local patch Ω, it is of the form

$$B(\xi,\lambda)u(y) = \frac{1}{(2\pi)^{n-1}} \iint e^{i(y-y')\eta} b(y,\xi,\eta,\lambda)u(y')\,dy'\,d\eta + D(\xi,\lambda)u(y)$$

for $y, y' \in \Omega$, $\eta \in \mathbb{R}^{n-1}$, where

$$D(\xi,\lambda)u(y) = \int c(y,y',\xi,\lambda)u(y')\,dy'$$

with integral kernel $c(y,y',\xi,\lambda) \in \mathscr{S}(\mathbb{R}\times\Lambda, C^\infty(\Omega\times\Omega))$, and where the symbol $b(y,\xi,\eta,\lambda)$ satisfies the symbol estimates of Definition 5.6 (but here of course independent of x).

As before, we do not consider general families of pseudodifferential operators on $(0,1)\times Y$ and restrict ourselves to operators in $\Psi^\mu(\Lambda)$ where the symbol $\tilde{a}(x,\xi,\lambda)$ in (5.12) is required to be of the form

$$\tilde{a}(x,\xi,\lambda) = x^{-\mu} a(x, x\xi, x^d\lambda),$$

where $a(x,\xi,\lambda)$ is smooth in $x \in [0,1)$ with values in $L^{\mu,(1,d)}(Y;\mathbb{R}\times\Lambda)$. Observe that this is precisely the class of operators that is obtained via globalizing the local classes from Definition 5.6 to the collar neighborhood $(0,1)\times Y$.

The parameter-dependent homogeneous principal symbol of an operator in $\Psi^\mu(\Lambda)$ extends to an anisotropic homogeneous section on $({}^cT^*([0,1)\times Y)\times\Lambda)\backslash 0$, and the global meaning of the c-ellipticity from Definition 5.10 is the invertibility of the principal symbol there. By patching together local parametrices from Proposition 5.11, we get the following:

Proposition 5.13. *There exists a parametrix $Q(\lambda) \in \Psi^{-m}(\Lambda)$ of $A-\lambda$ which is properly supported (uniformly in λ) and has the form*

$$Q(\lambda)u(x) = \frac{1}{2\pi} \iint e^{i(x-x')\xi} \tilde{p}(x,\xi,\lambda) u(x')\, dx'\, d\xi + C(\lambda)$$

for $x, x' \in (0,1)$, $\xi \in \mathbb{R}$, with $\tilde{p}(x,\xi,\lambda) = x^m p(x, x\xi, x^m\lambda)$ and $C(\lambda)$ as in (5.12).

We are finally ready to construct a parameter-dependent parametrix $B_1(\lambda)$ of $A-\lambda$ on M. The important aspect of the following theorem is the structure of the complete symbol of $B_1(\lambda)$ close to the boundary of M.

Theorem 5.14. *Let $Q_{\mathrm{int}}(\lambda)$ be a standard parameter-dependent parametrix of $A-\lambda$ on $\mathring{M}$ and let $Q(\lambda) \in \Psi^{-m}(\Lambda)$ be the parametrix of $A-\lambda$ from Proposition 5.13. Then for any cut-off functions $\omega, \omega_0, \omega_1 \in C_0^\infty([0,1))$ with $\omega_1 \prec \omega \prec \omega_0$, the properly supported pseudodifferential operator*

$$B_1(\lambda) = \omega Q(\lambda)\omega_0 + (1-\omega) Q_{\mathrm{int}}(\lambda)(1-\omega_1)$$

is a parametrix of $A-\lambda$ on M.

Recall that a cut-off function $\omega \in C_0^\infty([0,1))$ is a function which equals 1 in a neighborhood of the origin. Observe that these functions can also be considered as functions on M supported in the collar neighborhood $[0,1)\times Y$ of the boundary. Moreover, we use the notation $\varphi \prec \psi$ to indicate that the function ψ equals 1 in a neighborhood of the support of the function φ, in particular, $\varphi\psi = \varphi$.

The second step in our parametrix construction concerns the refinement of $B_1(\lambda)$ from Theorem 5.14 to a Fredholm inverse of $A-\lambda$. First of all, we want to modify $B_1(\lambda)$ in order to get a family of bounded operators

$$B_1(\lambda) : x^{-m/2} H_b^s(M;E) \to \mathcal{D}_{\min}^s(A)$$

for any $s \in \mathbb{R}$, where $\mathcal{D}_{\min}^s(A)$ denotes the minimal domain of A in $x^{-m/2}H_b^s(M;E)$, cf. Section 3. Recall that for every $t \in \mathbb{R}$,

$$x^{m/2} H_b^{t+m}(M;E) \hookrightarrow \mathcal{D}_{\min}^t \hookrightarrow x^{-m/2+\varepsilon} H_b^{t+m}(M;E).$$

Also, we use the notation $\mathcal{D}_{\min}(A) = \mathcal{D}_{\min}^0(A)$.

By Mellin quantization, one can easily modify $B_1(\lambda)$ in such a way that

$$B_1(\lambda) : x^{-m/2} H_b^s(M;E) \to x^{m/2} H_b^{s+m}(M;E)$$

is bounded for every $s \in \mathbb{R}$. Mellin representations of pseudodifferential operators are standard. The following proposition is a direct consequence of known results about the Mellin quantization that can be found for instance in [8].

Proposition 5.15. *Let $Q(\lambda)$ be the parametrix of $A-\lambda$ from Proposition 5.13 defined via the symbol $p(x,\xi,\lambda)$. Let*

$$h(x,\sigma,\lambda) = \frac{1}{2\pi} \iint e^{-i(r-1)\xi} r^{i\sigma} \varphi(r) p(x,\xi,\lambda)\, dr\, d\xi$$

for $r, x, \xi \in \mathbb{R}$*,* $\sigma \in \mathbb{C}$*, where* $\varphi \in C_0^\infty(\mathbb{R}_+)$ *is a function such that* $\varphi = 1$ *near* $r = 1$*. If we redefine* $Q(\lambda)$ *as*

$$Q(\lambda)u(x) = \frac{1}{2\pi i} \int\limits_{\Im\sigma = m/2} \int\limits_{(0,1)} \Big(\frac{x}{x'}\Big)^{i\sigma} x^m h(x,\sigma,x^m\lambda)u(x')\,\frac{dx'}{x'}\,d\sigma,$$

then the corresponding family $B_1(\lambda)$ *from Theorem 5.14 is again a properly supported parametrix of* $A - \lambda$ *such that, in addition,*

$$B_1(\lambda) : x^{-m/2}H_b^s(M;E) \to x^{m/2}H_b^{s+m}(M;E) \hookrightarrow \mathcal{D}^s_{\min}(A)$$

is bounded for every $s \in \mathbb{R}$*.*

Our goal in this second step is to refine this parameter-dependent parametrix in such a way that the remainders are elements of order zero in a suitable class of Green operators that will be defined below. To this end we consider scales of Hilbert spaces $\{\mathcal{E}^s\}_{s\in\mathbb{R}}$ on M and associated scales $\{\mathcal{E}^{s,\delta}_\wedge\}_{s,\delta\in\mathbb{R}}$ on $Y^\wedge$ as follows: Either $\mathcal{E}^s = x^\gamma H_b^s(M;E)$ for some weight $\gamma \in \mathbb{R}$, or $\mathcal{E}^s = \mathcal{D}^{s-m}_{\min}(A)$. With the Sobolev spaces $\mathcal{E} = x^\gamma H$ we associate

$$\mathcal{E}^{s,\delta}_\wedge = \omega\big(x^\gamma H_b^s(Y^\wedge;E)\big) + (1-\omega)\big(x^{\frac{n-m}{2}-\delta}H^s_{\mathrm{cone}}(Y^\wedge;E)\big),$$

and for the scale of minimal domains $\mathcal{E} = \mathcal{D}_{\min}$ we define

$$\mathcal{E}^{s,\delta}_\wedge = \omega\mathcal{D}^{s-m}_{\min}(A_\wedge) + (1-\omega)\big(x^{\frac{n-m}{2}-\delta}H^s_{\mathrm{cone}}(Y^\wedge;E)\big).$$

Here $\omega \in C_0^\infty([0,1))$ denotes, as usual, a cut-off function near the origin. Note that in the latter case we have $\mathcal{E}^{m,0}_\wedge = \mathcal{D}_{\min}(A_\wedge)$. Recall that $n = \dim M$.

Definition 5.16. An operator family $G(\lambda) : C_0^\infty(\mathring{M};E) \to C^\infty(\mathring{M};E)$ is called a *Green remainder* of order $\mu \in \mathbb{R}$ with respect to the scales $(\mathcal{E},\mathcal{F})$ if for all cut-off functions $\omega, \tilde\omega \in C_0^\infty([0,1))$ the following holds:

(*i*) $(1-\omega)G(\lambda),\ G(\lambda)(1-\tilde\omega) \in \bigcap\limits_{s,t\in\mathbb{R}} \mathscr{S}(\Lambda, \mathcal{K}(\mathcal{E}^s,\mathcal{F}^t))$;

(*ii*) $g(\lambda) = \omega G(\lambda)\tilde\omega : C_0^\infty(\mathring{Y}^\wedge;E) \to C^\infty(\mathring{Y}^\wedge;E)$ is a *Green symbol*, i.e., a classical operator-valued symbol of order $\mu \in \mathbb{R}$ in the sense that

$$g(\lambda) \in \bigcap_{s,t,\delta,\delta'\in\mathbb{R}} C^\infty(\Lambda, \mathcal{K}(\mathcal{E}^{s,\delta}_\wedge, \mathcal{F}^{t,\delta'}_\wedge)),$$

and for all multi-indices $\alpha \in \mathbb{N}_0^2$,

$$\Big\|\kappa^{-1}_{[\lambda]^{1/m}}\partial^\alpha_\lambda g(\lambda)\kappa_{[\lambda]^{1/m}}\Big\|_{\mathcal{K}(\mathcal{E}^{s,\delta}_\wedge,\mathcal{F}^{t,\delta'}_\wedge)} = O(|\lambda|^{\mu/m-|\alpha|}) \tag{5.17}$$

as $|\lambda| \to \infty$. Here $\mathcal{K}(\mathcal{E}^s,\mathcal{F}^t)$ denotes the space of compact operators from $\mathcal{E}^s$ to $\mathcal{F}^t$, and $[\cdot]$ is a strictly positive smoothing of the absolute value $|\cdot|$ near the origin. Without loss of generality we may assume $[\lambda] > 1$ for every λ.

Moreover, for $j \in \mathbb{N}_0$ there exist

$$g_{(\mu-j)}(\lambda) \in \bigcap_{s,t,\delta,\delta'\in\mathbb{R}} C^\infty(\Lambda\backslash\{0\}, \mathcal{K}(\mathcal{E}^{s,\delta}_\wedge, \mathcal{F}^{t,\delta'}_\wedge))$$

such that

$$g_{(\mu-j)}(\varrho^m\lambda) = \varrho^{\mu-j}\kappa_\varrho g_{(\mu-j)}(\lambda)\kappa_\varrho^{-1} \quad \text{for } \varrho > 0,$$

and for some function $\chi \in C^\infty(\Lambda)$ with $\chi = 0$ near zero and $\chi = 1$ near ∞, and all $j \in \mathbb{N}_0$, the symbol estimates (5.17) hold for $g(\lambda) - \sum_{k=0}^{j-1} \chi(\lambda) g_{(\mu-k)}(\lambda)$ with μ replaced by $\mu - j$.

As usual, the cut-off functions in $C_0^\infty([0,1))$ are considered as functions on both M and $Y^\wedge$, and $\{\kappa_\varrho\}_{\varrho\in\mathbb{R}_+}$ is the dilation group from (2.7). The κ-homogeneous components $g_{(\mu-j)}(\lambda)$ are well-defined for the Green remainder $G(\lambda)$, i.e., they do not depend on the particular choice of cut-off functions (see also Lemma 5.19 below). Hence a Green remainder is determined by an asymptotic expansion

$$G(\lambda) \sim \sum_{j=0}^{\infty} G_{(\mu-j)}(\lambda) \tag{5.18}$$

up to Green remainders of order $-\infty$, where $G_{(\mu-j)}(\lambda) = g_{(\mu-j)}(\lambda)$. The principal component of $G(\lambda)$ in this expansion will be denoted by

$$G_\wedge(\lambda) = G_{(\mu)}(\lambda).$$

Note that in view of Definition 5.16(i) every Green remainder $G(\lambda)$ is a parameter-dependent smoothing pseudodifferential operator over the manifold $\mathring{M}$.

It should be pointed out that the choice of the compact operators as operator ideal for the Green remainders is just for convenience; we could also pass to the Schatten classes $\ell^p(\mathcal{E}^s_\wedge, \mathcal{F}^t_\wedge)$ for arbitrary $p > 0$, or even to s-nuclear operators in $\bigcap_{p>0} \ell^p(\mathcal{E}^s_\wedge, \mathcal{F}^t_\wedge)$. This is useful for applications to index theory, especially the case of trace class remainders.

Lemma 5.19. *Let $g(\lambda)$ be a Green symbol of order $\mu \in \mathbb{R}$, and $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ a cut-off function near zero. Then $(1-\omega)g(\lambda)$ and $g(\lambda)(1-\omega)$ are Green symbols of order $-\infty$, i.e.,*

$$(1-\omega)g(\lambda),\;\; g(\lambda)(1-\omega) \in \mathscr{S}(\Lambda, \mathcal{K}(\mathcal{E}^{s,\delta}_\wedge, \mathcal{F}^{t,\delta'}_\wedge)).$$

Proof. We only need to prove that

$$(1-\omega)g(\lambda) = O([\lambda]^{-L}) \;\text{ as } |\lambda| \to \infty, \;\text{ for all } L \in \mathbb{R}.$$

The argument for higher derivatives and for $g(\lambda)(1-\omega)$ is analogous.

Write $(1-\omega(x)) = \varphi_k(x)x^k$ for every $k \in \mathbb{N}_0$. Note that $\varphi_k \in C^\infty(\mathbb{R}_+)$ is supported away from the origin, and $\varphi_k(x) = \frac{1}{x^k}$ for sufficiently large x. Then, for any given $s, t, \delta, \delta' \in \mathbb{R}$, and denoting the norms in $\mathscr{L}(\mathcal{E}^{s,\delta}_\wedge, \mathcal{F}^{t,\delta'}_\wedge)$ and $\mathscr{L}(\mathcal{F}^{t,\delta'}_\wedge)$ by $\|\cdot\|_{\delta,\delta'}$ and $\|\cdot\|_{\delta'}$, respectively, we have

$$\begin{aligned}
&\Big\|\kappa^{-1}_{[\lambda]^{1/m}}(1-\omega)g(\lambda)\kappa_{[\lambda]^{1/m}}\Big\|_{\delta,\delta'} \\
&\quad = \Big\|\varphi_k\big(\tfrac{x}{[\lambda]^{1/m}}\big)[\lambda]^{-k/m}x^k\kappa^{-1}_{[\lambda]^{1/m}}g(\lambda)\kappa_{[\lambda]^{1/m}}\Big\|_{\delta,\delta'} \\
&\quad \le C\Big\|\varphi_k\big(\tfrac{x}{[\lambda]^{1/m}}\big)\Big\|_{\delta'-k}\cdot\Big\|\kappa^{-1}_{[\lambda]^{1/m}}g(\lambda)\kappa_{[\lambda]^{1/m}}\Big\|_{\delta,\delta'-k}\cdot[\lambda]^{-k/m} \\
&\quad \le \tilde{C}\Big\|\varphi_k\big(\tfrac{x}{[\lambda]^{1/m}}\big)\Big\|_{\delta'-k}\cdot[\lambda]^{\frac{\mu-k}{m}}
\end{aligned}$$

for some constants C and $\tilde{C}$. As the norm of $\varphi_k(x/[\lambda]^{1/m})$ is $O(1)$ as $|\lambda| \to \infty$, the assertion follows for $(1-\omega)g(\lambda)$. $\square$

A direct consequence from Lemma 5.19 is that the space of Green remainders form an operator algebra. The homogeneous components of the product of two Green remainders are determined by formally multiplying the asymptotic sums (5.18). In particular,

$$(G_1G_2)_\wedge(\lambda) = G_{1,\wedge}(\lambda)G_{2,\wedge}(\lambda).$$

Lemma 5.20. *Let $G(\lambda)$ be a Green remainder of order $\mu \in \mathbb{R}$. Then*

(i) $(A-\lambda)G(\lambda)$ and $G(\lambda)(A-\lambda)$ are Green remainders of order $\mu+m$;

(ii) $B_1(\lambda)G(\lambda)$ and $G(\lambda)B_1(\lambda)$ are Green remainders of order $\mu-m$.

In all four cases the principal components are the composition of the principal components of the factors.

Recall that the principal component of $A-\lambda$ is $A_\wedge-\lambda$. On the other hand, the principal component of $B_1(\lambda)$ is given by

$$B_{1,\wedge}(\lambda)u(x) = x^m\Big(\frac{1}{2\pi i}\Big)\int\limits_{\Im\sigma=m/2}\int\limits_{\mathbb{R}_+}\Big(\frac{x}{x'}\Big)^{i\sigma}h(0,\sigma,x^m\lambda)u(x')\,\frac{dx'}{x'}\,d\sigma \tag{5.21}$$

for $u \in C_0^\infty(\mathbb{R}_+, C^\infty(Y;E))$, where $h(x,\sigma,\lambda)$ is the symbol from Proposition 5.15. For the above compositions to make sense, we are tacitly assuming that $G(\lambda)$ acts on corresponding scales.

Proof. Let us consider $(A-\lambda)G(\lambda)$. The product $G(\lambda)(A-\lambda)$ can be treated in a similar way. In the collar neighborhood $(0,1)\times Y$ we have

$$A = x^{-m}\sum_{j=0}^m a_j(x)(xD_x)^j,$$

where $a_j(x) \in C^\infty([0,1), \operatorname{Diff}^{m-j}(Y;E))$. We set $A_{(m)}(\lambda) = A_\wedge - \lambda$, and for $k \in \mathbb{N}$,

$$A_{(m-k)}(\lambda) = x^{-m+k}\sum_{j=0}^m \frac{1}{k!}\big(\partial_x^k a_j\big)(0)(xD_x)^j.$$

Observe that for each j, $A_{(j)}(\lambda) : C_0^\infty(\mathring{Y}^\wedge;E) \to C^\infty(\mathring{Y}^\wedge;E)$, and

$$\omega\Big((A-\lambda) - \sum_{k=0}^{N-1}A_{(m-k)}(\lambda)\Big)\tilde\omega \in x^{-m+N}\operatorname{Diff}_b^m(Y^\wedge;E)$$

for any cut-off functions ω, $\tilde\omega \in C_0^\infty([0,1))$.

Let $\omega \in C_0^\infty([0,1))$ be an arbitrary cut-off function. Then, as the operator norm of $A-\lambda$ grows polynomially, it follows immediately that $(A-\lambda)G(\lambda)(1-\omega)$ is rapidly decreasing in Λ. On the other hand, using a suitable cut-off function $\omega' \in C_0^\infty([0,1))$, we may write

$$(1-\omega)(A-\lambda)G(\lambda) = (1-\omega)(A-\lambda)(1-\omega')G(\lambda).$$

Thus also $(1-\omega)(A-\lambda)G(\lambda)$ is rapidly decreasing in Λ.

It remains to consider $\omega(A-\lambda)G(\lambda)\tilde\omega$ for cut-off functions ω, $\tilde\omega \in C_0^\infty([0,1))$. Choose cut-off functions ω_0 and ω_1 such that $\omega \prec \omega_1 \prec \omega_0$. Then

$$\begin{aligned}\omega(A-\lambda)G(\lambda)\tilde\omega &= \omega(A-\lambda)\omega_1\omega_0G(\lambda)\tilde\omega\\ &= \omega\Big(\sum_{k=0}^{N-1}A_{(m-k)}(\lambda)\Big)\omega_1\omega_0G(\lambda)\tilde\omega + \omega\tilde A_N\omega_1\omega_0G(\lambda)\tilde\omega\end{aligned}$$

for $N \in \mathbb{N}_0$, where $\tilde{A}_N \in x^{-m+N} \operatorname{Diff}_b^m(Y^\wedge; E)$. Since $g(\lambda) = \omega_0 G(\lambda)\tilde{\omega}$ is a Green symbol, it is easy to see that $\omega \tilde{A}_N \omega_1 g(\lambda)$ is an operator-valued symbol of order $\mu + m - N$, i.e., the estimates (5.17) hold with $\mu + m - N$ instead of μ. The argument here is to consider separately the terms $\omega(x)\omega(x[\lambda]^{1/m})\tilde{A}_N\omega_1 g(\lambda)$ and $\omega(x)(1-\omega(x[\lambda]^{1/m}))\tilde{A}_N\omega_1 g(\lambda)$.

Now, using the κ-homogeneity

$$A_{(m-k)}(\varrho^m \lambda) = \varrho^{m-k}\kappa_\varrho A_{(m-k)}(\lambda)\kappa_\varrho^{-1}$$

for $\varrho > 0$ and $\lambda \in \Lambda\backslash\{0\}$, and because of Lemma 5.19, we finally conclude that $(A-\lambda)G(\lambda)$ is a Green remainder of order $\mu + m$. Moreover, the homogeneous components of $(A-\lambda)G(\lambda)$ are given by

$$\big((A-\lambda)G(\lambda)\big)_{(\mu+m-j)} = \sum_{k+l=j} A_{(m-k)}(\lambda) G_{(\mu-l)}(\lambda).$$

The analysis for the products $G(\lambda)B_1(\lambda)$ and $B_1(\lambda)G(\lambda)$ follows the same lines. At the places where the locality of $(A-\lambda)$ was used, we can still draw the desired conclusions for $B_1(\lambda)$, noting that for cut-off functions $\omega \prec \tilde{\omega}$ in $C_0^\infty([0,1))$, the operator families $\omega B_1(\lambda)(1-\tilde{\omega})$ and $(1-\tilde{\omega})B_1(\lambda)\omega$ are Green remainders of order $-\infty$. Moreover, on $Y^\wedge$ we expand $B_1(\lambda)$ into components given by

$$u \mapsto x^{m+k}\frac{1}{2\pi i}\int\limits_{\Im\sigma = m/2}\int\limits_{\mathbb{R}_+}\Big(\frac{x}{x'}\Big)^{i\sigma}\frac{1}{k!}\big(\partial_x^k h\big)(0,\sigma,x^m\lambda)u(x')\,\frac{dx'}{x'}\,d\sigma, \quad k \in \mathbb{N}_0,$$

for $u \in C_0^\infty(\mathbb{R}_+, C^\infty(Y;E))$, and proceed as above. $\square$

Proposition 5.22. *For an operator family*

$$G(\lambda) : C_0^\infty(\mathring{M}; E) \to C^\infty(\mathring{M}; E)$$

the following are equivalent:

(i) *$G(\lambda)$ is a Green remainder of order $\mu \in \mathbb{R}$ in the scales $(\mathcal{E}, \mathcal{D}_{\min})$.*
(ii) *$G(\lambda)$ is a Green remainder of order $\mu \in \mathbb{R}$ in the scales $(\mathcal{E}, x^{m/2-\varepsilon}H)$ for every $\varepsilon > 0$, and $(A-\lambda)G(\lambda)$ is Green of order $\mu + m$ in $(\mathcal{E}, x^{-m/2}H)$.*

Proof. The direction (i) $\Rightarrow$ (ii) follows from Lemma 5.20 noting that

$$\mathcal{D}_{\min}^t(A) = \mathcal{D}_{\max}^t(A) \cap \Big(\bigcap_{\varepsilon>0} x^{m/2-\varepsilon}H_b^{t+m}(M;E)\Big).$$

Let us now assume (ii). Then it is evident that for every cut-off function $\omega \in C_0^\infty([0,1))$ the operator families $(1-\omega)G(\lambda)$ and $G(\lambda)(1-\omega)$ are rapidly decreasing in Λ with values in the scale $\mathcal{D}_{\min}$ of minimal domains. Hence it remains to consider $\omega G(\lambda)\tilde{\omega}$ for cut-off functions $\omega, \tilde{\omega} \in C_0^\infty([0,1))$.

Note first that the assertion of the proposition is obviously valid at the level of Green symbols, i.e., $g(\lambda)$ is a Green symbol of order $\mu \in \mathbb{R}$ with values in the $\mathcal{D}_{\min}$-scale on $Y^\wedge$ if and only if $g(\lambda)$ is a Green symbol of order $\mu \in \mathbb{R}$ with values in the scale $x^{m/2-\varepsilon}H$ of Sobolev spaces on $Y^\wedge$ for every $\varepsilon > 0$, and $(A_\wedge - \lambda)g(\lambda)$ is a Green symbol of order $\mu + m$ with values in the scale $x^{-m/2}H$ on $Y^\wedge$ (note that we are concerned with the associated scales on $Y^\wedge$ in the sense of Definition 5.16).

Now let ω_0 be another cut-off function such that $\omega \prec \omega_0$. Thus $\omega_0\omega = \omega$ and so

$$(A_\wedge - \lambda)\big(\omega G(\lambda)\tilde{\omega}\big) = \omega_0(A-\lambda)\omega_0\big(\omega G(\lambda)\tilde{\omega}\big) + \omega_0\tilde{A}\omega_0\big(\omega G(\lambda)\tilde{\omega}\big)$$

for some $\tilde{A} \in x^{-m+1}\operatorname{Diff}_b^m(Y^\wedge;E)$. Hence $\omega_0\tilde{A}\omega_0\big(\omega G(\lambda)\tilde{\omega}\big)$ is a Green symbol of order $\mu+m-1$ with values in the scale $x^{-m/2}H$ on $Y^\wedge$. Observe that this argument makes use of our assumption that $G(\lambda)$ is a Green remainder of order $\mu \in \mathbb{R}$ in the scales $(\mathcal{E}, x^{m/2-\varepsilon}H)$ for every $\varepsilon > 0$.

On the other hand, we may write

$$\begin{aligned}\omega_0(A-\lambda)\omega_0\big(\omega G(\lambda)\tilde{\omega}\big) &= \omega_0(A-\lambda)\omega G(\lambda)\tilde{\omega}\\ &= \omega_0\omega(A-\lambda)G(\lambda)\tilde{\omega} + \omega_0[(A-\lambda),\omega]G(\lambda)\tilde{\omega}\\ &= \omega(A-\lambda)G(\lambda)\tilde{\omega} + \omega_0[(A-\lambda),\omega]G(\lambda)\tilde{\omega},\end{aligned}$$

where $\omega_0[(A-\lambda),\omega]G(\lambda)\tilde{\omega}$ is rapidly decreasing in Λ. Thus we have proved

$$(A_\wedge - \lambda)\big(\omega G(\lambda)\tilde{\omega}\big) \equiv \omega(A-\lambda)G(\lambda)\tilde{\omega}$$

modulo a Green symbol of order $\mu+m-1$ with values in the scale of Sobolev spaces $x^{-m/2}H$ on $Y^\wedge$, and as $\omega(A-\lambda)G(\lambda)\tilde{\omega}$ is a Green symbol of order $\mu+m$ by our assumption (ii), the proposition follows. $\square$

Let $\hat{P}_0(\sigma) : C^\infty(Y;E|_Y) \to C^\infty(Y;E|_Y)$ be the conormal symbol of $A = x^{-m}P$, cf. (2.3). Since A is assumed to be c-elliptic, we know that the inverse $\hat{P}_0^{-1}(\sigma)$ of $\hat{P}_0(\sigma)$ is a finitely meromorphic Fredholm function on $\mathbb{C}$, and there exists a sufficiently small $\varepsilon_0 > 0$ such that $\hat{P}_0(\sigma)$ is invertible in

$$\{\sigma \in \mathbb{C} : -m/2-\varepsilon_0 < \Im\sigma < -m/2+\varepsilon_0,\ \Im\sigma \neq -m/2\},$$

with a holomorphic inverse there. Define

$$h_0(\sigma) = \hat{P}_0^{-1}(\sigma - im) - h(0,\sigma,0), \tag{5.23}$$

where h is the holomorphic Mellin symbol from Proposition 5.15. Then $h_0(\sigma)$ is finitely meromorphic in $\mathbb{C}$ taking values in $L^{-\infty}(Y)$ and it is rapidly decreasing as $|\Re\sigma| \to \infty$, uniformly for $\Im\sigma$ in compact intervals. Moreover, the strip

$$\{\sigma \in \mathbb{C} : m/2-\varepsilon_0 < \Im\sigma < m/2+\varepsilon_0,\ \Im\sigma \neq m/2\},$$

is free of poles of $h_0(\sigma)$.

For arbitrary $0 < \varepsilon < \varepsilon_0$ and cut-off function $\omega \in C_0^\infty([0,1))$ we define

$$M(\lambda) : C_0^\infty(\mathring{M};E) \to C^\infty(\mathring{M};E)$$

via

$$u \mapsto x^m\omega(x[\lambda]^{1/m})\bigg(\frac{1}{2\pi i}\int\limits_{\Im\sigma = m/2+\varepsilon}\int\limits_{\mathbb{R}_+}\Big(\frac{x}{x'}\Big)^{i\sigma} h_0(\sigma)\omega(x'[\lambda]^{1/m})u(x')\,\frac{dx'}{x'}\,d\sigma\bigg)$$

with the Mellin symbol $h_0(\sigma)$ from (5.23). $M(\lambda)$ is a parameter-dependent smoothing operator, and since the function $\omega(x[\lambda]^{1/m})$ is supported in the collar $[0,1)\times Y$, $M(\lambda)$ can be regarded as an operator on both M and $Y^\wedge$.

For $\lambda \neq 0$ we also define

$$M_\wedge(\lambda) : C_0^\infty(\mathring{Y}^\wedge;E) \to C^\infty(\mathring{Y}^\wedge;E)$$

via

$$u \mapsto x^m\omega(x|\lambda|^{1/m})\bigg(\frac{1}{2\pi i}\int\limits_{\Im\sigma = m/2+\varepsilon}\int\limits_{\mathbb{R}_+}\Big(\frac{x}{x'}\Big)^{i\sigma} h_0(\sigma)\omega(x'|\lambda|^{1/m})u(x')\,\frac{dx'}{x'}\,d\sigma\bigg).$$

Observe that $M_\wedge(\lambda)$ is κ-homogeneous of degree $-m$.

Theorem 5.24. *Set $B_2(\lambda) = B_1(\lambda) + M(\lambda)$. Then*

$$B_2(\lambda) : x^{-m/2}H_b^s(M;E) \to \mathcal{D}^s_{\min}(A)$$

is a parameter-dependent parametrix of $A-\lambda$, and the remainders

$$\begin{aligned}G_1(\lambda) &= (A-\lambda)B_2(\lambda) - 1 : x^{-m/2}H_b^s(M;E) \to x^{-m/2}H_b^t(M;E),\\ G_2(\lambda) &= B_2(\lambda)(A-\lambda) - 1 : \mathcal{D}^s_{\min}(A) \to \mathcal{D}^t_{\min}(A)\end{aligned}$$

are Green families of order zero in the sense of Definition 5.16 with principal components given by

$$G_{1,\wedge}(\lambda) = (A_\wedge - \lambda)B_{2,\wedge}(\lambda) - 1 \quad \textit{and} \quad G_{2,\wedge}(\lambda) = B_{2,\wedge}(\lambda)(A_\wedge - \lambda) - 1,$$

where

$$B_{2,\wedge}(\lambda) = B_{1,\wedge}(\lambda) + M_\wedge(\lambda) \tag{5.25}$$

with $B_{1,\wedge}(\lambda)$ as in (5.21).

Proof. Let us begin by noting that

$$B_2(\lambda) : x^{-m/2}H_b^s(M;E) \to \bigcap_{\varepsilon>0} x^{m/2-\varepsilon}H_b^{s+m}(M;E)$$

is continuous. Hence, in order to show that $B_2(\lambda)$ maps indeed into $\mathcal{D}^s_{\min}(A)$, it suffices to check that

$$(A-\lambda)B_2(\lambda) : x^{-m/2}H_b^s(M;E) \to x^{-m/2}H_b^s(M;E).$$

We will prove that this operator is in fact of the form $1 + G_1(\lambda)$.

By the standard composition rules for (parameter-dependent) cone operators in cone Sobolev spaces (see e.g. [4], [8], and [20]), we know that

$$(A-\lambda)B_1(\lambda) = 1 + \tilde{M}(\lambda) + G(\lambda),$$

where $G(\lambda)$ is a Green remainder of order zero in the scales $(x^{-m/2}H, x^{-m/2}H)$, and $\tilde{M}(\lambda)$ is a smoothing Mellin operator given by

$$\tilde{M}(\lambda)u(x) = \omega(x[\lambda]^{1/m})\bigg(\frac{1}{2\pi i}\int\limits_{\Im\sigma=m/2}\int\limits_{\mathbb{R}_+}\Big(\frac{x}{x'}\Big)^{i\sigma}\tilde{h}_0(\sigma)\omega(x'[\lambda]^{1/m})u(x')\,\frac{dx'}{x'}\,d\sigma\bigg)$$

with a holomorphic Mellin symbol

$$\tilde{h}_0(\sigma) = \hat{P}_0(\sigma - im)h(0,\sigma,0) - 1 = -\hat{P}_0(\sigma - im)h_0(\sigma) \tag{5.24a}$$

with h_0 as in (5.23). Moreover, the principal components satisfy the identity

$$(A_\wedge - \lambda)B_{1,\wedge}(\lambda) = 1 + \tilde{M}_\wedge(\lambda) + G_\wedge(\lambda),$$

where $\tilde{M}_\wedge(\lambda)$ is defined by replacing $[\lambda]$ by $|\lambda|$ in $\tilde{M}(\lambda)$.

Next we consider the composition $(A-\lambda)M(\lambda)$. As $M(\lambda)$ is a Green remainder of order $-m$ in the scales $(x^{-m/2}H, x^{m/2-\varepsilon}H)$ for every $\varepsilon > 0$, we conclude that up to a Green remainder of order 0 in $(x^{-m/2}H, x^{-m/2}H)$ we may write

$$\begin{aligned}(A-\lambda)M(\lambda) &\equiv \omega_0(x[\lambda]^{1/m})A_\wedge\omega_0(x[\lambda]^{1/m})M(\lambda) - \lambda M(\lambda)\\ &\equiv \omega_0(x[\lambda]^{1/m})A_\wedge\omega_0(x[\lambda]^{1/m})M(\lambda),\end{aligned}$$

where ω_0 is a cut-off function with $\omega \prec \omega_0$, so $\omega_0\omega = \omega$. Because of the relation (**??**a), and since the commutator $[A_\wedge, \omega(x[\lambda]^{1/m})] = [A_\wedge, \omega(x[\lambda]^{1/m})]\omega_0(x[\lambda]^{1/m})$ produces arbitrary flatness near the origin, we have

$$\omega_0(x[\lambda]^{1/m})A_\wedge\omega_0(x[\lambda]^{1/m})M(\lambda) \equiv -\tilde{M}(\lambda)$$

modulo a Green remainder of order zero in $(x^{-m/2}H, x^{-m/2}H)$.

Hence we have proved that $(A-\lambda)M(\lambda) = -\tilde{M}(\lambda) + \tilde{G}(\lambda)$ for some Green remainder $\tilde{G}(\lambda)$ of order zero in $(x^{-m/2}H, x^{-m/2}H)$. Consequently,

$$(A-\lambda)B_2(\lambda) = 1 + G_1(\lambda)$$

with $G_1(\lambda) = G(\lambda) + \tilde{G}(\lambda)$, and by κ-homogeneity the principal components necessarily satisfy $(A_\wedge - \lambda)B_{2,\wedge}(\lambda) = 1 + G_{1,\wedge}(\lambda)$. Thus the assertion of the theorem regarding the composition $(A-\lambda)B_2(\lambda)$ is proved.

It remains to analyze the composition $B_2(\lambda)(A-\lambda)$. Again, we first apply the standard composition rules of (parameter-dependent) cone operators in cone Sobolev spaces to see that $B_2(\lambda)(A-\lambda) = 1 + G_2(\lambda)$, where $G_2(\lambda)$ is a Green remainder of order zero in the scales $(\mathcal{D}_{\min}, x^{m/2-\varepsilon}H)$ for arbitrary $\varepsilon > 0$. Moreover, the principal components satisfy the desired identity $B_{2,\wedge}(\lambda)(A_\wedge - \lambda) = 1 + G_{2,\wedge}(\lambda)$. As $(A-\lambda)G_2(\lambda) = G_1(\lambda)(A-\lambda)$, we obtain from Lemma 5.20 that $(A-\lambda)G_2(\lambda)$ is a Green remainder of order m in $(\mathcal{D}_{\min}, x^{-m/2}H)$. Proposition 5.22 now implies that $G_2(\lambda)$ is a Green remainder of order zero in the scales $(\mathcal{D}_{\min}, \mathcal{D}_{\min})$. $\square$

Remark 5.26. The parametrix $B_2(\lambda)$ has the following properties.

(i) For $\lambda \in \Lambda\backslash\{0\}$,

$$A_\wedge - \lambda : \mathcal{D}_{\min}(A_\wedge) \to x^{-m/2}L_b^2(Y^\wedge; E)$$

is Fredholm and $B_{2,\wedge}(\lambda)$ is a Fredholm inverse.

(ii) The principal component $B_{2,\wedge}(\lambda)$ is κ-homogeneous of degree $-m$, i.e.,

$$B_{2,\wedge}(\varrho^m\lambda) = \varrho^{-m}\kappa_\varrho B_{2,\wedge}(\lambda)\kappa_\varrho^{-1} : C_0^\infty(\mathring{Y}^\wedge; E) \to C^\infty(\mathring{Y}^\wedge; E)$$

for $\varrho > 0$ and $\lambda \in \Lambda\backslash\{0\}$.

(iii) Let $G(\lambda)$ be a Green remainder of order $\mu \in \mathbb{R}$. Then $B_2(\lambda)G(\lambda)$ and $G(\lambda)B_2(\lambda)$ are both Green remainders of order $\mu - m$ with principal components $B_{2,\wedge}(\lambda)G_\wedge(\lambda)$ and $G_\wedge(\lambda)B_{2,\wedge}(\lambda)$, respectively.

(iv) For every $s \in \mathbb{R}$ the following equivalent norm estimates hold:

$$\|B_2(\lambda)\|_{\mathscr{L}(x^{-m/2}H_b^s)} \leq \text{const} \cdot [\lambda]^{2|s|/m-1}, \tag{5.27}$$

$$\|B_2(\lambda)\|_{\mathscr{L}(x^{-m/2}H_b^s, \mathcal{D}_{\min}^s(A))} \leq \text{const} \cdot [\lambda]^{2|s|/m}. \tag{5.28}$$

If $G(\lambda)$ is an arbitrary Green remainder of order $-m$, then $B_2(\lambda) + G(\lambda)$ is also an admissible parameter-dependent parametrix of $A - \lambda$ satisfying the same norm estimates as $B_2(\lambda)$.

Proof. The statement (i) is a consequence of Theorem 5.24, (ii) follows by construction. Let us prove (iii). By Lemma 5.20 we only need to deal with the terms $M(\lambda)G(\lambda)$ and $G(\lambda)M(\lambda)$. Now, since $M(\lambda) : C_0^\infty(\mathring{Y}^\wedge; E) \to C^\infty(\mathring{Y}^\wedge; E)$ satisfies

$$M(\varrho^m\lambda) = \varrho^{-m}\kappa_\varrho M(\lambda)\kappa_\varrho^{-1}$$

for $|\lambda| \gg 0$ and $\varrho \geq 1$, the assertion for these terms is evident.

We now prove (iv). The group action $\{\kappa_\varrho\}_{\varrho\in\mathbb{R}_+}$ satisfies the estimate

$$\big\|\kappa_{[\lambda]^{1/m}}\big\|_{\mathscr{L}(\mathcal{K}^{s,-m/2})} \le \text{const}\cdot[\lambda]^{|s|/m}$$

on the space $\mathcal{K}^{s,-m/2}(Y^\wedge;E)$. Recall that $\{\kappa_\varrho\}_{\varrho\in\mathbb{R}_+}$ is defined to be unitary in $x^{-m/2}L^2_b(Y^\wedge;E)$. Hence every Green remainder $G(\lambda)$ of order zero in the scales $(x^{-m/2}H, x^{-m/2}H)$ satisfies the norm estimate

$$\|G(\lambda)\|_{\mathscr{L}(x^{-m/2}H^s_b)} \le \text{const}\cdot[\lambda]^{2|s|/m}.$$

Together with Theorem 5.24 this implies that the asserted estimates are actually equivalent. Moreover, (5.27) follows from the estimates for the group action and the standard estimates for parameter-dependent pseudodifferential operators in Sobolev spaces, cf. Shubin [23, Section 9]. □

As outlined at the beginning of this section, our goal is the construction of a parametrix $B(\lambda)$ that is a left-inverse of $A-\lambda$ for λ sufficiently large. To achieve this, we additionally require that the family

$$A_\wedge - \lambda : \mathcal{D}_{\min}(A_\wedge) \to x^{-m/2}L^2_b(Y^\wedge;E)$$

be injective for all $\lambda\in\Lambda\backslash\{0\}$.

In the remaining part of this section we will prove the following theorem:

Theorem 5.29. *Let $B_2(\lambda)$ be the parametrix from Theorem 5.24. Then there exists a Green remainder $G(\lambda)$ of order $-m$ in the scales $(x^{-m/2}H,\mathcal{D}_{\min})$ such that*

$$B(\lambda) = B_2(\lambda) + G(\lambda)$$

is a parameter-dependent parametrix of $A-\lambda$ with $B(\lambda)(A-\lambda)=1$ for λ sufficiently large. In particular, for these values of λ, $(A-\lambda)B(\lambda)$ is a projection onto $\mathrm{rg}(A-\lambda)$, the range of

$$A-\lambda : \mathcal{D}^s_{\min}(A) \to x^{-m/2}H^s_b(M;E).$$

Thus the Green remainder

$$\Pi(\lambda) = 1-(A-\lambda)B(\lambda)$$

is a projection onto some complement of $\mathrm{rg}(A-\lambda)$ in $x^{-m/2}H^s_b(M;E)$ which is finite dimensional, is contained in $x^{-m/2}H^\infty_b(M;E)$, and is independent of s.

For the proof of this theorem we first introduce the following class of generalized Green remainders.

Definition 5.30. We consider scales of Hilbert spaces $\{\mathcal{E}^s\}_{s\in\mathbb{R}}$ on M and associated scales $\{\mathcal{E}^{s,\delta}_\wedge\}_{s,\delta\in\mathbb{R}}$ on $Y^\wedge$ as in Definition 5.16. Moreover, let $N_-,N_+\in\mathbb{N}_0$.

An operator family

$$G(\lambda) : \begin{array}{c} C_0^\infty(\mathring{M};E) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} C^\infty(\mathring{M};E) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}$$

is called a *generalized Green remainder* of order $\mu\in\mathbb{R}$ in the scales of spaces $\big(\mathcal{E}\oplus\mathbb{C}^{N_-},\mathcal{F}\oplus\mathbb{C}^{N_+}\big)$, if for any cut-off functions $\omega,\tilde\omega\in C_0^\infty([0,1))$ it holds:

(i) For every $s,t\in\mathbb{R}$ the families

$$\begin{pmatrix}(1-\omega) & 0\\ 0 & 0\end{pmatrix} G(\lambda) \;\text{ and }\; G(\lambda)\begin{pmatrix}(1-\tilde\omega) & 0\\ 0 & 0\end{pmatrix}$$

are rapidly decreasing in Λ with values in the compact operators mapping

$$\begin{matrix}\mathcal{E}^s\\ \oplus\\ \mathbb{C}^{N_-}\end{matrix} \to \begin{matrix}\mathcal{F}^t\\ \oplus\\ \mathbb{C}^{N_+}\end{matrix}.$$

(ii) The family $g(\lambda)$ given by

$$g(\lambda)=\begin{pmatrix}\omega & 0\\ 0 & 1\end{pmatrix} G(\lambda)\begin{pmatrix}\tilde\omega & 0\\ 0 & 1\end{pmatrix} : \begin{matrix}C_0^\infty(\mathring{Y}^\wedge;E)\\ \oplus\\ \mathbb{C}^{N_-}\end{matrix} \to \begin{matrix}C^\infty(\mathring{Y}^\wedge;E)\\ \oplus\\ \mathbb{C}^{N_+}\end{matrix}$$

is a *generalized Green symbol*, i.e., it is a classical operator-valued symbol of order $\mu\in\mathbb{R}$ in the sense that

$$g(\lambda)\in\bigcap_{s,t,\delta,\delta'\in\mathbb{R}} C^\infty\big(\Lambda,\mathcal{K}\big(\mathcal{E}_\wedge^{s,\delta}\oplus\mathbb{C}^{N_-},\mathcal{F}_\wedge^{t,\delta'}\oplus\mathbb{C}^{N_+}\big)\big),$$

and for all multi-indices $\alpha\in\mathbb{N}_0^2$,

$$\left\|\begin{pmatrix}\kappa_{[\lambda]^{1/m}} & 0\\ 0 & 1\end{pmatrix}^{-1}\partial_\lambda^\alpha g(\lambda)\begin{pmatrix}\kappa_{[\lambda]^{1/m}} & 0\\ 0 & 1\end{pmatrix}\right\| = O(|\lambda|^{\mu/m-|\alpha|}) \tag{5.31}$$

as $|\lambda|\to\infty$. Moreover, for $j\in\mathbb{N}_0$ there exist

$$g_{(\mu-j)}(\lambda)\in\bigcap_{s,t,\delta,\delta'\in\mathbb{R}} C^\infty\big(\Lambda\backslash\{0\},\mathcal{K}\big(\mathcal{E}_\wedge^{s,\delta}\oplus\mathbb{C}^{N_-},\mathcal{F}_\wedge^{t,\delta'}\oplus\mathbb{C}^{N_+}\big)\big),$$

such that

$$g_{(\mu-j)}(\varrho^m\lambda)=\varrho^{\mu-j}\begin{pmatrix}\kappa_\varrho & 0\\ 0 & 1\end{pmatrix} g_{(\mu-j)}(\lambda)\begin{pmatrix}\kappa_\varrho & 0\\ 0 & 1\end{pmatrix}^{-1}$$

for every $\varrho>0$, and for some function $\chi\in C^\infty(\Lambda)$ with $\chi=0$ near zero and $\chi=1$ near ∞, the symbol estimates (5.31) hold for $g(\lambda)-\sum_{k=0}^{j-1}\chi(\lambda)g_{(\mu-k)}(\lambda)$ with μ replaced by $\mu-j$.

Note that when $N_-=N_+=0$, we recover the class of Green remainders from Definition 5.16. Also for generalized Green remainders, the κ-homogeneous components $g_{(\mu-j)}(\lambda)$ are well-defined for $G(\lambda)$, i.e., they do not depend on the choice of the cut-off functions. Thus a generalized Green remainder is determined by an asymptotic expansion

$$G(\lambda)\sim\sum_{j=0}^\infty G_{(\mu-j)}(\lambda) \tag{5.32}$$

up to generalized Green remainders of order $-\infty$, where $G_{(\mu-j)}(\lambda)=g_{(\mu-j)}(\lambda)$. The principal component will again be denoted by $G_\wedge(\lambda)=G_{(\mu)}(\lambda)$.

We will be particularly concerned with the operators

$$\begin{pmatrix}A-\lambda & 0\\ 0 & 0\end{pmatrix}+G(\lambda),\qquad \begin{pmatrix}B_2(\lambda) & 0\\ 0 & 0\end{pmatrix}+G'(\lambda)$$

for generalized Green remainders $G(\lambda)$ and $G'(\lambda)$ of order m and $-m$, respectively. We will also need their κ-homogeneous principal components

$$\begin{pmatrix} A_\wedge - \lambda & 0 \\ 0 & 0 \end{pmatrix} + G_\wedge(\lambda), \quad \begin{pmatrix} B_{2,\wedge}(\lambda) & 0 \\ 0 & 0 \end{pmatrix} + G'_\wedge(\lambda).$$

Lemma 5.20 (as well as (iii) in Remark 5.26) continues to hold in this more general framework, and Theorem 5.24 implies

$$\left(\begin{pmatrix} A - \lambda & 0 \\ 0 & 0 \end{pmatrix} + G(\lambda)\right)\left(\begin{pmatrix} B_2(\lambda) & 0 \\ 0 & 0 \end{pmatrix} + G'(\lambda)\right) = 1 + G_1(\lambda),$$
$$\left(\begin{pmatrix} B_2(\lambda) & 0 \\ 0 & 0 \end{pmatrix} + G'(\lambda)\right)\left(\begin{pmatrix} A - \lambda & 0 \\ 0 & 0 \end{pmatrix} + G(\lambda)\right) = 1 + G_2(\lambda)$$

with generalized Green remainders $G_1(\lambda)$ and $G_2(\lambda)$ of order zero, provided the scales are such that the composition makes sense. Moreover, the principal components satisfy the same relations.

Lemma 5.33. *Let $G(\lambda)$ be a generalized Green remainder of order zero in the scales $(\mathcal{E} \oplus \mathbb{C}^N, \mathcal{E} \oplus \mathbb{C}^N)$ for some $N \in \mathbb{N}_0$. If*

$$1 + G_\wedge(\lambda) : \begin{matrix} \mathcal{E}^{s,\delta}_\wedge \\ \oplus \\ \mathbb{C}^N \end{matrix} \to \begin{matrix} \mathcal{E}^{s,\delta}_\wedge \\ \oplus \\ \mathbb{C}^N \end{matrix}$$

is invertible for all $\lambda \in \Lambda \backslash \{0\}$ and some $s, \delta \in \mathbb{R}$, then there exists a generalized Green remainder $\tilde{G}(\lambda)$ of order zero such that

$$(1 + G(\lambda))(1 + \tilde{G}(\lambda)) - 1 \ \text{ and } \ (1 + \tilde{G}(\lambda))(1 + G(\lambda)) - 1$$

are generalized Green remainders of order $-\infty$. Moreover, $\tilde{G}(\lambda)$ can be arranged in such a way that these remainders are compactly supported in Λ, thus $(1 + \tilde{G}(\lambda))$ inverts $(1 + G(\lambda))$ for every λ sufficiently large.

Proof. The inverse of $1 + G_\wedge(\lambda)$ can be written as

$$(1 + G_\wedge(\lambda))^{-1} = 1 + \tilde{G}_\wedge(\lambda)$$

where $\tilde{G}_\wedge(\lambda) = G_\wedge(\lambda)(1+G_\wedge(\lambda))^{-1}G_\wedge(\lambda) - G_\wedge(\lambda)$ is a homogeneous Green symbol of order zero. For $\lambda \in \Lambda$ set

$$G'(\lambda) = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \chi(\lambda)\tilde{G}_\wedge(\lambda) \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix},$$

where $\omega \in C_0^\infty([0,1))$ is a cut-off function and $\chi \in C^\infty(\Lambda)$ is a function with $\chi = 0$ near 0 and $\chi = 1$ near ∞. Hence $G'(\lambda)$ is a generalized Green remainder of order zero, and by construction we obtain

$$(1 + G(\lambda))(1 + G'(\lambda)) = 1 + \tilde{G}_1(\lambda), \quad (1 + G'(\lambda))(1 + G(\lambda)) = 1 + \tilde{G}_2(\lambda)$$

with generalized Green remainders $\tilde{G}_1(\lambda)$ and $\tilde{G}_2(\lambda)$ of order -1.

As the class of generalized Green remainders is asymptotically complete, there exists a generalized Green remainder $\tilde{G}_R(\lambda)$ of order -1 with

$$\tilde{G}_R(\lambda) \sim \sum_{k=1}^\infty (-1)^k \tilde{G}_1^k(\lambda).$$

This asymptotic expansion holds up to generalized Green remainders of order $-\infty$. Hence

$$(1+G(\lambda))(1+G'(\lambda))(1+\tilde{G}_R(\lambda)) = 1+\tilde{G}_{(-\infty)}(\lambda)$$

with a generalized Green remainder $\tilde{G}_{(-\infty)}(\lambda)$ of order $-\infty$. In particular, the operator norm of $\tilde{G}_{(-\infty)}(\lambda)$ is decreasing as $|\lambda|\to\infty$ and therefore $1+\tilde{G}_{(-\infty)}(\lambda)$ is invertible for λ large. Moreover, the inverse can be written as

$$\big(1+\tilde{G}_{(-\infty)}(\lambda)\big)^{-1} = 1+\tilde{G}^{(-\infty)}(\lambda),$$

where $\tilde{G}^{(-\infty)}(\lambda) = \tilde{G}_{(-\infty)}(\lambda)\big(1+\tilde{G}_{(-\infty)}(\lambda)\big)^{-1}\tilde{G}_{(-\infty)}(\lambda) - \tilde{G}_{(-\infty)}(\lambda)$. Note that if $\chi\in C^\infty(\Lambda)$ is a suitable function with $\chi=0$ near 0 and $\chi=1$ near ∞, then $\chi(\lambda)\tilde{G}^{(-\infty)}(\lambda)$ is a generalized Green remainder of order $-\infty$. Summing up, we have proved that

$$(1+G(\lambda))(1+G'(\lambda))(1+\tilde{G}_R(\lambda))(1+\chi(\lambda)\tilde{G}^{(-\infty)}(\lambda)) - 1$$

is compactly supported in Λ. Finally, we define $\tilde{G}(\lambda)$ by

$$1+\tilde{G}(\lambda) = (1+G'(\lambda))(1+\tilde{G}_R(\lambda))(1+\chi(\lambda)\tilde{G}^{(-\infty)}(\lambda)).$$

By construction, $\tilde{G}(\lambda)$ is a generalized Green remainder of order zero and $1+\tilde{G}(\lambda)$ inverts $1+G(\lambda)$ from the right for large values of λ.

In the same way, we can prove that $1+G(\lambda)$ has a left-inverse for λ sufficiently large. This inverse must be necessarily $1+\tilde{G}(\lambda)$ and the lemma is proved. □

The following theorem implies Theorem 5.29.

Theorem 5.34. *For $\lambda\in\Lambda\backslash\{0\}$ let $d''=-\operatorname{ind}(A_{\wedge,\mathcal{D}_{\min}}-\lambda)$, There exists a generalized Green remainder $\begin{pmatrix}0 & K(\lambda)\end{pmatrix}$ of order m in the scales $(\mathcal{D}_{\min}\oplus\mathbb{C}^{d''}, x^{-m/2}H)$ such that*

$$\begin{pmatrix}A-\lambda & K(\lambda)\end{pmatrix} : \begin{matrix}\mathcal{D}^s_{\min}(A)\\ \oplus\\ \mathbb{C}^{d''}\end{matrix} \to x^{-m/2}H^s_b(M;E)$$

is invertible for λ sufficiently large. Moreover, the inverse can be written as

$$\begin{pmatrix}A-\lambda & K(\lambda)\end{pmatrix}^{-1} = \begin{pmatrix}B_2(\lambda)+G(\lambda)\\ T(\lambda)\end{pmatrix},$$

where $\begin{pmatrix}G(\lambda)\\ T(\lambda)\end{pmatrix}$ is a generalized Green remainder of order $-m$ in the corresponding scales $(x^{-m/2}H,\mathcal{D}_{\min}\oplus\mathbb{C}^{d''})$. In particular, the parameter-dependent parametrix

$$B(\lambda) = B_2(\lambda)+G(\lambda)$$

satisfies the conditions of Theorem 5.29.

Proof. From Theorem A.1 (see also Remark A.2 and Corollary A.3) we conclude that there exists $k_\wedge(\lambda)$ such that

$$\begin{pmatrix}A_\wedge-\lambda & k_\wedge(\lambda)\end{pmatrix} : \begin{matrix}\mathcal{D}_{\min}(A_\wedge)\\ \oplus\\ \mathbb{C}^{d''}\end{matrix} \to x^{-m/2}L^2_b(Y^\wedge;E)$$

is invertible for $\lambda\in\Lambda\backslash\{0\}$, and $k_\wedge(\lambda)$ can be arranged to be a κ-homogeneous principal Green symbol of order m.

Let $\omega \in C_0^\infty([0,1))$ be a cut-off function and let $\chi \in C^\infty(\Lambda)$ be a function with $\chi = 0$ near 0 and $\chi = 1$ near ∞. If we set $K(\lambda) = \omega\chi(\lambda)k_\wedge(\lambda)$, then $\begin{pmatrix} 0 & K(\lambda)\end{pmatrix}$ is a generalized Green remainder of order m. We will prove that the theorem holds with this particular choice for $K(\lambda)$.

As $B_{2,\wedge}(\lambda)$ is a Fredholm inverse of $A_\wedge - \lambda$ for $\lambda \in \Lambda\backslash\{0\}$, we may apply once again the results from Appendix A to conclude the existence of families $\tilde{k}_\wedge(\lambda)$, $\tilde{t}_\wedge(\lambda)$, and $\tilde{q}_\wedge(\lambda)$ such that

$$\begin{pmatrix} B_{2,\wedge}(\lambda) & \tilde{k}_\wedge(\lambda) \\ \tilde{t}_\wedge(\lambda) & \tilde{q}_\wedge(\lambda) \end{pmatrix} : \begin{matrix} x^{-m/2}L_b^2(Y^\wedge;E) \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \begin{matrix} \mathcal{D}_{\min}(A_\wedge) \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix}$$

is invertible for $\lambda \in \Lambda\backslash\{0\}$, and $\begin{pmatrix} \mathbf{0} & \tilde{k}_\wedge(\lambda) \\ \tilde{t}_\wedge(\lambda) & \tilde{q}_\wedge(\lambda) \end{pmatrix}$ is a homogeneous principal Green symbol of order $-m$. Note that by construction $N_+ - N_- = \operatorname{ind} B_{2,\wedge}(\lambda) = d''$. According to $\mathbb{C}^{N_+} = \mathbb{C}^{d''} \oplus \mathbb{C}^{N_-}$ we decompose (arbitrarily)

$$\tilde{t}_\wedge(\lambda) = \begin{pmatrix} \tilde{t}_{\wedge,1}(\lambda) \\ \tilde{t}_{\wedge,2}(\lambda) \end{pmatrix} \quad \text{and} \quad \tilde{q}_\wedge(\lambda) = \begin{pmatrix} \tilde{q}_{\wedge,1}(\lambda) \\ \tilde{q}_{\wedge,2}(\lambda) \end{pmatrix},$$

and let

$$G'(\lambda) = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \chi(\lambda) \begin{pmatrix} \mathbf{0} & \tilde{k}_\wedge(\lambda) \\ \tilde{t}_{\wedge,1}(\lambda) & \tilde{q}_{\wedge,1}(\lambda) \\ \tilde{t}_{\wedge,2}(\lambda) & \tilde{q}_{\wedge,2}(\lambda) \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix},$$

where ω and χ are as above. Then $G'(\lambda)$ is a generalized Green remainder of order $-m$ in the scales $(x^{-m/2}H \oplus \mathbb{C}^{N_-}, \mathcal{D}_{\min} \oplus \mathbb{C}^{N_+})$. We now let

$$\mathcal{A}(\lambda) = \left(\begin{array}{cc|c} A-\lambda & K(\lambda) & 0 \\ \hline 0 & 0 & [\lambda] \end{array}\right) \quad \text{and} \quad \mathcal{B}(\lambda) = \left(\begin{array}{c|c} B_2(\lambda) & 0 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right) + G'(\lambda),$$

and consider the compositions

$$\begin{aligned} \mathcal{A}(\lambda)\mathcal{B}(\lambda) &= 1 + G_1(\lambda) \ \text{ on } x^{-m/2}L_b^2(M;E) \oplus \mathbb{C}^{N_-}, \\ \mathcal{B}(\lambda)\mathcal{A}(\lambda) &= 1 + G_2(\lambda) \ \text{ on } \big(\mathcal{D}_{\min}(A) \oplus \mathbb{C}^{d''}\big) \oplus \mathbb{C}^{N_-}. \end{aligned}$$

Note that $\left(\begin{array}{cc|c} 0 & K(\lambda) & 0 \\ \hline 0 & 0 & [\lambda] \end{array}\right)$ is a generalized Green remainder of order m with principal component

$$\left(\begin{array}{cc|c} 0 & k_\wedge(\lambda) & 0 \\ \hline 0 & 0 & |\lambda| \end{array}\right).$$

Hence $G_1(\lambda)$ and $G_2(\lambda)$ are generalized Green remainders of order zero, and by construction both $1 + G_{1,\wedge}(\lambda)$ and $1 + G_{2,\wedge}(\lambda)$ are invertible for $\lambda \in \Lambda\backslash\{0\}$.

Lemma 5.33 now implies the invertibility of $\mathcal{A}(\lambda)$ for λ large. Consequently, the diagonal matrix structure of $\mathcal{A}(\lambda)$ gives the invertibility of $\begin{pmatrix} A-\lambda & K(\lambda) \end{pmatrix}$. Moreover,

$$\mathcal{A}(\lambda)^{-1} = \left(\begin{array}{cc|c} A-\lambda & K(\lambda) & 0 \\ \hline 0 & 0 & [\lambda] \end{array}\right)^{-1} = \mathcal{B}(\lambda)(1 + \tilde{G}(\lambda))$$

for some generalized Green remainder $\tilde{G}(\lambda)$ of order $-m$. Thus $\begin{pmatrix} A-\lambda & K(\lambda)\end{pmatrix}^{-1}$ must be of the form

$$\begin{pmatrix} B_2(\lambda)+G(\lambda) \\ T(\lambda)\end{pmatrix}$$

which proves the theorem. □

Corollary 5.35. *For $\lambda \in \Lambda\backslash\{0\}$ we have* $\operatorname{ind}(A_{\wedge,\mathcal{D}_{\min}}-\lambda) = \operatorname{ind} A_{\mathcal{D}_{\min}}$.

As stated above, the parameter-dependent family $B(\lambda) = B_2(\lambda) + G(\lambda)$ is a parametrix of $(A-\lambda)$ satisfying the conditions of Theorem 5.29. Let us draw some consequences of that theorem.

Corollary 5.36. *There exists a discrete set $\Delta \subset \mathbb{C}$ such that*

$$A-\lambda : \mathcal{D}^s_{\min}(A) \to x^{-m/2}H^s_b(M;E)$$

is injective for $\lambda \in \mathbb{C}\backslash\Delta$, and it has a finitely meromorphic left-inverse.

Proof. Due to Theorem 5.29,

$$A-\lambda : \mathcal{D}^s_{\min}(A) \to x^{-m/2}H^s_b(M;E)$$

is injective for $\lambda \in \Lambda$ sufficiently large, and the parametrix $B(\lambda)$ is a left-inverse.

Fix some large $\lambda_0 \in \Lambda$ and consider the operator function

$$F : \mathbb{C} \ni \lambda \mapsto B(\lambda_0)(A-\lambda) \in \mathscr{L}(\mathcal{D}^s_{\min}(A)).$$

Then F is a holomorphic Fredholm family on $\mathbb{C}$, and $F(\lambda_0) = 1$ is invertible. The well known theorem on the inversion of holomorphic Fredholm families now implies that the inverse $\mathbb{C}\backslash\Delta \ni \lambda \mapsto F(\lambda)^{-1}$ is a finitely meromorphic operator function, where $\Delta \subset \mathbb{C}$ is discrete. Hence $A-\lambda$ is injective for $\lambda \in \mathbb{C}\backslash\Delta$, and $F(\lambda)^{-1}B(\lambda_0)$ is a finitely meromorphic left-inverse. □

Corollary 5.37. *Let $\lambda_0 \in \Lambda$ and assume there exists some domain $\mathcal{D}^s$ such that*

$$A-\lambda_0 : \mathcal{D}^s \to x^{-m/2}H^s_b(M;E)$$

is invertible. Then it is invertible for all $s \in \mathbb{R}$, and we have

$$(A-\lambda_0)^{-1} = B(\lambda_0) + (A-\lambda_0)^{-1}\Pi(\lambda_0)$$

with the parametrix $B(\lambda)$ and the projection $\Pi(\lambda)$ from Theorem 5.29.

6. Resolvents

The elements of the quotient

$$\tilde{\mathcal{E}}_{\max} = \mathcal{D}_{\max}/\mathcal{D}_{\min}$$

can be conveniently identified with singular functions as follows. Let $u \in \mathcal{D}_{\max}$. Then there is a finite sum of the form

$$\tilde{u} = \sum_{-\frac{m}{2}\le\Im(\sigma)<\frac{m}{2}} \Big(\sum_{k=0}^{m_\sigma} c_{\sigma,k}(y)\log^k x\Big)x^{i\sigma} \tag{6.1}$$

with $c_{\sigma,k}(y) \in C^\infty(Y;E)$ such that $u-\omega\tilde{u} \in \mathcal{D}_{\min}$, where $\omega \in C_0^\infty([0,1))$ is a cut-off function near zero. The function $\tilde{u}$ is uniquely determined by the equivalence class $u+\mathcal{D}_{\min}$, and in this way we may identify $\tilde{\mathcal{E}}_{\max}$ with a finite dimensional subspace

of $C^\infty(\mathring{Y}^\wedge;E)$ consisting of singular functions (6.1). Analogously, we also obtain an identification of
$$\tilde{\mathcal{E}}_{\wedge,\max} = \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$$
with a finite dimensional space of functions of the form (6.1).

In order to prove the existence of sectors of minimal growth for a given extension $A_{\mathcal{D}}$, we are led to consider a particular extension $A_{\wedge,\mathcal{D}_\wedge}$ of the model operator. Thereby, the domain $\mathcal{D}_\wedge$ is associated to $\mathcal{D}$ via
$$\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} = \theta\big(\mathcal{D}/\mathcal{D}_{\min}\big), \tag{6.2}$$
where
$$\theta : \tilde{\mathcal{E}}_{\max} \to \tilde{\mathcal{E}}_{\wedge,\max}$$
is the natural isomorphism introduced in [5].

Using the identification of the quotients with spaces of singular functions, we briefly recall the definition of θ. To this end, we split
$$A = x^{-m}\sum_{k=0}^{m-1} P_k x^k + \tilde{P}_m \tag{6.3}$$
near Y, where each $P_k \in \mathrm{Diff}_b^m(Y^\wedge;E)$ has coefficients independent of x, and $\tilde{P}_m \in \mathrm{Diff}_b^m(Y^\wedge;E)$. Let $\hat{P}_k(\sigma)$ be the conormal symbol associated with P_k. In this section, all arguments involving (6.3) will refer to functions that are supported near Y, so we may assume that the coefficients of $\tilde{P}_m$ vanish near infinity. In slight abuse of the notation from [5] we now write
$$\tilde{\mathcal{E}}_{\max} = \bigoplus_{\sigma_0\in\Sigma}\tilde{\mathcal{E}}_{\sigma_0} \ \text{ and } \ \tilde{\mathcal{E}}_{\wedge,\max} = \bigoplus_{\sigma_0\in\Sigma}\tilde{\mathcal{E}}_{\wedge,\sigma_0},$$
where
$$\Sigma = \mathrm{spec}_b(A)\cap\{\sigma\in\mathbb{C} : -m/2 < \Im(\sigma) < m/2\}. \tag{6.4}$$

The space $\tilde{\mathcal{E}}_{\wedge,\sigma_0}$ consists of all singular functions of the form
$$\Big(\sum_{k=0}^{m_{\sigma_0}} c_{\sigma_0,k}(y)\log^k x\Big)x^{i\sigma_0}$$
that are associated with elements of $\tilde{\mathcal{E}}_{\wedge,\max}$. The operator θ acts isomorphically between $\tilde{\mathcal{E}}_{\sigma_0} \to \tilde{\mathcal{E}}_{\wedge,\sigma_0}$. Both, the space $\tilde{\mathcal{E}}_{\sigma_0}$ and the operator itself, are easiest understood from its inverse
$$\theta^{-1}|_{\tilde{\mathcal{E}}_{\wedge,\sigma_0}} = \sum_{k=0}^{N(\sigma_0)} \mathrm{e}_{\sigma_0,k} : \tilde{\mathcal{E}}_{\wedge,\sigma_0} \to \tilde{\mathcal{E}}_{\sigma_0}, \tag{6.5}$$
where $N(\sigma_0)\in\mathbb{N}_0$ is the largest integer such that $\Im\sigma_0 - N(\sigma_0) \geq -m/2$, and the operators
$$\mathrm{e}_{\sigma_0,k} : \tilde{\mathcal{E}}_{\wedge,\sigma_0} \to C^\infty(\mathring{Y}^\wedge;E)$$
are inductively defined as follows:

- $\mathrm{e}_{\sigma_0,0} = I$, the identity map.
- Given $\mathrm{e}_{\sigma_0,0},\dots,\mathrm{e}_{\sigma_0,\vartheta-1}$ for some $\vartheta\in\{1,\dots,N(\sigma_0)\}$, we define $\mathrm{e}_{\sigma_0,\vartheta}(\psi)$ for $\psi\in\tilde{\mathcal{E}}_{\wedge,\sigma_0}$ to be the unique singular function of the form
$$\Big(\sum_{k=0}^{m_{\sigma_0-i\vartheta}} c_{\sigma_0-i\vartheta,k}(y)\log^k x\Big)x^{i(\sigma_0-i\vartheta)}$$

such that

$$(\omega \mathrm{e}_{\sigma_0,\vartheta}(\psi))^\wedge(\sigma) + \hat{P}_0(\sigma)^{-1}\Big(\sum_{k=1}^{\vartheta} \hat{P}_k(\sigma) \mathrm{s}_{\sigma_0 - i\vartheta}(\omega \mathrm{e}_{\sigma_0,\vartheta-k}(\psi))^\wedge(\sigma+ik)\Big)$$

is holomorphic at $\sigma = \sigma_0 - i\vartheta$, where $(\omega \mathrm{e}_{\sigma_0,\vartheta-k}(\psi))^\wedge(\sigma)$ is the Mellin transform of the function $\omega \mathrm{e}_{\sigma_0,\vartheta-k}(\psi)$, and $\mathrm{s}_{\sigma_0-i\vartheta}(\omega \mathrm{e}_{\sigma_0,\vartheta-k}(\psi))^\wedge(\sigma+ik)$ is the singular part of its Laurent expansion at $\sigma_0 - i\vartheta$. Here, $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ is an arbitrary cut-off function near zero. Recall that the Mellin transform of $\omega \mathrm{e}_{\sigma_0,\vartheta-k}(\psi)$ is meromorphic in $\mathbb{C}$ with only one pole at $\sigma_0 - i(\vartheta - k)$.

It is of interest to note that this construction yields

$$\sum_{k=0}^{\vartheta} \big(P_k x^k\big)(\mathrm{e}_{\sigma_0,\vartheta-k}(\psi)) = 0$$

for every $\psi \in \tilde{\mathcal{E}}_{\wedge,\sigma_0}$ and every $\vartheta = 0,\dots,N(\sigma_0)$.

In conclusion, every space $\tilde{\mathcal{E}}_{\sigma_0}$ consists of singular functions of the form

$$\tilde{u} = \sum_{\vartheta=0}^{N(\sigma_0)} \Big(\sum_{k=0}^{m_{\sigma_0-i\vartheta}} c_{\sigma_0-i\vartheta,k}(y) \log^k x\Big) x^{i(\sigma_0-i\vartheta)},$$

and we have

$$\theta\tilde{u} = \Big(\sum_{k=0}^{m_{\sigma_0}} c_{\sigma_0,k}(y)\log^k x\Big)x^{i\sigma_0}. \tag{6.6}$$

The main result of this section concerns the existence of sectors of minimal growth for closed extensions of a c-elliptic cone operator A. Recall that a sector

$$\Lambda = \{\lambda \in \mathbb{C} : \lambda = re^{i\theta} \text{ for } r \geq 0,\ \theta \in \mathbb{R},\ |\theta - \theta_0| \leq a\},$$

with $\theta_0 \in \mathbb{R}$ and $a > 0$, is called a sector of minimal growth for the extension

$$A_{\mathcal{D}} : \mathcal{D} \subset x^{-m/2}L_b^2(M;E) \to x^{-m/2}L_b^2(M;E)$$

if for $\lambda \in \Lambda$ with $|\lambda| > R$ sufficiently large,

$$A_{\mathcal{D}} - \lambda : \mathcal{D} \to x^{-m/2}L_b^2(M;E)$$

is invertible, and the resolvent $(A_{\mathcal{D}} - \lambda)^{-1}$ satisfies the equivalent norm estimates

$$\begin{aligned} \big\|(A_{\mathcal{D}} - \lambda)^{-1}\big\|_{\mathscr{L}(x^{-m/2}L_b^2)} &= O(|\lambda|^{-1}) \ \text{ as } |\lambda| \to \infty,\\ \big\|(A_{\mathcal{D}} - \lambda)^{-1}\big\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathcal{D}_{\max})} &= O(1) \ \text{ as } |\lambda| \to \infty. \end{aligned} \tag{6.7}$$

Analogously, we call Λ a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$ if

$$A_{\wedge,\mathcal{D}_\wedge} - \lambda : \mathcal{D}_\wedge \to x^{-m/2}L_b^2(Y^\wedge;E)$$

is invertible for large $|\lambda| > 0$ in Λ, and the inverse satisfies the equivalent estimates

$$\begin{aligned} \big\|(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}\big\|_{\mathscr{L}(x^{-m/2}L_b^2)} &= O(|\lambda|^{-1}) \ \text{ as } |\lambda| \to \infty,\\ \big\|(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}\big\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathcal{D}_{\max})} &= O(1) \ \text{ as } |\lambda| \to \infty. \end{aligned} \tag{6.8}$$

Theorem 6.9. *Let $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ be c-elliptic with parameter in Λ. Let $\mathcal{D} \subset x^{-m/2}L_b^2(M;E)$ be a domain such that $A_{\mathcal{D}}$ is closed and let $\mathcal{D}_\wedge$ be the associated domain defined via (6.2). Assume that Λ is a sector of minimal growth for the extension $A_{\wedge,\mathcal{D}_\wedge}$. Then Λ is a sector of minimal growth for the operator $A_{\mathcal{D}}$. Moreover, the resolvent of $A_{\mathcal{D}}$ satisfies the equation*

$$(A_{\mathcal{D}} - \lambda)^{-1} = B(\lambda) + (A_{\mathcal{D}} - \lambda)^{-1}\Pi(\lambda) \tag{6.10}$$

with the parametrix $B(\lambda)$ and the projection $\Pi(\lambda)$ from Theorem 5.29.

Before we prove this theorem, we discuss some interesting properties of the resolvent conditions on $A_\wedge$. For more details see [5].

Proposition 6.11. *If $\mathcal{D}_\wedge$ is κ-invariant, then the invertibility of $A_{\wedge,\mathcal{D}_\wedge} - \lambda$ for $\lambda \in \Lambda$ with $|\lambda| > R$ implies the invertibility of $A_{\wedge,\mathcal{D}_\wedge} - \lambda$ for all $\lambda \in \Lambda\backslash\{0\}$, and Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$.*

Proposition 6.12. *If Λ is a sector of minimal growth for the operator $A_\wedge$ with domain $\mathcal{D}_\wedge$, then Λ is also a sector of minimal growth for $A_\wedge$ with domain $\kappa_\varrho\mathcal{D}_\wedge$ for any $\varrho > 0$. In particular, the resolvent $B_{\varrho,\wedge}(\lambda)$ of $A_{\wedge,\kappa_\varrho\mathcal{D}_\wedge}$ satisfies*

$$B_{\varrho,\wedge}(\lambda) = \varrho^{-m}\kappa_\varrho(A_{\wedge,\mathcal{D}_\wedge} - \varrho^{-m}\lambda)^{-1}\kappa_\varrho^{-1}.$$

In general, the norm estimates (6.8) are not easy to check. However, the following proposition shows that this resolvent condition only needs to be verified for the projection of $(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}$ onto the finite dimensional space $\tilde{\mathcal{E}}_{\wedge,\max} = \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$.

Proposition 6.13. *Let A be c-elliptic with parameter in Λ. The sector Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$ if and only if*

$$A_{\wedge,\mathcal{D}_\wedge} - \lambda : \mathcal{D}_\wedge \to x^{-m/2}L_b^2(Y^\wedge;E)$$

is invertible for large $|\lambda| > 0$, and the inverse satisfies the estimate

$$\left\|\kappa_{|\lambda|^{1/m}}^{-1} q_\wedge (A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}\right\|_{\mathscr{L}(x^{-m/2}L_b^2,\tilde{\mathcal{E}}_{\wedge,\max})} = O(|\lambda|^{-1}) \ \text{ as } |\lambda| \to \infty. \tag{6.14}$$

Here $q_\wedge : \mathcal{D}_{\wedge,\max} \to \tilde{\mathcal{E}}_{\wedge,\max}$ denotes the canonical projection.

Proof. We first observe that the κ-homogeneity of $A_\wedge$ implies

$$A_\wedge \kappa_{|\lambda|^{1/m}}^{-1}(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1} = \kappa_{|\lambda|^{1/m}}^{-1}|\lambda|^{-1}A_\wedge(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}$$

as operators in $\mathscr{L}(x^{-m/2}L_b^2)$. Using this identity and the fact that κ_ϱ is an isometry in $\mathscr{L}(x^{-m/2}L_b^2)$, one can easily see that the estimates (6.8) are equivalent to

$$\|\kappa_{|\lambda|^{1/m}}^{-1}(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathcal{D}_{\wedge,\max})} = O(|\lambda|^{-1}) \ \text{ as } |\lambda| \to \infty, \tag{6.15}$$

and therefore (6.14) holds. Note that $\kappa_\varrho q_\wedge = q_\wedge \kappa_\varrho$.

Conversely, assume that we have (6.14). Let $B_\wedge(\lambda)$ be the principal part of the parametrix $B(\lambda)$ from Theorem 5.29. Then, for $\lambda \in \Lambda\backslash\{0\}$, we have

$$1 - B_\wedge(\lambda)(A_\wedge - \lambda) = 0 \ \text{ on } \mathcal{D}_{\wedge,\min},$$

and we may write

$$(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1} = B_\wedge(\lambda) + (1 - B_\wedge(\lambda)(A_\wedge - \lambda))q_\wedge(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}$$

as operators in $\mathscr{L}(x^{-m/2}L^2_b, \mathcal{D}_{\wedge,\max})$. Since $B_\wedge(\lambda)$ and $(A_\wedge - \lambda)$ are κ-homogeneous of degree $-m$ and m, respectively, we have the identities

$$\kappa^{-1}_{|\lambda|^{1/m}} B_\wedge(\lambda) = |\lambda|^{-1} B_\wedge\big(\tfrac{\lambda}{|\lambda|}\big)\kappa^{-1}_{|\lambda|^{1/m}},$$
$$\kappa^{-1}_{|\lambda|^{1/m}}(A_\wedge - \lambda) = |\lambda|\big(A_\wedge - \tfrac{\lambda}{|\lambda|}\big)\kappa^{-1}_{|\lambda|^{1/m}},$$

which imply

$$\begin{aligned}\kappa^{-1}_{|\lambda|^{1/m}}(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1} &= |\lambda|^{-1} B_\wedge\big(\tfrac{\lambda}{|\lambda|}\big)\kappa^{-1}_{|\lambda|^{1/m}} \\ &\quad + \Big(1 - B_\wedge\big(\tfrac{\lambda}{|\lambda|}\big)\big(A_\wedge - \tfrac{\lambda}{|\lambda|}\big)\Big)\kappa^{-1}_{|\lambda|^{1/m}} q_\wedge (A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}.\end{aligned}$$

Passing to the norm in $\mathscr{L}(x^{-m/2}L^2_b, \mathcal{D}_{\wedge,\max})$ and using (6.14) we obtain (6.15) which is equivalent to the estimates (6.8). □

For the proof of Theorem 6.9 we need further ingredients. First of all, using the operator θ defined via (6.5) and (6.6), we now define on $\tilde{\mathcal{E}}_{\max}$ the group action

$$\tilde{\kappa}_\varrho = \theta^{-1}\kappa_\varrho\theta \quad \text{for } \varrho > 0. \tag{6.16}$$

We may write $\tilde{\kappa}_\varrho = \kappa_\varrho L_\varrho$, where

$$L_\varrho = \kappa_\varrho^{-1}\theta^{-1}\kappa_\varrho\theta : \tilde{\mathcal{E}}_{\max} \to C^\infty(\mathring{Y}^\wedge; E)$$

is the direct sum of the operators $L_\varrho|_{\tilde{\mathcal{E}}_{\sigma_0}}$ given by

$$L_\varrho \tilde{u} = \sum_{\vartheta=0}^{N(\sigma_0)} \varrho^{-\vartheta} \mathrm{e}_{\sigma_0,\vartheta}(\varrho)(\theta\tilde{u}) \quad \text{for } \tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}, \tag{6.17}$$

where $\mathrm{e}_{\sigma_0,\vartheta}(\varrho)$ is defined as

$$\mathrm{e}_{\sigma_0,\vartheta}(\varrho) = \varrho^\vartheta \kappa_\varrho^{-1} \mathrm{e}_{\sigma_0,\vartheta}\kappa_\varrho : \tilde{\mathcal{E}}_{\wedge,\sigma_0} \to C^\infty(\mathring{Y}^\wedge; E).$$

In particular, $\mathrm{e}_{\sigma_0,0}(\varrho)(\tilde{u}) = \tilde{u}$ for all $\varrho \in \mathbb{R}_+$ and $\tilde{u} \in \tilde{\mathcal{E}}_{\wedge,\sigma_0}$.

Lemma 6.18.

(i) For every $\psi \in \tilde{\mathcal{E}}_{\wedge,\sigma_0}$ and every $\vartheta \in \{0,\dots,N(\sigma_0)\}$ there exists a polynomial $q_\vartheta(y,\log x,\log\varrho)$ in $(\log x,\log\varrho)$ with coefficients in $C^\infty(Y;E)$ such that

$$\mathrm{e}_{\sigma_0,\vartheta}(\varrho)(\psi) = q_\vartheta(y,\log x,\log\varrho)x^{i(\sigma_0 - i\vartheta)}, \tag{6.19}$$

and the degree of q_ϑ with respect to $(\log x,\log\varrho)$ is bounded by some $\mu \in \mathbb{N}_0$ which is independent of $\sigma_0 \in \Sigma$, $\psi \in \tilde{\mathcal{E}}_{\wedge,\sigma_0}$, and $\vartheta \in \{0,\dots,N(\sigma_0)\}$.

(ii) Let $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ be any cut-off function near the origin, i.e., $\omega = 1$ near zero and $\omega = 0$ near infinity. Then the operator family

$$\omega\big(L_\varrho - \theta\big) : \tilde{\mathcal{E}}_{\max} \to \mathcal{K}^{\infty,-m/2}(Y^\wedge; E)$$

satisfies for every $s \in \mathbb{R}$ the norm estimate

$$\|\omega\big(L_\varrho - \theta\big)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},\mathcal{K}^{s,-m/2})} = O(\varrho^{-1}\log^\mu\varrho) \quad \text{as } \varrho \to \infty,$$

where $\mu \in \mathbb{N}_0$ is the bound for the degrees of the polynomials q_ϑ in (i), and $\mathcal{K}^{s,-m/2}(Y^\wedge; E)$ is the weighted Sobolev space defined in Section 2.

Proof. As Σ is a finite set and all spaces $\tilde{\mathcal{E}}_{\wedge,\sigma_0}$ are finite dimensional, it suffices to show that (6.19) holds for a basis of $\tilde{\mathcal{E}}_{\wedge,\sigma_0}$. We pick a basis $\{\psi_0,\dots,\psi_K\}\subset\tilde{\mathcal{E}}_{\wedge,\sigma_0}$ which is a Jordan basis for the infinitesimal generator $(\frac{m}{2}+x\partial_x)$ of the group $\kappa_\varrho|_{\tilde{\mathcal{E}}_{\wedge,\sigma_0}}\in\mathscr{L}(\tilde{\mathcal{E}}_{\wedge,\sigma_0})$. Recall that $\tilde{\mathcal{E}}_{\wedge,\max}$ is κ-invariant, and so are necessarily all the spaces $\tilde{\mathcal{E}}_{\wedge,\sigma_0}$. Note that the only eigenvalue of $(\frac{m}{2}+x\partial_x)$ on $\tilde{\mathcal{E}}_{\wedge,\sigma_0}$ is $m/2+i\sigma_0$.

Consequently, for each j we may write

$$\kappa_\varrho\psi_j=\varrho^{m/2+i\sigma_0}\sum_{k=0}^K p_{jk}(\log\varrho)\psi_k,$$

where p_{jk} is a polynomial, and thus

$$\mathrm{e}_{\sigma_0,\vartheta}(\varrho)(\psi_j)=\varrho^{\vartheta}\kappa_\varrho^{-1}\mathrm{e}_{\sigma_0,\vartheta}(\kappa_\varrho\psi_j)=\sum_{k=0}^K p_{jk}(\log\varrho)\varrho^{i(\sigma_0-i\vartheta)}\varrho^{m/2}\kappa_\varrho^{-1}\mathrm{e}_{\sigma_0,\vartheta}(\psi_k).$$

Every $\mathrm{e}_{\sigma_0,\vartheta}(\psi_k)$ is a singular function of the form

$$\Big(\sum_{\nu=0}^{m^{(k)}_{\sigma_0-i\vartheta}} c^{(k)}_{\sigma_0-i\vartheta,\nu}(y)\log^\nu x\Big)x^{i(\sigma_0-i\vartheta)},$$

and so

$$\varrho^{i(\sigma_0-i\vartheta)}\varrho^{m/2}\kappa_\varrho^{-1}\mathrm{e}_{\sigma_0,\vartheta}(\psi_k)=\Big(\sum_{\nu=0}^{m^{(k)}_{\sigma_0-i\vartheta}} c^{(k)}_{\sigma_0-i\vartheta,\nu}(y)(\log x-\log\varrho)^\nu\Big)x^{i(\sigma_0-i\vartheta)}.$$

Hence (i) is proved.

For the proof of (ii) note that according to (6.17) and (i), we have for $\tilde{u}\in\tilde{\mathcal{E}}_{\sigma_0}$

$$\begin{aligned}\omega\big(L_\varrho-\theta\big)\tilde{u}&=\omega\sum_{\vartheta=1}^{N(\sigma_0)}\varrho^{-\vartheta}\mathrm{e}_{\sigma_0,\vartheta}(\varrho)(\theta\tilde{u})\\&=\varrho^{-1}\sum_{\vartheta=1}^{N(\sigma_0)}\varrho^{1-\vartheta}\omega q_\vartheta(y,\log x,\log\varrho)x^{i(\sigma_0-i\vartheta)},\end{aligned}$$

and consequently

$$\|\omega\big(L_\varrho-\theta\big)\tilde{u}\|_{\mathcal{K}^{s,-m/2}}\le\text{const}\cdot\big(\varrho^{-1}\log^\mu\varrho\big)$$

for $\varrho\ge1$, which then in fact holds for all $\tilde{u}\in\tilde{\mathcal{E}}_{\max}$. As

$$\omega\big(L_\varrho-\theta\big):\tilde{\mathcal{E}}_{\max}\to\mathcal{K}^{s,-m/2}(Y^\wedge;E)$$

is continuous for every $\varrho>0$, we obtain (ii) from the Banach-Steinhaus theorem. $\square$

Lemma 6.20. *Fix a cut-off function $\omega\in C_0^\infty([0,1))$ near 0. For $\varrho\ge1$ consider the operator family*

$$\tilde{\mathbf{K}}(\varrho)=\omega_\varrho\tilde{\kappa}_\varrho:\tilde{\mathcal{E}}_{\max}\to\mathcal{D}^\infty_{\max}(A)=\bigcap_{t\in\mathbb{R}}\mathcal{D}^t_{\max}(A),$$

where $\omega_\varrho(x)=\omega(\varrho x)$. If $q:\mathcal{D}_{\max}(A)\to\tilde{\mathcal{E}}_{\max}$ is the canonical projection, then

$$q\circ\tilde{\mathbf{K}}(\varrho)=\tilde{\kappa}_\varrho,$$

and we have the norm estimates

$$\|\tilde{\mathbf{K}}(\varrho)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},x^{-m/2}L^2_b)} = O(1) \quad \text{as } \varrho\to\infty, \tag{6.21}$$

$$\|\tilde{\mathbf{K}}(\varrho)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},\mathcal{D}_{\max})} = O(\varrho^m) \quad \text{as } \varrho\to\infty. \tag{6.22}$$

Moreover, for every $t\in\mathbb{R}$ there exists $M_t\in\mathbb{R}$ such that

$$\|\tilde{\mathbf{K}}(\varrho)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},\mathcal{D}^t_{\max})} = O(\varrho^{M_t}) \quad \text{as } \varrho\to\infty. \tag{6.23}$$

Proof. That $\tilde{\mathbf{K}}(\varrho)$ is a lift of $\tilde{\kappa}_\varrho$ to $\mathcal{D}^\infty_{\max}(A)$ is evident from the definition. In order to show the norm estimates, it is sufficient to consider for each $\sigma_0\in\Sigma$ the restriction

$$\tilde{\mathbf{K}}_{\sigma_0}(\varrho) = \tilde{\mathbf{K}}(\varrho)|_{\tilde{\mathcal{E}}_{\sigma_0}} : \tilde{\mathcal{E}}_{\sigma_0}\to\mathcal{D}^\infty_{\max}(A)$$

and prove the estimates for this operator. Recall that $\tilde{\kappa}_\varrho=\kappa_\varrho L_\varrho$ so that for $\tilde{u}\in\tilde{\mathcal{E}}_{\sigma_0}$ we have $\tilde{\mathbf{K}}_{\sigma_0}(\varrho)\tilde{u}=\kappa_\varrho(\omega L_\varrho\tilde{u})$. On the other hand, by Lemma 6.18, $\omega L_\varrho\to\omega\theta$ in $\mathscr{L}(\tilde{\mathcal{E}}_{\max},x^{-m/2}L^2_b)$ as $\varrho\to\infty$, so the family ωL_ϱ is uniformly bounded for $\varrho\geq 1$. Thus

$$\|\tilde{\mathbf{K}}_{\sigma_0}(\varrho)\tilde{u}\|_{x^{-m/2}L^2_b(M;E)} \leq \text{const}\|\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big)\|_{x^{-m/2}L^2_b(Y^\wedge;E)} \leq \text{const}\|\omega\tilde{u}\|_{\mathcal{D}_{\max}}$$

since the norm $\|\omega\cdot\|_{\mathcal{D}_{\max}}$ is an admissible norm on the finite dimensional space $\tilde{\mathcal{E}}_{\sigma_0}$. Recall that κ_ϱ is an isometry in $x^{-m/2}L^2_b$. Finally, the above estimate gives (6.21).

For proving (6.22) we need to show that

$$\|A\tilde{\mathbf{K}}_{\sigma_0}(\varrho)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\sigma_0},x^{-m/2}L^2_b)} = O(\varrho^m) \text{ as } \varrho\to\infty.$$

Thus we will prove that there exists a constant $C>0$, independent of $\tilde{u}\in\tilde{\mathcal{E}}_{\sigma_0}$ and $\varrho\geq 1$, such that

$$\|A(\kappa_\varrho(\omega L_\varrho\tilde{u}))\|_{x^{-m/2}L^2_b(Y^\wedge;E)} \leq C\varrho^m\|\omega\tilde{u}\|_{\mathcal{D}_{\max}}.$$

To this end we split A near the boundary as in (6.3) and use (6.17) to obtain

$$\begin{aligned} &A(\kappa_\varrho(\omega L_\varrho\tilde{u}))\\ &\quad= \Big(x^{-m}\sum_{k=0}^{m-1}P_kx^k\Big)\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big) + \tilde{P}_m\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big)\\ &\quad= \varrho^m\kappa_\varrho\Big(x^{-m}\sum_{k=0}^{m-1}\varrho^{-k}P_kx^k\Big)\Big(\omega\sum_{j=0}^{N(\sigma_0)}\varrho^{-j}\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})\Big) + \tilde{P}_m\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big)\\ &\quad= \sum_{\vartheta=0}^{2m-2}\varrho^{m-\vartheta}\kappa_\varrho\Big(x^{-m}\sum_{\substack{k+j=\vartheta\\0\leq k,j\leq m-1}}\big(P_kx^k\big)\big(\omega\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})\big)\Big) + \tilde{P}_m\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big) \end{aligned} \tag{6.24}$$

with the convention that $e_{\sigma_0,j}(\varrho)=0$ for $j>N(\sigma_0)$.

For every $\vartheta\in\{0,\dots,2m-2\}$ we consider the family of linear maps

$$\tilde{u}\mapsto x^{-m}\sum_{\substack{k+j=\vartheta\\0\leq k,j\leq m-1}}\big(P_kx^k\big)\big(\omega\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})\big) : \tilde{\mathcal{E}}_{\sigma_0}\to x^{-m/2}L^2_b(Y^\wedge;E). \tag{6.25}$$

We will prove that (6.25) is well-defined, i.e., every $\tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}$ is indeed mapped into $x^{-m/2}L^2_b(Y^\wedge;E)$, and that the operator norm of each map is bounded by a constant times $\log^\mu \varrho$ as $\varrho \to \infty$ with μ as in Lemma 6.18. Then, for every $\vartheta \in \{1,\dots,2m-2\}$,

$$\Big\| \varrho^{m-\vartheta}\kappa_\varrho\Big(x^{-m}\sum_{\substack{k+j=\vartheta\\ 0\le k,j\le m-1}}\big(P_k x^k\big)\big(\omega \mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})\big)\Big)\Big\|_{x^{-m/2}L^2_b} \\ \le \mathrm{const}\cdot\big(\varrho^{m-\vartheta}\log^\mu\varrho\big)\|\omega\tilde{u}\|_{\mathcal{D}_{\max}},$$

while for $\vartheta = 0$,

$$\varrho^m\kappa_\varrho x^{-m}P_0\omega \mathrm{e}_{\sigma_0,0}(\varrho)(\theta\tilde{u}) = \varrho^m\kappa_\varrho A_\wedge\omega(\theta\tilde{u}) = A_\wedge\kappa_\varrho\big(\omega\theta\tilde{u}\big), \tag{6.26}$$

so for this term we have a norm estimate without log.

Let $\tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function near 0 with $\omega \prec \tilde{\omega}$. Then there are suitable $\varphi,\tilde{\varphi} \in C_0^\infty(\mathbb{R}_+)$ such that for all $\tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}$,

$$x^{-m}\sum_{\substack{k+j=\vartheta\\ 0\le k,j\le m-1}}\big(P_k x^k\big)\big(\omega \mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})\big) \\ = \tilde{\omega}x^{-m}\sum_{\substack{k+j=\vartheta\\ 0\le k,j\le m-1}}\big(P_k x^k\big)\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u}) + \tilde{\varphi}x^{-m}\sum_{\substack{k+j=\vartheta\\ 0\le k,j\le m-1}}\big(P_k x^k\big)\varphi \mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u}). \tag{6.27}$$

Using Lemma 6.18 we get that the second sum in (6.27) is a polynomial in $\log\varrho$ of degree at most μ with coefficients in $x^{-m/2}L^2_b(Y^\wedge;E)$. As both $A(\kappa_\varrho(\omega L_\varrho\tilde{u}))$ and $\tilde{P}_m(\kappa_\varrho(\omega L_\varrho\tilde{u}))$ belong to $x^{-m/2}L^2_b(Y^\wedge;E)$, we get from the equations (6.24) and (6.27) that necessarily

$$x^{-m}\sum_{\substack{k+j=\vartheta\\ 0\le k,j\le m-1}}\big(P_k x^k\big)\big(\omega \mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})\big) \in x^{-m/2}L^2_b(Y^\wedge;E)$$

for all $\varrho \in \mathbb{R}_+$ and all $\tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}$, and, moreover, that

$$\tilde{\omega}x^{-m}\sum_{\substack{k+j=\vartheta\\ 0\le k,j\le m-1}}\big(P_k x^k\big)\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u}) = 0$$

for $\Im\sigma_0 - \vartheta \ge -m/2$. Observe that these functions are actually of the form

$$\tilde{\omega}\Big(\sum_\nu c_{\sigma_0-i(\vartheta-m),\nu}(y)\log^\nu x\Big)x^{i(\sigma_0-i(\vartheta-m))}.$$

For $\Im\sigma_0 - \vartheta < -m/2$ every single summand $\tilde{\omega}x^{-m}\big(P_k x^k\big)\mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})$ clearly belongs to $x^{-m/2}L^2_b(Y^\wedge;E)$, and by Lemma 6.18 it is a polynomial in $\log\varrho$ of degree at most μ with coefficients in $x^{-m/2}L^2_b(Y^\wedge;E)$.

Summing up, we have shown that for every $\tilde{u} \in \tilde{\mathcal{E}}_{\sigma_0}$ the function

$$x^{-m}\sum_{\substack{k+j=\vartheta\\ 0\le k,j\le m-1}}\big(P_k x^k\big)\big(\omega \mathrm{e}_{\sigma_0,j}(\varrho)(\theta\tilde{u})\big)$$

is a polynomial in $\log\varrho$ of degree at most μ with coefficients in $x^{-m/2}L^2_b(Y^\wedge;E)$. The desired norm estimates for (6.25) follow from the Banach-Steinhaus theorem.

On the other hand,

$$\begin{aligned}\|\tilde{P}_m\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big)\|_{x^{-m/2}L^2_b} &= \|\kappa_\varrho^{-1}\tilde{P}_m\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big)\|_{x^{-m/2}L^2_b}\\ &\le \|\omega_0\kappa_\varrho^{-1}\tilde{P}_m\kappa_\varrho\omega_1\|_{\mathscr{L}(\mathcal{K}^{m,-m/2},x^{-m/2}L^2_b)}\|\omega L_\varrho\tilde{u}\|_{\mathcal{K}^{m,-m/2}}\end{aligned}$$

for cut-off functions $\omega_0,\omega_1\in C_0^\infty([0,1))$ with $\omega\prec\omega_1\prec\omega_0$. Lemma 6.18 implies $\|\omega L_\varrho\tilde{u}\|_{\mathcal{K}^{m,-m/2}}\le\text{const}\,\|\omega\tilde{u}\|_{\mathcal{D}_{\max}}$, so

$$\|\tilde{P}_m\kappa_\varrho\big(\omega L_\varrho\tilde{u}\big)\|_{x^{-m/2}L^2_b}\le\text{const}\|\omega\tilde{u}\|_{\mathcal{D}_{\max}}$$

since $\|\omega_0\kappa_\varrho^{-1}\tilde{P}_m\kappa_\varrho\omega_1\|_{\mathscr{L}(\mathcal{K}^{m,-m/2},x^{-m/2}L^2_b)}=O(1)$ as $\varrho\to\infty$. Thus (6.22) is proved.

Finally, an inspection of the proof reveals that for $t\in\mathbb{R}$ we obtain

$$\begin{aligned}\|\tilde{\mathbf{K}}(\varrho)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},x^{-m/2}H^t_b)} &= O(\|\kappa_\varrho\|_{\mathscr{L}(\mathcal{K}^{t,-m/2})})\ \text{ as } \varrho\to\infty,\\ \|\tilde{\mathbf{K}}(\varrho)\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},\mathcal{D}^t_{\max})} &= O(\varrho^m\|\kappa_\varrho\|_{\mathscr{L}(\mathcal{K}^{t,-m/2})})\ \text{ as } \varrho\to\infty,\end{aligned}$$

and consequently (6.23) follows because the norm $\|\kappa_\varrho\|_{\mathscr{L}(\mathcal{K}^{t,-m/2})}$ behaves polynomially as $\varrho\to\infty$. □

Proof of Theorem 6.9. Fix some complement $\mathcal{E}_{\max}$ of $\mathcal{D}_{\min}$ in $\mathcal{D}_{\max}$ and let $\mathcal{E}\subset\mathcal{E}_{\max}$ be a subspace such that $\mathcal{D}=\mathcal{D}_{\min}\oplus\mathcal{E}$. With respect to this decomposition the operator $A_{\mathcal{D}}-\lambda$ can be written as

$$(A_{\mathcal{D}}-\lambda)=\big((A-\lambda)|_{\mathcal{D}_{\min}}\quad(A-\lambda)|_{\mathcal{E}}\big):\begin{array}{c}\mathcal{D}_{\min}\\ \oplus\\ \mathcal{E}\end{array}\to x^{-m/2}L^2_b(M;E).$$

Let $d''=\dim\mathcal{E}$. Under the ellipticity condition on $A-\lambda$ and the injectivity of $A_\wedge-\lambda$ on $\mathcal{D}_{\wedge,\min}$ we already proved in Theorem 5.34 the existence of a parametrix $B(\lambda)$ of $A-\lambda$ on $\mathcal{D}_{\min}$ and a generalized Green remainder $\big(0\quad K(\lambda)\big)$ of order m such that

$$\big((A-\lambda)|_{\mathcal{D}_{\min}}\quad K(\lambda)\big):\begin{array}{c}\mathcal{D}_{\min}\\ \oplus\\ \mathbb{C}^{d''}\end{array}\to x^{-m/2}L^2_b(M;E)$$

is invertible for λ sufficiently large with inverse

$$\big((A-\lambda)|_{\mathcal{D}_{\min}}\quad K(\lambda)\big)^{-1}=\begin{pmatrix}B(\lambda)\\ T(\lambda)\end{pmatrix},\tag{6.28}$$

where $\begin{pmatrix}0\\ T(\lambda)\end{pmatrix}$ is a generalized Green remainder of order $-m$. Since

$$I=\begin{pmatrix}B(\lambda)\\ T(\lambda)\end{pmatrix}\big((A-\lambda)|_{\mathcal{D}_{\min}}\quad K(\lambda)\big)=\begin{pmatrix}B(\lambda)(A-\lambda)|_{\mathcal{D}_{\min}} & B(\lambda)K(\lambda)\\ T(\lambda)(A-\lambda)|_{\mathcal{D}_{\min}} & T(\lambda)K(\lambda)\end{pmatrix},$$

we have $B(\lambda)(A-\lambda)|_{\mathcal{D}_{\min}}=1$ and $T(\lambda)(A-\lambda)|_{\mathcal{D}_{\min}}=0$. Then

$$\begin{pmatrix}B(\lambda)\\ T(\lambda)\end{pmatrix}\big((A-\lambda)|_{\mathcal{D}_{\min}}\quad(A-\lambda)|_{\mathcal{E}}\big)=\begin{pmatrix}1 & B(\lambda)(A-\lambda)|_{\mathcal{E}}\\ 0 & T(\lambda)(A-\lambda)|_{\mathcal{E}}\end{pmatrix}\tag{6.29}$$

which implies that $\big((A-\lambda)|_{\mathcal{D}_{\min}}\quad(A-\lambda)|_{\mathcal{E}}\big)$ is invertible if and only if

$$F(\lambda)=T(\lambda)(A-\lambda):\mathcal{E}\to\mathbb{C}^{d''}\tag{6.30}$$

is invertible. Moreover, we get the explicit representation

$$(A_{\mathcal{D}}-\lambda)^{-1}=B(\lambda)+\big(1-B(\lambda)(A-\lambda)\big)F(\lambda)^{-1}T(\lambda),\tag{6.31}$$

and (6.10) follows from Corollary 5.37.

As $F(\lambda)$ and $1-B(\lambda)(A-\lambda)$ vanish on $\mathcal{D}_{\min}$ for large λ, they descend to operators $F(\lambda):\tilde{\mathcal{E}}_{\max}\to\mathbb{C}^{d''}$ and $1-B(\lambda)(A-\lambda):\tilde{\mathcal{E}}_{\max}\to\mathcal{D}_{\max}$. If $\tilde{\mathcal{E}}=\mathcal{D}/\mathcal{D}_{\min}$, then the invertibility of (6.30) is equivalent to the invertibility of

$$F(\lambda):\tilde{\mathcal{E}}\to\mathbb{C}^{d''},$$

and in this case, (6.31) still makes sense in this context.

Let $q:\mathcal{D}_{\max}\to\tilde{\mathcal{E}}_{\max}$ be the canonical projection. The resolvent $(A_{\mathcal{D}}-\lambda)^{-1}$ and $F(\lambda)^{-1}:\mathbb{C}^{d''}\to\tilde{\mathcal{E}}_{\max}$ are related by the formulas

$$\begin{aligned} F(\lambda)^{-1}&=q(A_{\mathcal{D}}-\lambda)^{-1}K(\lambda):\mathbb{C}^{d''}\to\tilde{\mathcal{E}}_{\max},\\ q(A_{\mathcal{D}}-\lambda)^{-1}&=F(\lambda)^{-1}T(\lambda):x^{-m/2}L^2_b\to\tilde{\mathcal{E}}_{\max} \end{aligned}$$

in view of $T(\lambda)K(\lambda)=1$, cf. (6.28).

Under the assumptions of Theorem 6.9 we will prove that $F(\lambda):\tilde{\mathcal{E}}\to\mathbb{C}^{d''}$ is invertible for large λ, and that the inverse satisfies the estimate

$$\|\tilde{\kappa}^{-1}_{[\lambda]^{1/m}}F(\lambda)^{-1}\|_{\mathscr{L}(\mathbb{C}^{d''},\tilde{\mathcal{E}}_{\max})}=O(1)\ \text{ as } |\lambda|\to\infty. \tag{6.32}$$

Observe that the parametrix construction from Theorem 5.34 gives the relation

$$\begin{pmatrix}(A_\wedge-\lambda)|_{\mathcal{D}_{\wedge,\min}} & K_\wedge(\lambda)\end{pmatrix}^{-1}=\begin{pmatrix}B_\wedge(\lambda)\\ T_\wedge(\lambda)\end{pmatrix}$$

for the κ-homogeneous principal parts of (6.28). Thus with the same reasoning as above we conclude that

$$A_\wedge-\lambda:\mathcal{D}_\wedge\to x^{-m/2}L^2_b(Y^\wedge;E)$$

is invertible if and only if the restriction of the induced operator

$$F_\wedge(\lambda)=T_\wedge(\lambda)(A_\wedge-\lambda):\tilde{\mathcal{E}}_{\wedge,\max}\to\mathbb{C}^{d''}$$

to $\tilde{\mathcal{E}}_\wedge=\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}$ is invertible. Let $q_\wedge:\mathcal{D}_{\wedge,\max}\to\tilde{\mathcal{E}}_{\wedge,\max}$ be the canonical projection. From the relations

$$\begin{aligned} F_\wedge(\lambda)^{-1}&=q_\wedge(A_{\wedge,\mathcal{D}_\wedge}-\lambda)^{-1}K_\wedge(\lambda):\mathbb{C}^{d''}\to\tilde{\mathcal{E}}_{\wedge,\max},\\ q_\wedge(A_{\wedge,\mathcal{D}_\wedge}-\lambda)^{-1}&=F_\wedge(\lambda)^{-1}T_\wedge(\lambda):x^{-m/2}L^2_b\to\tilde{\mathcal{E}}_{\wedge,\max}, \end{aligned}$$

and Proposition 6.13, we deduce that our assumption on $A_\wedge$ is equivalent to

$$\|\kappa^{-1}_{|\lambda|^{1/m}}F_\wedge(\lambda)^{-1}\|_{\mathscr{L}(\mathbb{C}^{d''},\tilde{\mathcal{E}}_{\wedge,\max})}=O(1)\ \text{ as } |\lambda|\to\infty. \tag{6.33}$$

Note that $\|K_\wedge(\lambda)\|=O(|\lambda|)$ and $\|T_\wedge(\lambda)\|=O(|\lambda|^{-1})$ as $|\lambda|\to\infty$ when considered as operators $\mathbb{C}^{d''}\to x^{-m/2}L^2_b$ and $x^{-m/2}L^2_b\to\mathbb{C}^{d''}$, respectively.

Write the operator $F(\lambda)\theta^{-1}F_\wedge(\lambda)^{-1}:\mathbb{C}^{d''}\to\mathbb{C}^{d''}$ as

$$F(\lambda)\theta^{-1}F_\wedge(\lambda)^{-1}=1+\big(F(\lambda)-F_\wedge(\lambda)\theta\big)\tilde{\kappa}_{|\lambda|^{1/m}}\theta^{-1}\kappa^{-1}_{|\lambda|^{1/m}}F_\wedge(\lambda)^{-1},$$

and let

$$R(\lambda)=\big(F(\lambda)-F_\wedge(\lambda)\theta\big)\tilde{\kappa}_{|\lambda|^{1/m}}\theta^{-1}\kappa^{-1}_{|\lambda|^{1/m}}F_\wedge(\lambda)^{-1}.$$

We will prove in Lemma 6.34 that

$$\|(F(\lambda)-F_\wedge(\lambda)\theta)\tilde{\kappa}_{|\lambda|^{1/m}}\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},\mathbb{C}^{d''})}\to 0\ \text{ as } |\lambda|\to\infty.$$

Thus together with (6.33) we obtain that $\|R(\lambda)\|\to 0$ as $|\lambda|\to\infty$. Hence $1+R(\lambda)$ is invertible for large $|\lambda|>0$, and the inverse is of the form $1+\tilde{R}(\lambda)$ with $\|\tilde{R}(\lambda)\|\to 0$

as $|\lambda| \to \infty$. This shows that $F(\lambda) : \tilde{\mathcal{E}} \to \mathbb{C}^{d''}$ is invertible from the right for large λ, and by (6.33) the right-inverse $\theta^{-1}F_\wedge(\lambda)^{-1}(1+\tilde{R}(\lambda))$ satisfies the estimate (6.32). Since

$$\dim \tilde{\mathcal{E}} = \dim \tilde{\mathcal{E}}_\wedge = d'',$$

we conclude that $F(\lambda)$ is also injective, hence the invertibility of $F(\lambda)$ is proved. In particular, the operator

$$A_{\mathcal{D}} - \lambda : \mathcal{D} \to x^{-m/2}L_b^2(M;E)$$

is invertible for large λ. It remains to show the estimates (6.7).

In order to prove (6.7) we make use of the family $\tilde{\mathbf{K}}(\varrho)$ from Lemma 6.20 and the representation (6.31) of the resolvent. Thus we may write

$$\begin{aligned}(A_{\mathcal{D}} - \lambda)^{-1} &= B(\lambda) + (1 - B(\lambda)(A-\lambda))\tilde{\mathbf{K}}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}T(\lambda)\\ &= B(\lambda) + \tilde{\mathbf{K}}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}T(\lambda)\\ &\qquad\qquad - B(\lambda)(A-\lambda)\tilde{\mathbf{K}}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}T(\lambda).\end{aligned}$$

By Remark 5.26 we have $\|B(\lambda)\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathcal{D}_{\max})} = O(1)$ as $|\lambda| \to \infty$. In view of $\|T(\lambda)\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathbb{C}^{d''})} = O(|\lambda|^{-1})$ and (6.32) we further obtain

$$\|\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}T(\lambda)\|_{\mathscr{L}(x^{-m/2}L_b^2,\tilde{\mathcal{E}}_{\max})} = O(|\lambda|^{-1}) \ \text{ as } |\lambda| \to \infty,$$

and consequently, using (6.22) we get

$$\|\tilde{\mathbf{K}}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}T(\lambda)\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathcal{D}_{\max})} = O(1) \ \text{ as } |\lambda| \to \infty.$$

On the other hand, by (6.32) and the estimates (6.21) and (6.22) we have

$$\|(A-\lambda)\tilde{\mathbf{K}}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}\|_{\mathscr{L}(\mathbb{C}^{d''},x^{-m/2}L_b^2)} = O(|\lambda|) \ \text{ as } |\lambda| \to \infty.$$

In view of $\|B(\lambda)\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathcal{D}_{\max})} = O(1)$ and $\|T(\lambda)\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathbb{C}^{d''})} = O(|\lambda|^{-1})$, we conclude that, as $|\lambda| \to \infty$,

$$\|B(\lambda)(A-\lambda)\tilde{\mathbf{K}}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}F(\lambda)^{-1}T(\lambda)\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathcal{D}_{\max})} = O(1).$$

Summing up, we have proved

$$\|(A_{\mathcal{D}} - \lambda)^{-1}\|_{\mathscr{L}(x^{-m/2}L_b^2,\mathcal{D}_{\max})} = O(1) \ \text{ as } |\lambda| \to \infty,$$

and the estimates (6.7) follow. □

The following lemma completes the proof of Theorem 6.9.

Lemma 6.34. *With the notation of the proof of Theorem 6.9, let*

$$\begin{aligned}F(\lambda) &= T(\lambda)(A-\lambda) : \tilde{\mathcal{E}}_{\max} \to \mathbb{C}^{d''},\\ F_\wedge(\lambda) &= T_\wedge(\lambda)(A_\wedge - \lambda) : \tilde{\mathcal{E}}_{\wedge,\max} \to \mathbb{C}^{d''}.\end{aligned}$$

Then

$$\|(F(\lambda) - F_\wedge(\lambda)\theta)\tilde{\kappa}_{|\lambda|^{1/m}}\|_{\mathscr{L}(\tilde{\mathcal{E}}_{\max},\mathbb{C}^{d''})} \to 0 \ \text{ as } |\lambda| \to \infty. \tag{6.35}$$

Proof. For proving (6.35) it is sufficient to consider the restrictions

$$(F(\lambda) - F_\wedge(\lambda)\theta)\tilde{\kappa}_{|\lambda|^{1/m}} : \tilde{\mathscr{E}}_{\sigma_0} \to \mathbb{C}^{d''}$$

for all $\sigma_0 \in \Sigma$. First of all, observe that

$$\begin{aligned} F(\lambda)\tilde{\kappa}_{|\lambda|^{1/m}} &= T(\lambda)(A-\lambda)\tilde{\mathbf{K}}(|\lambda|^{1/m}), \quad \text{and} \\ F_\wedge(\lambda)\theta\tilde{\kappa}_{|\lambda|^{1/m}} &= F_\wedge(\lambda)\kappa_{|\lambda|^{1/m}}\theta = T_\wedge(\lambda)(A_\wedge - \lambda)\kappa_{|\lambda|^{1/m}}\omega\theta \end{aligned}$$

with the operator family $\tilde{\mathbf{K}}(\varrho) = \omega(\varrho x)\tilde{\kappa}_\varrho$ from Lemma 6.20. If $\omega_0 \in C_0^\infty([0,1))$ is a cut-off function near zero with $\omega \prec \omega_0$, then

$$\begin{aligned} (F(\lambda) - F_\wedge(\lambda)\theta)\tilde{\kappa}_{|\lambda|^{1/m}} &= T(\lambda)(A-\lambda)\tilde{\mathbf{K}}(|\lambda|^{1/m}) - T_\wedge(\lambda)(A_\wedge - \lambda)\kappa_{|\lambda|^{1/m}}\omega\theta \\ &= T(\lambda)\omega_0(A-\lambda)\tilde{\mathbf{K}}(|\lambda|^{1/m}) - T_\wedge(\lambda)\omega_0(A_\wedge - \lambda)\kappa_{|\lambda|^{1/m}}\omega\theta \\ &= T(\lambda)\omega_0\Big((A-\lambda)\tilde{\mathbf{K}}(|\lambda|^{1/m}) - (A_\wedge - \lambda)\kappa_{|\lambda|^{1/m}}\omega\theta\Big) \\ &\qquad + (T(\lambda) - T_\wedge(\lambda))\omega_0(A_\wedge - \lambda)\kappa_{|\lambda|^{1/m}}\omega\theta \\ &= T(\lambda)\omega_0\Big(A\tilde{\mathbf{K}}(|\lambda|^{1/m}) - A_\wedge\kappa_{|\lambda|^{1/m}}\omega\theta\Big) \\ &\qquad - T(\lambda)\omega_0\lambda\Big(\tilde{\mathbf{K}}(|\lambda|^{1/m}) - \kappa_{|\lambda|^{1/m}}\omega\theta\Big) \\ &\qquad\qquad + (T(\lambda) - T_\wedge(\lambda))\omega_0(A_\wedge - \lambda)\kappa_{|\lambda|^{1/m}}\omega\theta. \end{aligned}$$

By (6.24), (6.26), and Lemma 6.18 it follows that the norm of

$$A\tilde{\mathbf{K}}(|\lambda|^{1/m}) - A_\wedge\kappa_{|\lambda|^{1/m}}\omega\theta = A\tilde{\mathbf{K}}(|\lambda|^{1/m}) - |\lambda|\kappa_{|\lambda|^{1/m}}A_\wedge\omega\theta$$

and

$$\lambda\big(\tilde{\mathbf{K}}(|\lambda|^{1/m}) - \kappa_{|\lambda|^{1/m}}\omega\theta\big) = \lambda\kappa_{|\lambda|^{1/m}}\omega(L_{|\lambda|^{1/m}} - \theta)$$

in $\mathscr{L}(\tilde{\mathscr{E}}_{\sigma_0}, x^{-m/2}L_b^2)$ are both $O(|\lambda|^{1-1/m}\log^\mu|\lambda|)$ as $|\lambda| \to \infty$. Finally, because of the norm estimates $\|T(\lambda)\omega_0\| = O(|\lambda|^{-1})$, $\|(A_\wedge - \lambda)\kappa_{|\lambda|^{1/m}}\omega\theta\| = O(|\lambda|)$, and also $\|\big(T(\lambda) - T_\wedge(\lambda)\big)\omega_0\| = O(|\lambda|^{-1-1/m})$ as $|\lambda| \to \infty$, the lemma follows. □

Finally, we want to point out that under the assumptions of Theorem 6.9 we get the existence of the resolvent with polynomial bounds for the norm also for closed extensions in Sobolev spaces of arbitrary smoothness.

Theorem 6.36. *Let $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ be c-elliptic with parameter in $\Lambda \subset \mathbb{C}$, and let $\mathcal{D}^s \subset x^{-m/2}H_b^s(M;E)$ be a domain such that $A_{\mathcal{D}^s}$ is closed. Assume that Λ is a sector of minimal growth for the closed extension $A_{\wedge,\mathcal{D}^0_\wedge}$ of $A_\wedge$ in $x^{-m/2}L_b^2$, where $\mathcal{D}^0_\wedge \subset x^{-m/2}L_b^2$ is the domain associated with $\mathcal{D}^0$ according to (6.2). Then for $\lambda \in \Lambda$ sufficiently large,*

$$A_{\mathcal{D}^s} - \lambda : \mathcal{D}^s \to x^{-m/2}H_b^s(M;E)$$

is invertible and the resolvent satisfies the equation

$$(A_{\mathcal{D}^s} - \lambda)^{-1} = B(\lambda) + (A_{\mathcal{D}^s} - \lambda)^{-1}\Pi(\lambda)$$

with the parametrix $B(\lambda)$ and the projection $\Pi(\lambda)$ from Theorem 5.29. Moreover, for every $s \in \mathbb{R}$ there exists $M(s) \in \mathbb{R}$ such that

$$\begin{aligned} \big\|(A_{\mathcal{D}^s} - \lambda)^{-1}\big\|_{\mathscr{L}(x^{-m/2}H_b^s)} &= O(|\lambda|^{M(s)-1}) \quad \text{as } |\lambda| \to \infty, \\ \big\|(A_{\mathcal{D}^s} - \lambda)^{-1}\big\|_{\mathscr{L}(x^{-m/2}H_b^s, \mathcal{D}^s_{\max})} &= O(|\lambda|^{M(s)}) \quad \text{as } |\lambda| \to \infty. \end{aligned} \tag{6.37}$$

Proof. We know from Proposition 3.12 that the spectrum does not depend on the regularity $s \in \mathbb{R}$. Consequently, from Theorem 6.9 we obtain the existence of the resolvent $(A_{\mathcal{D}^s} - \lambda)^{-1}$ for large λ.

Moreover, as in the proof of Theorem 6.9 we may write

$$(A_{\mathcal{D}^s} - \lambda)^{-1} = B(\lambda) + (1 - B(\lambda)(A - \lambda))\tilde{\mathbf{K}}(|\lambda|^{1/m})\tilde{\kappa}^{-1}_{|\lambda|^{1/m}} F(\lambda)^{-1} T(\lambda).$$

According to what we have proved in this and the previous section we obtain that the norms of all operators

$$\begin{aligned} B(\lambda) &: x^{-m/2} H_b^s \to \mathcal{D}^s_{\max}, \\ T(\lambda) &: x^{-m/2} H_b^s \to \mathbb{C}^{d''}, \\ \tilde{\kappa}^{-1}_{|\lambda|^{1/m}} F(\lambda)^{-1} &: \mathbb{C}^{d''} \to \tilde{\mathcal{E}}_{\max}, \\ \tilde{\mathbf{K}}(|\lambda|^{1/m}) &: \tilde{\mathcal{E}}_{\max} \to \mathcal{D}^s_{\max}, \\ (1 - B(\lambda)(A - \lambda)) &: \mathcal{D}^s_{\max} \to \mathcal{D}^s_{\max} \end{aligned}$$

behave polynomially as $|\lambda| \to \infty$. This proves the theorem. □

Appendix A. Invertibility of Fredholm families

The theorem of this section is essential for the existence of extra conditions in order to make the family $A_\wedge - \lambda$ invertible on the model cone $Y^\wedge$. The main application of Theorem A.1 concerns the Fredholm family

$$a(\lambda) = A_\wedge - \lambda : \mathcal{D}_{\min}(A_\wedge) \to x^{-m/2} L^2_b(Y^\wedge; E),$$

where $\lambda \in \Omega = \{z \in \Lambda : |z| = 1\}$ (see also Corollary A.3).

Theorem A.1 is rather standard and widely used throughout the literature. However, since several of our key arguments in the parametrix construction given in Theorem 5.34 rely on this result, we decide to give here an independent proof.

Theorem A.1. *Let Ω be a compact connected space (C^∞-manifold), and let $a : \Omega \to \mathscr{L}(H_1, H_2)$ be a continuous (smooth) Fredholm family in the Hilbert bundles H_1 and H_2. Then there exist (smooth) vector bundles J_-, $J_+ \in \mathrm{Vect}(\Omega)$ and continuous (smooth) sections t, k, q such that*

$$\begin{pmatrix} a & k \\ t & q \end{pmatrix} : \Omega \to \mathscr{L}\left(\begin{matrix} H_1 \\ \oplus \\ J_- \end{matrix}, \begin{matrix} H_2 \\ \oplus \\ J_+ \end{matrix} \right)$$

is a family of isomorphisms. The difference $[J_+] - [J_-] \in K(\Omega)$ equals the index $\mathrm{ind}_K(a)$ of a. If a is onto or one-to-one, we can choose $J_- = 0$ or $J_+ = 0$, respectively. If Ω is contractible, then we have $J_\pm = \mathbb{C}^{N_\pm}$ with $N_\pm \in \mathbb{N}_0$.

Proof. Let $x \in \Omega$ be arbitrary. Choose (smooth) sections $s_1, \ldots, s_{N(x)}$ of H_2 such that $\{s_1(x), \ldots, s_{N(x)}(x)\}$ forms a basis of a complement of $\mathrm{rg}(a(x))$ in H_2. Define

$$k_x : \Omega \to \mathscr{L}(\mathbb{C}^{N(x)}, H_2), \qquad \begin{pmatrix} c_1 \\ \vdots \\ c_{N(x)} \end{pmatrix} \mapsto \sum_{j=1}^{N(x)} c_j s_j.$$

It follows that

$$\begin{pmatrix} a(x) & k_x(x) \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ \mathbb{C}^{N(x)} \end{matrix} \to H_2$$

is surjective and so $\begin{pmatrix} a & k_x \end{pmatrix}$ is surjective in an open neighborhood $U(x) \subset \Omega$. Let $\Omega = \bigcup_{k=1}^{M} U(x_k)$ be a covering of Ω by such neighborhoods and set

$$k = \begin{pmatrix} k_{x_1} \dots k_{x_M} \end{pmatrix} : \Omega \to \mathscr{L}\left(\bigoplus_{k=1}^{M} \mathbb{C}^{N(x_k)}, H_2 \right).$$

Then

$$\begin{pmatrix} a(x) & k(x) \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to H_2$$

is surjective for all $x \in \Omega$, where $N_- = \sum_{k=1}^{M} N(x_k)$.

So suppose without loss of generality that $a(x)$ is a surjective Fredholm family. Then $\dim \ker a(x)$ is independent of x and the disjoint union

$$J_+ = \bigsqcup_{x \in \Omega} \ker a(x)$$

is a locally trivial finite rank continuous (smooth) vector bundle. Let $\pi_x : H_1 \to J_+$ be the orthogonal projection. Then

$$\begin{pmatrix} a \\ \pi \end{pmatrix} : H_1 \to \begin{matrix} H_2 \\ \oplus \\ J_+ \end{matrix}$$

is invertible.

If a is pointwise injective, we obtain from the above argument, applied to a^*, that we may choose $J_+ = 0$. This finishes the proof of the theorem. $\square$

Remark A.2. Let H_1, H_2 be Hilbert spaces and let

$$\begin{pmatrix} a & k \\ t & q \end{pmatrix} : \Omega \to \mathscr{L}\left(\begin{matrix} H_1 \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix}, \begin{matrix} H_2 \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix} \right)$$

be a smooth family of isomorphisms as in Theorem A.1. Moreover, let $D_1' \subset H_1'$ and $D_2 \subset H_2$ be dense subspaces. Then we can modify t and k such that

$$k \in C^\infty(\Omega) \otimes (\mathbb{C}^{N_-})^* \otimes D_2 \ \text{ and } \ t \in C^\infty(\Omega) \otimes D_1' \otimes \mathbb{C}^{N_+}.$$

Corollary A.3. *Let Λ be a closed sector in $\mathbb{C}$ as defined in Section 5. Let H_1 and H_2 be Hilbert spaces with strongly continuous groups $\{\kappa_\varrho\}_{\varrho \in \mathbb{R}_+}$ and $\{\tilde{\kappa}_\varrho\}_{\varrho \in \mathbb{R}_+}$, and let $a \in C^\infty(\Lambda \backslash \{0\}, \mathscr{L}(H_1, H_2))$ be a Fredholm family that satisfies*

$$a(\varrho^d \lambda) = \varrho^\mu \tilde{\kappa}_\varrho a(\lambda) \kappa_\varrho^{-1}$$

for every $\varrho > 0$, where $d \in \mathbb{N}_0$ and $\mu \in \mathbb{R}$ are given numbers. Then there exist t, k, and q such that

$$\begin{pmatrix} a & k \\ t & q \end{pmatrix} \in C^\infty\left(\Lambda \backslash \{0\}, \mathscr{L}\left(\begin{matrix} H_1 \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix}, \begin{matrix} H_2 \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix} \right)\right)$$

is pointwise an isomorphism, and it satisfies

$$\begin{pmatrix} a(\varrho^d\lambda) & k(\varrho^d\lambda) \\ t(\varrho^d\lambda) & q(\varrho^d\lambda) \end{pmatrix} = \varrho^\mu \begin{pmatrix} \tilde{\kappa}_\varrho & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(\lambda) & k(\lambda) \\ t(\lambda) & q(\lambda) \end{pmatrix} \begin{pmatrix} \kappa_\varrho^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

for every $\varrho > 0$. If a is onto or one-to-one, then we may choose $N_- = 0$ or $N_+ = 0$, respectively.

Proof. Let $\Omega = \{z \in \Lambda : |z| = 1\}$ and let $\hat{a} = a|_\Omega$. According to Theorem A.1 there exist $\hat{t}$, $\hat{k}$, and $\hat{q}$ such that the operator function

$$\begin{pmatrix} \hat{a} & \hat{k} \\ \hat{t} & \hat{q} \end{pmatrix} \in C^\infty\left(\Omega, \mathscr{L}\begin{pmatrix} H_1 & & H_2 \\ \oplus & , & \oplus \\ \mathbb{C}^{N_-} & & \mathbb{C}^{N_+} \end{pmatrix}\right)$$

is pointwise bijective and we may choose $N_- = 0$ or $N_+ = 0$ provided that a is everywhere surjective or injective, respectively. We will be done if we can show that the extension by κ-homogeneity

$$\begin{pmatrix} a(\lambda) & k(\lambda) \\ t(\lambda) & q(\lambda) \end{pmatrix} = |\lambda|^{\frac{\mu}{d}} \begin{pmatrix} \tilde{\kappa}_{|\lambda|^{1/d}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}\big(\frac{\lambda}{|\lambda|}\big) & \hat{k}\big(\frac{\lambda}{|\lambda|}\big) \\ \hat{t}\big(\frac{\lambda}{|\lambda|}\big) & \hat{q}\big(\frac{\lambda}{|\lambda|}\big) \end{pmatrix} \begin{pmatrix} \kappa_{|\lambda|^{1/d}}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \tag{A.4}$$

for $\lambda \in \Lambda \backslash \{0\}$ depends smoothly on λ; note that the group actions are assumed to be only strongly continuous.

In fact, q is clearly C^∞ and a was assumed to be smooth. Thus we only have to check the smoothness of t and k. According to Remark A.2 we may take $\hat{k} \in C^\infty(\Omega) \otimes (\mathbb{C}^{N_-})^* \otimes D_2$ and $\hat{t} \in C^\infty(\Omega) \otimes D_1' \otimes \mathbb{C}^{N_+}$, where $D_1' \subset H_1'$ is the space of C^∞-elements of the dual group action $\{\kappa_\varrho'\}$ on H_1', and D_2 is the space of C^∞-elements of the group action $\{\tilde{\kappa}_\varrho\}$ on H_2. With these choices the operator function defined in (A.4) is smooth, as desired. $\square$

Remark A.5. In our applications the group action involved is always the dilation group defined in (2.7). The space of compactly supported smooth functions is then an admissible choice for the spaces D_1' and D_2 in the proof of Corollary A.3 (see also Remark A.2).

References

[1] J. Brüning and R. Seeley, *The expansion of the resolvent near a singular stratum of conical type*, J. Funct. Anal. **95** (1991), no. 2, 255–290.

[2] J. Cheeger, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. USA **76** (1979), 2103–2106.

[3] H. Falomir, M.A. Muschietti, P.A.G. Pisani, and R. Seeley, *Unusual poles of the ζ-functions for some regular singular differential operators*, J. Phys. A **36** (2003), no. 39, 9991–10010.

[4] J. Gil, *Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators*, Math. Nachr. **250** (2003), 25–57.

[5] J. Gil, T. Krainer, and G. Mendoza, *Geometry and spectra of closed extensions of elliptic cone operators*, to appear in Canad. J. Math.

[6] J. Gil, P. Loya, and G. Mendoza, *On the index of elliptic cone operators*, preprint 2002.

[7] J. Gil and G. Mendoza, *Adjoints of elliptic cone operators*, Amer. J. Math. **125** (2003) 2, 357–408.

[8] J. Gil, B.-W. Schulze, and J. Seiler, *Cone pseudodifferential operators in the edge symbolic calculus*, Osaka J. Math. **37** (2000), no. 1, 221–260.

[9] V.A. Kondrat'ev, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Mosc. Math. Soc. **16** (1967), 227–313.

[10] H. Kumano-go, *Pseudodifferential operators*, MIT Press, Cambridge, Mass.-London, 1981.

[11] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Math. vol 136, B.G. Teubner, Stuttgart, Leipzig, 1997.
[12] P. Loya, *On the resolvent of differential operators on conic manifolds*, Comm. Anal. Geom. **10** (2002), no. 5, 877–934.
[13] R. Melrose, *Transformation of boundary value problems*, Acta Math. **147** (1981), no. 3-4, 149–236.
[14] ———, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, A K Peters, Ltd., Wellesley, MA, 1993.
[15] R. Melrose and G. Mendoza, *Elliptic operators of totally characteristic type*, MSRI Preprint, 1983.
[16] E. Mooers, *Heat kernel asymptotics on manifolds with conic singularities*, J. Anal. Math. **78** (1999), 1–36.
[17] E. Schrohe and J. Seiler, *The resolvent of closed extensions of cone differential operators*, Canad. J. Math. **57** (2005), no. 4, 771–811.
[18] B.-W. Schulze, *Pseudo-differential operators on manifolds with edges*, Proc. Symp. 'Partial Differential Equations', Holzhau 1988 (Leipzig), Teubner-Texte zur Math., vol. 112, Teubner, 1989, pp. 259–288.
[19] ———, *Pseudo-differential operators on manifolds with singularities*, Studies in Mathematics and its Applications, 24. North-Holland Publishing Co., Amsterdam, 1991.
[20] ———, *Boundary value problems and singular pseudo-differential operators*, Pure and Applied Mathematics (New York). John Wiley & Sons, Ltd., Chichester, 1998.
[21] R. Seeley, *Complex powers of an elliptic operator*, Singular Integrals, AMS Proc. Symp. Pure Math. X, 1966, Amer. Math. Soc., Providence, 1967, pp. 288–307.
[22] ———, *Topics in Pseudo-Differential Operators*, CIME Conference of Pseudo-Differential Operators 1968, Edizione Cremonese, Roma, 1969, pp. 169–305.
[23] M.A. Shubin, *Pseudodifferential operators and spectral theory*, Springer-Verlag, Berlin, 1987.

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