The trace bundle of an elliptic wedge operator

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(joint work with Thomas Krainer)

Let $\mathrm{Diff}^m_e(\mathcal{M};E,F)$ be the class of edge differential operators of order m of Mazzeo [3], associated to a manifold $\mathcal{M}$ with boundary $\mathcal{N}=\partial\mathcal{M}$ and fibration $\wp:\mathcal{N}\to\mathcal{Y}$ with typical (compact) fiber $\mathcal{Z}$; E and F are vector bundles over $\mathcal{M}$. Let x be a defining function for $\mathcal{N}$, positive in $\mathring{\mathcal{M}}$. Let $A\in x^{-m}\,\mathrm{Diff}^m_e(\mathcal{M};E,F)$, $m>0$, be a wedge differential operator. The operator A has a natural principal symbol ${}^w\!\boldsymbol{\sigma}(A)$ off the zero section of the w-cotangent bundle ${}^w\pi:{}^wT^*\mathcal{M}\to\mathcal{M}$ of $\mathcal{M}$, a section of $\mathrm{Hom}({}^w\pi^*E,{}^w\pi^*F)$, see [1]. Suppose throughout the rest of this note that A is w-elliptic, which of course means that ${}^w\!\boldsymbol{\sigma}(A)$ is invertible everywhere. Equivalently, we assume that the principal symbol of x^mA is elliptic in the sense of [3].

Denote by $\pi_\wedge:\mathcal{N}^\wedge\to\mathcal{N}$ the inward pointing normal bundle of $\mathcal{N}$ including the zero section and by $\wp_\wedge$ the composition $\mathcal{N}^\wedge\to\mathcal{N}\to\mathcal{Y}$, so $\wp_\wedge:\mathcal{N}^\wedge\to\mathcal{Y}$ is a fibration with typical fiber $\mathcal{Z}\times\overline{\mathbb{R}}_+$. We write x also for the function $dx:\mathcal{N}^\wedge\to\mathbb{R}$ determined by the defining function for $\mathcal{N}$. This x is of course a defining function for $\partial\mathcal{N}^\wedge$, the zero section of $\mathcal{N}^\wedge$.

The indicial operator of $P=x^mA$, an operator in $\mathrm{Diff}^m_e(\mathcal{N}^\wedge;\pi_\wedge^*E,\pi_\wedge^*F)$, commutes with multiplication by elements of $\wp_\wedge^*C^\infty(\mathcal{Y})$ so it can be viewed as a family of elliptic b-operators ${}^bP_y$ on the fibers $\mathcal{N}^\wedge_y$ of $\wp_\wedge$. See [5] for the meaning of b-operators and the concept of ellipticity in that context, as well as the notion of boundary spectrum, $\mathrm{spec}_b({}^bP_y)$, to be used in the statement of the theorem below.

Pick $\mu\in\mathbb{R}$ arbitrarily. For $y\in\mathcal{Y}$ let $\mathcal{T}_y$ be the space whose elements u are functions on $\mathring{\mathcal{N}}^\wedge_y$ of the form

$$u=\sum_{\substack{\sigma\in\mathrm{spec}_b({}^bP_y)\\ \mu-m<\Im\sigma<\mu}}\sum_{\ell=0}^{N_\sigma}a_{\sigma,\ell}x^{i\sigma}\log^\ell x,\quad a_{\sigma,\ell}\in C^\infty(\wp^{-1}(y);E|_{\wp^{-1}(y)})$$

and satisfy ${}^bP_yu=0$. Define

$$\mathcal{T}=\bigsqcup_{y\in\mathcal{Y}}\mathcal{T}_y,\quad \pi:\mathcal{T}\to\mathcal{Y}\text{ the natural map.}$$

If $U\subset\mathcal{Y}$ is open and u is a section of $\mathcal{T}$ over U (the meaning of which is clear), then u can be viewed as a section of $\pi_\wedge^*E$ over $\mathring{\wp}_\wedge^{-1}(U)\subset\mathcal{N}^\wedge\backslash\partial\mathcal{N}^\wedge$. Define $\mathcal{B}^\infty(U;\mathcal{T})$ as the space of sections of $\mathcal{T}$ which viewed thus are smooth over $\mathring{\wp}_\wedge^{-1}(U)$. Then $\mathcal{B}^\infty(U;\mathcal{T})$ is a module over $C^\infty(U)$, in particular, $\mathcal{B}^\infty(\mathcal{Y};\mathcal{T})$ is a module over $C^\infty(\mathcal{Y})$.

Theorem 1. *Suppose that the set*

$$\mathrm{spec}_e(A)=\{(y,\sigma)\in\mathcal{Y}\times\mathbb{C}:\sigma\in\mathrm{spec}_b({}^bP_y)\}.$$

is disjoint from $\{(y,\sigma)\in\mathcal{Y}\times\mathbb{C}:\Im\sigma=\mu,\ \mu-m\}$. *Then* $\mathcal{T}\to\mathcal{Y}$ *is a smooth vector bundle whose space of* C^∞ *sections is* $\mathcal{B}^\infty(\mathcal{Y};\mathcal{T})$.

The proof is given elsewhere (see [2]). The relevancy of the theorem lies in the well known interpretation of the indicial roots of the ${}^bP_y$ as being (after multiplication by the imaginary unit i) the leading powers of the (generalized) Taylor expansion at $x = 0$ of solutions of $Au = f$ when $u \in x^{-\mu}L^2_b(\mathcal{M}; E)$ and $f \in x^{-\mu}L^2_b(\mathcal{M}; F)$. Of course the difficulty of implementing this interpretation in a general setting lies in the possibility that the indicial roots in the range $\mu - m < \Im\sigma < \mu$ vary with y without constant multiplicity (this may even be the case if the location of the indicial roots is constant). Our theorem, or rather its proof, is one of the tools we use to handle this difficulty.

Complementing Theorem 1, we also show in [2], under the assumption that the normal family of A is invertible on its minimal domain (see [1] for the effect of this hypothesis on the nature of $\mathcal{D}_{\min}(A)$), how to construct a continuous operator $\mathcal{P} : H^m(\mathcal{Y}; \mathcal{T}) \to \mathcal{D}_{\max}(A)$ with range complementary to $\mathcal{D}_{\min}(A)$ that admits a left inverse γ modulo smoothing a smoothing operator. These two tools allow us to give solid meaning to boundary value problems of the form

$$\begin{cases} Au = f \in x^{-\mu}L^2_b(\mathcal{M}; F), \quad u \in H^m_A, \\ B\gamma u = g \end{cases}$$

where $H^m_A = \mathcal{D}_{\min}(A) \oplus \mathcal{P}(H^m(\mathcal{Y}; \mathcal{T}))$, g is a section of some other given vector bundle over $\mathcal{Y}$, B is, for instance, a pseudodifferential operator acting on sections of $\mathcal{T}$, and of course the unkown is u.

We point out that Mazzeo and Vertman [4] have results starting with a slightly different set-up concerning boundary value problems for elliptic edge operators but assuming constancy of the indicial roots.

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References

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