The resolvent trace of an elliptic cone operator

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(joint work with Juan B. Gil and Thomas Krainer)

This note discusses some aspects of the analysis leading to the proof of the main theorem in [10] (stated here as Theorem 1) on the structure of the asymptotics of the resolvent trace of a general elliptic cone operator as the spectral parameter tends to infinity, under suitable minimal growth assumptions on the principal symbols of the operator.

We deal with an elliptic cone differential operator

$$A : C_c^\infty(\mathring{\mathcal{M}};E) \subset x^\gamma L_b^2(\mathcal{M};E) \to x^\gamma L_b^2(\mathcal{M};E) \tag{1}$$

of positive order m, an element of $x^{-m}\operatorname{Diff}_b^m(\mathcal{M};E)$; $\mathcal{M}$ is a compact n-manifold with boundary, $E \to \mathcal{M}$ is a smooth complex Hermitian vector bundle, and L_b^2 is defined using some fixed smooth positive b-density $\mathfrak{m}$. As usual, x is a smooth defining function for $\mathcal{Y} = \partial\mathcal{M}$, positive in the interior of $\mathcal{M}$. The number $\gamma \in \mathbb{R}$ is arbitrary. Ellipticity means that $P = x^m A$ is a b-elliptic differential operator. Somewhat unnaturally (see [6]), we write here ${}^c\!\boldsymbol{\sigma}(A) = {}^b\!\boldsymbol{\sigma}(P)$ (see [16] for the notation ${}^b\!\boldsymbol{\sigma}(P)$ and the notion of b-ellipticity).

Details of the following setup can be found in [6]. Let $\mathcal{D}_{\min}$, resp. $\mathcal{D}_{\max}$, be the domains of the minimal, resp. maximal, extensions of the operator (1):

$$\mathcal{D}_{\max} = \{u \in x^\gamma L_b^2(\mathcal{M};E) : Au \in x^\gamma L_b^2(\mathcal{M};E)\}$$

is a Hilbert space with respect to the inner product $(u,v)_A = (Au,Av) + (u,v)$, $u,v \in \mathcal{D}_{\max}$, and $\mathcal{D}_{\min}$ is the closure of $C_c^\infty(\mathring{\mathcal{M}};E)$ in $\mathcal{D}_{\max}$. From [13] we know that $\mathcal{D}_{\min}$ has finite codimension in $\mathcal{D}_{\max}$, hence every closed extension of (1) has as domain a subspace $\mathcal{D} \subset \mathcal{D}_{\max}$ of the form $\mathcal{D} = D + \mathcal{D}_{\min}$ where D is uniquely determined by the condition that D is orthogonal to $\mathcal{D}_{\min}$. We let $\mathcal{E}$ be the orthogonal complement of $\mathcal{D}_{\min}$ in $\mathcal{D}_{\max}$. Domains of closed extensions then correspond to the points of the various complex Grassmannian varieties associated with $\mathcal{E}$.

There is an operator

$$A_\wedge : C_c^\infty(\mathring{\mathcal{Y}}^\wedge;\wp^*E) \subset x_\wedge^\gamma L_b^2(\mathcal{Y}^\wedge;\wp^*E) \to x_\wedge^\gamma L_b^2(\mathcal{Y}^\wedge;\wp^*E) \tag{2}$$

canonically associated with A. Here $\wp : \mathcal{Y}^\wedge \to \mathcal{Y}$ is the inward pointing normal bundle of $\mathcal{Y}$ in $\mathcal{M}$ (the zero section is included), $x_\wedge = dx|_{\mathcal{Y}^\wedge}$ and the L^2 space is defined using the density $x_\wedge^{-1}dx_\wedge \otimes \mathfrak{m}_\mathcal{Y}$ where $\mathfrak{m} = x^{-1}dx \otimes \mathfrak{m}_\mathcal{Y}$ along $\mathcal{Y}$. The operator (2) has its own maximal and minimal domains $\mathcal{D}_{\wedge,\max}$ and $\mathcal{D}_{\wedge,\min}$. Lesch's result on the codimension of the latter in the former still holds. We let $\mathcal{E}_\wedge$ be the orthogonal complement of $\mathcal{D}_{\wedge,\min}$ in $\mathcal{D}_{\wedge,\max}$. There is a natural vector space isomorphism

$$\theta : \mathcal{E} \to \mathcal{E}_\wedge$$

which allows passage from domains of closed extensions of (1) to those of (2) and back, namely

$$\mathcal{D} = D + \mathcal{D}_{\min} \longleftrightarrow \mathcal{D}_\wedge = \theta(D) + \mathcal{D}_{\wedge,\min}. \tag{3}$$

Let Λ be a closed sector in $\mathbb{C}$. The main result of [7] asserts that if ${}^c\boldsymbol{\sigma}(A)-\lambda$ is invertible when $\lambda\in\Lambda$ and if in addition Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$ ($A_\wedge$with domain $\mathcal{D}_\wedge$), then Λ is a sector of minimal growth for $A_{\mathcal{D}}$, where $\mathcal{D}$ and $\mathcal{D}_\wedge$ are related by (3). In [10] we show:

Theorem 1. *Let Λ be a sector of minimal growth both for ${}^c\boldsymbol{\sigma}(A)$ and for $A_{\wedge,\mathcal{D}_\wedge}$. For any $\varphi\in C^\infty(M;\operatorname{End}(E))$ and $\ell\in\mathbb{N}$ with $m\ell>n$,*

$$\operatorname{Tr}\big(\varphi(A_{\mathcal{D}}-\lambda)^{-\ell}\big)\sim\sum\nolimits_{j=0}^{n-1}\alpha_j\lambda^{\frac{n-\ell m-j}{m}}+\alpha_n\log(\lambda)\lambda^{-\ell}+s_{\mathcal{D}}(\lambda)$$

with coefficients $\alpha_j\in\mathbb{C}$ that are independent of the choice of $\mathcal{D}$, and

$$s_{\mathcal{D}}(\lambda)\sim\sum\nolimits_{j=0}^{\infty}r_j(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)\lambda^{\nu_j/m}\ \text{ as } |\lambda|\to\infty,\tag{4}$$

where each r_j is a rational function in $N+1$ variables, $N\in\mathbb{N}_0$, with real numbers μ_k, $k=1,\dots,N$, and $0\ge\nu_j\searrow-\infty$ as $j\to\infty$. We have $r_j=p_j/q_j$ with $p_j,\ q_j\in\mathbb{C}[z_1,\dots,z_{N+1}]$ such that $q_j(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)$ is uniformly bounded away from zero for large λ.

A number of references at the end of this note, needless to say incomplete, point to earlier related work by various other authors in special cases.

Let $\operatorname{bg-res}(A_\wedge)$ be the set of $\lambda\in\mathbb{C}$ such that $A_\wedge-\lambda$ is injective on $\mathcal{D}_{\wedge,\min}$ and surjective on $\mathcal{D}_{\wedge,\max}$. For $\lambda\in\operatorname{bg-res}(A_\wedge)$ set $\mathcal{K}_\lambda=\ker(A_{\wedge,\mathcal{D}_{\wedge,\max}}-\lambda)$. Then $\lambda\in\operatorname{res}(A_{\wedge,\mathcal{D}_\wedge})$ iff $\lambda\in\operatorname{bg-res}(A_\wedge)$ and $\mathcal{K}_\lambda\oplus\mathcal{D}_\wedge=\mathcal{D}_{\wedge,\max}$. Let $\mathcal{R}_\lambda$ be the range of $A_\wedge-\lambda$ on $\mathcal{D}_{\wedge,\min}$. There exist

$$B_{\wedge,\min}(\lambda):x_\wedge^\gamma L_b^2\to\mathcal{D}_{\min}\text{ with kernel equal to }\mathcal{R}_\lambda^\perp$$

such that $B_{\wedge,\min}(\lambda)(A_\wedge-\lambda)=I$ on $\mathcal{D}_{\wedge,\min}$, and

$$B_{\wedge,\max}(\lambda):x_\wedge^\gamma L_b^2\to\mathcal{D}_{\max}\text{ with range equal to }\mathcal{K}_\lambda^\perp\cap\mathcal{D}_{\wedge,\max}$$

(the orthogonal in the space $x_\wedge^\gamma L_b^2$) such that $(A_\wedge-\lambda)B_{\wedge,\max}(\lambda)=I$ on $x_\wedge^\gamma L_b^2$. The resolvent of $A_{\wedge,\mathcal{D}_\wedge}$ is (see [6])

$$B_{\wedge,\mathcal{D}_\wedge}(\lambda)=B_{\wedge,\max}(\lambda)-\big[I-B_{\wedge,\min}(\lambda)(A_\wedge-\lambda)\big]\pi_{\wedge,\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}_\wedge}\pi_{\wedge,\max}B_{\wedge,\max}(\lambda)$$

in which $\pi_{\mathcal{K}_\lambda,\mathcal{D}_\wedge}:\mathcal{D}_{\wedge,\max}\to\mathcal{D}_{\wedge,\max}$ is the projection on $\mathcal{K}_\lambda$ according to the decomposition $\mathcal{D}_{\wedge,\max}=\mathcal{K}_\lambda\oplus\mathcal{D}_\wedge$ (this holds when $\lambda\in\operatorname{res}(A_\wedge)$) and $\pi_{\wedge,\max}$ is the orthogonal projection on $\mathcal{E}_\wedge$. Altogether, $\pi_{\wedge,\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}_\wedge}|_{\mathcal{E}_\wedge}$ is equal to the projection $\pi_{K_\lambda,D_\wedge}:\mathcal{E}_\wedge\to\mathcal{E}_\wedge$ on $K_\lambda=\pi_{\max}\mathcal{K}_\lambda$ according to $\mathcal{E}_\wedge=K_\lambda\oplus D_\wedge$.

The multiplicative group $\mathbb{R}_+$ acts canonically on $\mathcal{Y}^\wedge$. Define κ_ϱ on $C_c^\infty(\mathcal{Y}^\wedge;E)$ for $\varrho\in\mathbb{R}_+$ by $\kappa_\varrho(u)(\nu)=\varrho^{-\gamma}u(\varrho\nu)$. The operators κ_ϱ extend to give a strongly continuous unitary action of $\mathbb{R}_+$ on $x_\wedge^\gamma L_b^2(\mathcal{Y}^\wedge;\wp^*E)$. The operator $A_\wedge$ has the property $\kappa_\varrho^{-1}(A_\wedge-\varrho^m\lambda)\kappa_\varrho=\varrho^m(A_\wedge-\lambda)$, which in turn produces $\kappa_\varrho\mathcal{K}_\lambda=\mathcal{K}_{\varrho^m\lambda}$ as well as $\kappa_\varrho^{-1}B_{\wedge,\min}(\varrho^m\lambda)\kappa_\varrho=\varrho^{-m}B_{\wedge,\min}(\lambda)$ and the same formula with min replaced by max.

As a consequence, $B_{\wedge,\mathcal{D}_\wedge}(\varrho^m\lambda)$ is equal to

$$\varrho^{-m}\kappa_\varrho\big\{B_{\wedge,\max}(\lambda)-\big[I-B_{\wedge,\min}(\lambda)(A_\wedge-\lambda)\big]\pi_{K_\lambda,\kappa_\varrho^{-1}D_\wedge}\pi_{\wedge,\max}B_{\wedge,\max}(\lambda)\big\}\kappa_\varrho^{-1}.$$

This formula brings to the forefront the role played by the dynamical system $\varrho\mapsto\kappa_\varrho^{-1}D_\wedge$ in the Grassmannian $\operatorname{Gr}_k(\mathcal{E}_\wedge)$, $k=\dim D_\wedge$, especially the limiting sets, on the behavior of the resolvent. One can show that the ray through $\lambda \neq 0$ is a ray of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$ if and only if the set

$$\Omega^-(\mathcal{D}_\wedge) = \{D \in \mathrm{Gr}_k(\mathcal{E}_\wedge) : \exists \{\varrho_k\}_{k=1}^\infty,\ \varrho \to \infty,\ \lim \kappa_\varrho^{-1} D_\wedge = D\}$$

is disjoint from $\mathcal{V}_{K_\lambda} = \{D \in \mathrm{Gr}_k(\mathcal{E}_\wedge) : D \cap K_\lambda = 0\}$, see [8]. The hypotheses of Theorem 1 imply this is the case for λ in a closed arc in Λ. In [10] we elucidate the asymptotics of $\pi_{K_\lambda,\kappa_\varrho^{-1}D_\wedge}$ and use it to determine the asymptotics of the trace of the resolvent of $A_{\mathcal{D}}$.

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