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Anisotropic blowup and compactification

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Abstract. The anisotropic blowup along the zero section and the fiberwise compactification of a real vector bundle $F \to \mathcal{X}$ are manifolds with boundary for which the natural structure bundle is the b-tangent bundle. We give a necessary and sufficient condition for these procedures to give a b-complex structure when applied to a holomorphic vector bundle over a complex manifold, and analyze some aspects of the structure induced on the boundary.

1. Introduction

The idea of consistently using the b-tangent bundle ${}^bT\mathcal{M}$ on a manifold $\mathcal{M}$ with boundary instead of the standard tangent bundle as the structure bundle originated with Melrose [**3**] and, together with a number of variants, has proved to be quite useful in the study of geometric and analytic problems on such manifolds, manifolds with conical and other types of singularities, and certain noncompact manifolds. It is thus natural to translate the idea to the context of complex manifold, more generally, of CR manifolds, in the following form:

Definition 1.1. Let $\mathcal{M}$ be a smooth manifold with boundary. A b-CR structure on $\mathcal{M}$ is an involutive subbundle $\mathcal{V}$ of the complexification $\mathbb{C}{}^bT\mathcal{M}$ of ${}^bT\mathcal{M}$ such that

$$\mathcal{V} \cap \overline{\mathcal{V}} = 0. \tag{1.1}$$

The corank of $\mathcal{V} \oplus \overline{\mathcal{V}}$ in $\mathbb{C}{}^bT\mathcal{M}$ is the CR codimension of the structure. If the CR codimension is 0, then $\mathcal{V}$ is a b-complex structure and we write ${}^bT^{1,0}\mathcal{M}$ instead of $\mathcal{V}$ and let ${}^bT^{0,1}\mathcal{M} = \overline{\mathcal{V}}$.

For the definition of ${}^bT\mathcal{M}$ see Melrose [**4**]. The notion of b-CR structure was introduced in [**6**] with a focus on strictly pseudoconvex structures (necessarily of CR codimension 1). Some aspects of b-complex structures were studied in [**7**] and generalized in [**8**]. Section 2 of the last mentioned paper contains a complete description of general aspects of b-complex structures and a brief account of relevant facts concerning the b-tangent bundle. In this note we give a necessary and sufficient condition in order for a real fiberwise anisotropic blowup along the zero section, or compactification, of a holomorphic vector bundle $E \to \mathcal{X}$ over a complex manifold

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$\mathcal{X}$ to induce a b-complex structure on the resulting manifold and analyze some aspects of the structure induced on the boundary. The present note serves, in particular, as a source of examples for the structures discussed in [**7, 8**].

The blowup and compactification in the real context will be discussed in Section 2. In Section 3 we analyze the conditions on the real blowup or compactification that lead to the existence of a b-complex structure.

Some of the features of the structure induced on the boundary of the blowup or compactification will be discussed in Section 4. In general, whatever the CR codimension, the boundary of a b-CR manifold inherits a rich structure. The b-CR structure $\mathcal{V}$ determines an involutive vector subbundle $\overline{\mathcal{V}}_b \subset \mathbb{C}T\partial\mathcal{M}$, canonically isomorphic to $\overline{\mathcal{V}}|_{\partial\mathcal{M}}$. Associated with this vector subbundle there is a canonical complex of first order differential operators $\overline{\mathbb{D}}$ on sections of the exterior powers of the dual of $\overline{\mathcal{V}}_b$. In addition to $\mathcal{V}_b$ and the complex, the b-CR structure determines a class $\boldsymbol{\beta}$ of $\overline{\mathbb{D}}$-closed sections of $\overline{\mathcal{V}}_b^*$ with elements $\beta_{\mathfrak{r}}$ labeled by the (smooth) defining functions $\mathfrak{r} : \mathcal{M} \to \mathbb{R}$ of the boundary (taken to be positive in the interior), in such a way that if $\mathfrak{r}$ and $\mathfrak{r}'$ are defining functions for $\partial\mathcal{M}$, so $\mathfrak{r}' = \mathfrak{r}e^u$ for some smooth real-valued function, then $\beta_{\mathfrak{r}'} = \beta_{\mathfrak{r}} + \overline{\mathbb{D}}u$. The elements of $\boldsymbol{\beta}$ encode the relation between $\overline{\mathcal{V}}_b$ and $\overline{\mathcal{V}}|_{\partial\mathcal{M}}$. In the b-complex case, if ${}^b\overline{\partial}$ denotes the $\overline{\partial}$-operators, then $\beta_{\mathfrak{r}}$ is the section $\mathfrak{r}^{-1}\,{}^b\overline{\partial}\mathfrak{r}|_{\partial\mathcal{M}}$ of ${}^b\bigwedge^{0,1}\mathcal{M}|_{\partial\mathcal{M}}$ regarded as a section of $\overline{\mathcal{V}}_b^*$ via the isomorphism $\overline{\mathcal{V}}_b \leftrightarrow {}^bT^{0,1}\mathcal{M}|_{\partial\mathcal{M}}$, cf. [**8**] ($\beta_{\mathfrak{r}}$ is define in a similar way in the general case, cf. [**6**]). Still in the b-complex case the vector bundle $\overline{\mathcal{V}}$ is an elliptic structure, which means that $\mathcal{V} \oplus \overline{\mathcal{V}} = \mathbb{C}T\partial\mathcal{M}$. The intersection $\mathcal{V} \cap \overline{\mathcal{V}}$ is a rank 1 bundle globally spanned by the real vector field

$$\mathcal{T} = J(\mathfrak{r}\partial_{\mathfrak{r}}); \tag{1.2}$$

$\mathfrak{r}\partial_{\mathfrak{r}}$ denotes a certain canonical section of ${}^bT\mathcal{M}$, so $\mathcal{T}$ is a canonical vector field on $\partial\mathcal{M}$ determined by the b-complex structure. Whatever the element $\beta_{\mathfrak{r}} \in \boldsymbol{\beta}$, its kernel is a CR structure on $\partial\mathcal{M}$.

2. Blowup and compactification of real vector bundles

Let $\mathcal{X}$ be a smooth n-manifold and $\pi : F \to \mathcal{X}$ a smooth real vector bundle. Assume given an isomorphism $A : F \to F$ (covering the identity) such that for each $p \in \mathcal{X}$, all eigenvalues of $A : F_p \to F_p$ are positive. With A we get an $\mathbb{R}^+$-action on $F\backslash 0$,

$$t \cdot u = t^A u \tag{2.1}$$

whose set of orbits, $\mathcal{N}$, is a Hausdorff space homeomorphic to the sphere bundle of F (if all the eigenvalues of A are real but not all of the same sign, then the space of orbits is not Hausdorff). Let $\pi_A : F\backslash 0 \to \mathcal{N}$ be the quotient map.

Following [**1**], define the A-blowup of F along its zero section to be the set $F_A = (F\backslash 0) \sqcup \mathcal{N}$ together with the map $\wp_A : F_A \to F$ which is the identity on $F\backslash 0$ and sends the orbit $p \in \mathcal{N}$ to $0 \in F_x$ if p lies in F_x. The topology of F_A is the smallest topology containing the open sets of $F\backslash 0$ and the sets $\pi_A^{-1}(U) \cup U$ for each open set $U \subset \mathcal{N}$.

Still following [**1**], we give F_A the structure of a smooth manifold with boundary as follows. First, define a function $F\backslash 0 \to \mathbb{R}$ to be A-homogeneous of degree s if $f(t \cdot u) = t^s f(u)$ for all $t \in \mathbb{R}$ and $u \in F\backslash 0$. Suppose

$$\rho : F\backslash 0 \to \mathbb{R} \text{ is smooth, } A\text{-homogeneous of degree 1, and } \rho > 0. \tag{2.2}$$

Then

$$d\rho \neq 0, \quad \rho(t\cdot u) \to 0 \text{ as } t \to 0, \qquad \text{and } \rho(t\cdot u) \to \infty \text{ as } t \to \infty. \tag{2.3}$$

For example, let h be a metric on the vector bundle F, let SF be the unit sphere bundle of F, and let R be the infinitesimal generator of the action (2.1). Then the solution $\rho : F\backslash 0 \to \mathbb{R}$ of $R\rho = \rho$ with $\rho = 1$ on SF satisfies (2.2). With arbitrary ρ satisfying (2.2) let

$$\mathcal{N}_\rho = \{\theta \in F : \rho(\theta) = 1\}, \quad F_{A,\rho} = \mathcal{N}_\rho \times [0,\infty).$$

Because of (2.3), the set $\mathcal{N}_\rho$ is a smooth hypersurface intersecting each orbit of the action exactly once. Denoting by $(\theta, \mathfrak{r})$ the points in $F_{A,\rho}$, define

$$\wp_{A,\rho} : F_{A,\rho} \to F, \quad \wp_{A,\rho}(\theta,\mathfrak{r}) = \mathfrak{r}\cdot\theta.$$

This map lifts to a homeomorphism $\hat{\wp}_{A,\rho} : F_{A,\rho} \to F_A$, and we define the C^∞ structure of F_A to be that defined by $\hat{\wp}_{A,\rho}$ and the standard structure on $F_{A,\rho}$.

Suppose $\tilde{\rho} : F\backslash 0 \to \mathbb{R}$ is another smooth function satisfying (2.2). Write $(\tilde{\theta}, \tilde{\mathfrak{r}})$ for the points in $F_{A,\tilde{\rho}} = \mathcal{N}_{\tilde{\rho}} \times [0,\infty)$. Then the function $\Phi : F_{A,\rho} \to F_{A,\tilde{\rho}}$ defined by

$$\Phi : (\theta,\mathfrak{r}) \mapsto (\tilde{\theta},\tilde{\mathfrak{r}}) = (\tilde{\rho}(\theta)^{-1}\cdot\theta, \tilde{\rho}(\theta)\mathfrak{r})$$

is a diffeomorphism, and

$$\wp_{A,\tilde{\rho}} \circ \Phi = \wp_{A,\rho}.$$

So the C^∞ structure on F_A is independent of the choice of ρ. The argument also shows that $\mathcal{N}$ is diffeomorphic to the sphere bundle of F.

The motivation for the definition of the A-compactification F^A of F comes from [**5**]. The compactification is the set $F^A = F \sqcup \mathcal{N}$ topologized with the smallest topology that contains all open sets of F and the sets $(\pi_A^{-1}(U) \cap K^c) \cup U$ with $U \subset \mathcal{N}$ open and $K \subset F$ compact, together with the inclusion map $F \hookrightarrow F^A$.

To give a C^∞ structure to F^A we again pick an auxiliary function $\rho : F\backslash 0 \to \mathbb{R}$ satisfying (2.2). Let $\phi \in C^\infty(F)$ be such that $0 \le \phi \le 1$, $\phi(u) = 0$ when $\rho(u)$ is small and $\phi(u) = 1$ when $\rho(u) > 1-\varepsilon$ for some $\varepsilon > 0$. Let

$$F^A_\rho = \{(w,\mathfrak{r}) \in F\times\mathbb{R} : \mathfrak{r} \ge 0,\ |\phi(w)\rho(w)|^2 + \mathfrak{r}^2 = 1\}. \tag{2.4}$$

Any two of these manifolds with boundary, obtained with the same function ρ but two choices of ϕ, are diffeomorphic. Define $\wp^A_\rho : \mathring{F}^A_\rho \to F$ by

$$\wp^A_\rho(w,\mathfrak{r}) = \mathfrak{r}^{-1}\cdot w$$

This map lifts to a homeomorphism $\hat{\wp}^A_\rho : F^A_\rho \to F^A$. As before, if $\tilde{\rho}$ also satisfies (2.2), then F^A_ρ and $F^A_{\tilde{\rho}}$ are diffeomorphic, so there is a well defined C^∞ structure on F^A. The boundary of F^A is diffeomorphic to the sphere bundle of F.

Note that $\wp_A : F_A \to F$, while giving a smooth diffeomorphism $F_A\backslash\partial F_A \to F\backslash 0$, need not be smooth at ∂F_A. It is smooth if and only if A is diagonalizable and the eigenvalues are (positive) integers, in which case F splits as a direct sum. We wish to allow $\wp_A$ to be singular, but if A is not diagonalizable, then the action involves logarithms, and if A is diagonalizable but the eigenvalues are nonconstant then derivatives of the action in horizontal directions will involve logarithms. To avoid these logarithmic terms we will assume that

(2.5) A is diagonalizable and all of its eigenvalues are constant and positive;

the eigenvalues need not be integers. Therefore F splits as a direct sum of subbundles according to the eigenspaces of A. Order the eigenvalues as an increasing sequence and label them by λ_a, $a = 1, \dots, N$. Denote by F_a the eigensubbundle corresponding to λ_a. Let $\iota_a : F_a \to F$ be the inclusion map and let $\pi_a : F \to F_a$ be the projection map according to the decomposition $F = \bigoplus F_a$.

3. Blowup and compactification of complex vector bundles

Suppose now that $\mathcal{X}$ is a connected complex n-manifold and that $F \to \mathcal{X}$ is the underlying real vector bundle of a holomorphic vector bundle $E \to \mathcal{X}$ of complex rank κ. We assume given an isomorphism $A : F \to F$ satisfying (2.5), and use the notation of the last paragraph of the previous section for the eigenvalues of A, eigensubbundles of F, and inclusion and projection maps.

Write $J : F \to F$ for the complex structure of E and define $J_a^b : F_a \to F_b$ by

$$J_a^b = \pi_b \circ J \circ \iota_a.$$

The almost complex structure $J_{\mathcal{X}}$ of $\bigwedge^{0,1}\mathcal{X}$ gives an almost complex structure $J_\otimes = \mathrm{Id} \otimes_{\mathbb{R}} J_{\mathcal{X}}$ for the vector bundle $F \otimes_{\mathbb{R}} \bigwedge^{0,1}\mathcal{X}$. The latter, viewed as a complex vector bundle using $J_\otimes$, is isomorphic to $\mathbb{C}F \otimes_{\mathbb{C}} \bigwedge^{0,1}\mathcal{X}$. This assertion, while elementary, is fundamental in the arguments that follow, so we will prove it.

Lemma 3.1. *Let F be a real vector space and let W be a complex vector space. Write $W_{\mathbb{R}}$ for the underlying real vector space of W and J for its almost complex structure. Then $F \otimes_{\mathbb{R}} W_{\mathbb{R}}$ with the almost complex structure $J_\otimes = \mathrm{Id} \otimes_{\mathbb{R}} J$is canonically isomorphic to $\mathbb{C}F \otimes_{\mathbb{C}} W$.*

Proof. Identify $F \otimes_{\mathbb{R}} W_{\mathbb{R}}$ with the space of $\mathbb{R}$-bilinear maps $\beta : F^* \times W_{\mathbb{R}}^* \to \mathbb{R}$. The complex structure is given by $(J_\otimes\beta)(f^*, w^*) = \beta(f^*, J^*w^*)$. Likewise identify $\mathbb{C}F \otimes_{\mathbb{C}} W$ with the space of $\mathbb{C}$-bilinear maps $\beta : \mathbb{C}F^* \times W^* \to \mathbb{C}$. Define

$$T : \mathbb{C}F \otimes_{\mathbb{C}} W \to F \otimes_{\mathbb{R}} W_{\mathbb{R}}, \quad (T\beta)(f^*, w^*) = \Re[\beta(f^*, \gamma w^*)]$$

where $\gamma : W_{\mathbb{R}}^* \to W^*$ is the map $\gamma(w^*) = w^* - iw^* \circ J^*$; γ is an $\mathbb{R}$-linear map. It is immediate that $T\beta$ is $\mathbb{R}$-linear. To see that T is $\mathbb{C}$-linear, let $\beta \in \mathbb{C}F \otimes W$. Then $T(i\beta)$ is the $\mathbb{R}$-bilinear map whose value at $(f^*, w^*) \in F^* \times W_{\mathbb{R}}^*$ is

$$\Re(i\beta(f^*, \gamma w^*)) = \Re[\beta(f^*, i\gamma(w^*))] = \Re[\beta(f^*, \gamma(J^*w^*))] = (J_\otimes T\beta)(f^*, w^*).$$

It is easy to see that T is injective. Since $\dim_{\mathbb{C}}(F \otimes_{\mathbb{R}} W_{\mathbb{R}}, J_\otimes) = \dim_{\mathbb{R}} F \cdot \dim_{\mathbb{R}} W_{\mathbb{R}} = \dim_{\mathbb{C}}(\mathbb{C}F \otimes_{\mathbb{C}} W)$, T is a isomorphism of complex vector spaces. $\square$

Let

$$F^{1,0} = \{v - iJv \in \mathbb{C}F : v \in F\}, \qquad F^{0,1} = \{v + iJv \in \mathbb{C}F : v \in F\}.$$

These are complex vector bundles, and E is canonically isomorphic to $F^{1,0}$. Therefore, since $\mathbb{C}F = F^{1,0} \oplus F^{0,1}$, we have an embedding

$$E \otimes_{\mathbb{C}} \textstyle\bigwedge^{0,1}\mathcal{X} \hookrightarrow F \otimes_{\mathbb{R}} \bigwedge^{0,1}\mathcal{X}. \tag{3.1}$$

Let

$$\overline{\partial} : C^\infty(\mathcal{X}; E) \to C^\infty(\mathcal{X}; E \otimes_{\mathbb{C}} \textstyle\bigwedge^{0,1}\mathcal{X})$$

be the $\overline{\partial}$ operator of E. Composing with the homomorphism in (3.1) we get a first order differential operator

$$\overline{D} : C^\infty(\mathcal{X}, F) \to C^\infty(\mathcal{X}; F \otimes_{\mathbb{R}} \textstyle\bigwedge^{0,1}\mathcal{X})$$

(a section of F is a section of E). Define $\overline{D}^b_a : C^\infty(\mathcal{X}; F_a) \to C^\infty(\mathcal{X}; F_b \otimes_{\mathbb{R}} \bigwedge^{0,1}\mathcal{X})$ by

$$\overline{D}^b_a = \pi_b \circ \overline{D} \circ \iota_a.$$

Here π_b is the operator $F \otimes_{\mathbb{R}} \bigwedge^{0,1}\mathcal{X} \to F_b \otimes_{\mathbb{R}} \bigwedge^{0,1}\mathcal{X}$ defined by $\pi_b : F \to F_b$. Let $\mathbb{N}_0$ be the set of nonnegative integers.

THEOREM 3.2. *The complex structure of the complex manifold $E\backslash 0$ induces a smooth b-complex structure on F_A if and only if*

$$\forall a,\ b:\ J^b_a \neq 0 \text{ or } \overline{D}^b_a \neq 0 \implies \lambda_a - \lambda_b \in \mathbb{N}_0. \tag{3.2}$$

In particular, (3.2) gives that if $a < b$, then J^b_a and $\overline{D}^b_a$ both vanish because $\lambda_a - \lambda_b < 0$. Thus, if J and $\overline{D}$ are written as matrices of operators reflecting the decomposition $F = \bigoplus F_a$, then these matrices are triangular. Additionally in each matrix the block corresponding to the pair (a, b), $a > b$, vanishes if $\lambda_a - \lambda_b$ is not an integer.

If (3.2) holds, we will write E_A for the manifold F_A with ${}^bT^{0,1}E_A$, the induced structure, as structure bundle.

In the following we fix ρ satisfying (2.2) and define $\mathfrak{r} = \rho \circ \wp_F$.

PROOF. Fix an arbitrary frame η_μ of E over an open set $U \subset \mathcal{X}$ and let ζ^μ denote the dual frame. Define ${\omega''}^\nu_\mu$ by $\overline{D}\eta_\mu = \sum_\nu \eta_\nu \otimes {\omega''}^\nu_\mu$; the ${\omega''}^\nu_\mu$ are $(0,1)$-forms. Suppose $z^1, \dots, z^n$ are holomorphic coordinates on U. Using the z^j and the ζ^μ as coordinates on $\pi^{-1}(U)$, the bundle of anti-holomorphic vectors of E is spanned by

$$\partial_{\overline{\zeta}^\mu},\ \mu = 1, \dots, \kappa, \quad \partial_{\overline{z}^j} - \sum_{\mu,\nu} \zeta^\mu \langle {\omega''}^\nu_\mu, \partial_{\overline{z}^j}\rangle \partial_{\zeta^\mu}\ j = 1 \dots, n.$$

We will show that (3.2) is equivalent to the statement that the certain smooth frames of $T^{0,1}(E\backslash 0)$ lift to frames of $\mathbb{C}{}^bT\mathring{F}_A$ that extend smoothly to the boundary so as to become frames of a b-complex structure. We will first deal with the real vertical vector fields.

Let $\operatorname{Vert}(F) \to F$ be the bundle of vertical vectors of F. There is a canonical isomorphism

$$V : \pi^*F \to \operatorname{Vert}(F) \tag{3.3}$$

which associates to an element $(u, v) \in \pi^*F$ the vector v at u. If η_μ denotes a (real) local frame of F and ξ^μ is the dual frame, then the ξ^μ give coordinates on each of the fibers of F and V sends (u, v) to $\sum \xi^\mu(v)\partial_{\xi_\mu}|_u$.

LEMMA 3.3. *Let $\eta_1, \dots, \eta_{2\kappa}$ be a smooth (real) local frame of $F \to \mathcal{X}$ with domain U, consisting of eigensections of A, so $A\eta_\mu = \tau_\mu\eta_\mu$ for some τ_μ. Denote by $V_{\rho^{\tau_\mu}\eta_\mu}$ the vertical vector field on $\pi^{-1}(U) \subset F$ corresponding to $u \mapsto (u, \rho(u)^{\tau_\mu}\eta_\mu)$. The vector fields*

$$d\wp_A^{-1}(V_{\rho^{\tau_\mu}\eta_\mu}) \tag{3.4}$$

initially defined on $\mathring{F}_A$, extend to F_A as part of a smooth frame of ${}^bTF_A$ near ∂F_A.

PROOF. Write ξ^μ for the frame dual to η_μ. These functions are linear coordinates on the fibers of F, and as a vector field, V_{η_μ} is $\rho^{\tau_\mu}\partial_{\xi^\mu}$. Since the η_μ are eigensections, $A\eta_\mu = \tau_\mu\eta_\mu$ where the τ_μ are the eigenvalues repeated according to multiplicity, in arbitrary order.

Suppose $p_0 \in \mathcal{N}_\rho$ and $\partial\rho/\partial\xi^{2\kappa}(p_0) \neq 0$. Let θ^μ be the function ξ^μ restricted to $\mathcal{N}_\rho$. Then the functions $\theta^1, \dots, \theta^{2\kappa-1}$ together with coordinates on U provide coordinates on $\mathcal{N}_\rho$ in a neighborhood of p_0, and they, with $\mathfrak{r}$, give coordinates on $F_{A,\rho}$ also near p_0. We will write the vector fields (3.4) using these coordinates. If $u \in \pi^{-1}(U)\backslash 0$, then $u = \wp_{A,\rho}(\theta, \mathfrak{r})$ with $\mathfrak{r} = \rho(u)$ and $\theta = \rho(u)^{-1} \cdot u$. Using $t \cdot \sum u^\mu \eta_\mu = \sum t^{\tau_\mu} u^\mu \eta_\mu$ and

$$t\frac{\partial\rho}{\partial\xi^\mu}(u) = \frac{\partial}{\partial\xi^\mu}\rho(t\cdot u) = t^{\tau_\mu}\frac{\partial\rho}{\partial\xi^\mu}(t\cdot u)$$

we get

$$\frac{\partial\rho}{\partial\xi^\mu}(u) = \mathfrak{r}^{1-\tau_\mu}\frac{\partial\rho}{\partial\xi^\mu}(\theta)$$

if $u = \mathfrak{r}\cdot\theta$. So

$$\frac{\partial\theta^\nu}{\partial\xi^\mu}(u) = \frac{\partial}{\partial\xi^\mu}\bigg|_u \frac{\xi^\nu}{\rho^{\tau_\nu}} = \frac{\delta^\nu_\mu}{\mathfrak{r}^{\tau_\nu}} - \tau_\nu\frac{\xi^\nu}{\mathfrak{r}^{\tau_\nu+1}}\mathfrak{r}^{1-\tau_\mu}\frac{\partial\rho}{\partial\xi^\mu}(\theta) = \frac{1}{\mathfrak{r}^{\tau_\mu}}(\delta^\nu_\mu - \tau_\nu\theta^\nu\frac{\partial\rho}{\partial\xi^\mu}(\theta)).$$

Thus

$$\begin{aligned} d\wp^{-1}_{A,\rho}(\partial_{\xi^\mu}) &= \frac{\partial\rho}{\partial\xi^\mu}(u)\partial_\mathfrak{r} + \sum_{\nu=1}^{2\kappa-1}\frac{\partial\theta^\nu}{\partial\xi^\mu}(u)\partial_{\theta^\nu} \\ &= \mathfrak{r}^{-\tau_\mu}[\frac{\partial\rho}{\partial\xi^\mu}(\theta)\mathfrak{r}\partial_\mathfrak{r} + \sum_{\nu=1}^{2\kappa-1}(\delta^\nu_\mu - \tau_\nu\theta^\nu\frac{\partial\rho}{\partial\xi^\mu}(\theta))\partial_{\theta^\nu}], \end{aligned}$$

which shows that $\wp^{-1}_{A,\rho}(\rho^{\tau_\mu}\partial_{\xi^\mu})$ is a smooth section of ${}^bTF_{A,\rho}$. The pointwise independence of these sections is easy to prove. □

With the notation of the lemma, let η be a smooth section of F over U. So $\eta = \sum f^\mu\eta_\mu$, and $\rho(u)^A\eta = \sum\rho(u)^{\tau_\mu}f^\mu\eta^\mu$. Let $V_{\rho(u)^A\eta}$ be the vector field that corresponds to $u \mapsto (u, \rho(u)^A\eta)$. By the lemma,

$$d\wp_A^{-1}(V_{\rho(u)^A\eta})$$

is a smooth section of ${}^bTF_A$. In the next lemma, $V_{J\rho^A\eta}$ means the vector field associated with $u \mapsto (u, J\rho(u)^A\eta)$.

Lemma 3.4. *The statement that $d\wp_A^{-1}(V_{J\rho^A\eta})$ is a smooth section of ${}^bTF_A$ for every smooth section η of F is equivalent to the statement that*

$$\forall a,\ b:\ J^b_a \neq 0 \implies \lambda_a - \lambda_b \in \mathbb{N}_0. \tag{3.5}$$

Thus, if (3.5) holds, then for any smooth section η of F,

$$d\wp_A^{-1}(V_{\rho^A\eta} - iV_{J\rho^A\eta}) \text{ and } d\wp_A^{-1}(V_{\rho^A\eta} + iV_{J\rho^A\eta})$$

are both smooth sections of $\mathbb{C}{}^bTF_A$.

Proof. Suppose that η is a smooth section of F. Using

$$t^{-A}Jt^A = \sum_{a,b} t^{\lambda_a-\lambda_b}\iota_b J^b_a\pi_a$$

and the smoothness of the $d\wp_A^{-1}(V_{\rho^A\iota_bJ^b_a\pi_a\eta})$ we get that

$$d\wp_A^{-1}(V_{J\rho^A\eta}) = \sum_{a,b}\mathfrak{r}^{\lambda_a-\lambda_b}d\wp_A^{-1}(V_{\rho^A\iota_bJ^b_a\pi_a\eta})$$

is smooth if (3.5) holds. Conversely, if $d\wp_A^{-1}(V_{J\rho^A\eta})$ is smooth for any smooth η, then J_a^b must vanish if $\lambda_a - \lambda_b$ is not a nonnegative integer, since by Lemma 3.3, $d\wp_A^{-1}(V_{\rho^A\iota_b J_a^b \pi_a \eta})$ is part of a frame, in particular nonvanishing, over the points of $\mathcal{X}$ where $\pi_a\eta$ does not vanish. □

Identify F_A with $F_{A,\rho} = \mathcal{N}_\rho \times [0,\infty)$. Suppose, as in Lemma 3.3, that $\eta_1, \ldots, \eta_{2\kappa}$ is a frame of eigensections of F, $A\eta^\mu = \tau_\mu \eta^\mu$, with dual frame ξ^μ. Let $\theta^\mu = \xi^\mu|_{\mathcal{N}_\rho}$. Fix arbitrary real coordinates x^j in the domain of the frame. Suppose that at $p_0 \in \mathcal{N}_0$, $\partial\rho/\partial\xi^{2\kappa} \neq 0$, so x^j, $j = 1, \ldots, 2n$, θ^μ, $\mu = 1, \ldots, 2\kappa - 1$, $\mathfrak{r}$ are coordinates on $E_{A,\rho}$ near p_0. Then

$$d\wp_{A,\rho}(\partial_{\theta^\mu}) = \rho^{\tau_\mu}\partial_{\xi^\mu} + \rho^{\tau_{2\kappa}}\frac{\partial\theta^{2\kappa}}{\partial\theta^\mu}\partial_{\xi^{2\kappa}}$$

and

$$d\wp_{A,\rho}(\mathfrak{r}\partial_{\mathfrak{r}}) = \sum_\mu \tau_\mu \rho^{\tau_\mu}\partial_{\xi^\mu}.$$

The liftings of

$$J(d\wp_{A,\rho}(\partial_{\theta^\mu})) = \rho^{\tau_\mu}J\partial_{\xi^\mu} + \rho^{\tau_{2\kappa}}\frac{\partial\theta^{2\kappa}}{\partial\theta^\mu}J\partial_{\xi^{2\kappa}}$$

and

$$Jd\wp_{A,\rho}(\mathfrak{r}\partial_{\mathfrak{r}}) = \sum_\mu \tau_\mu \rho^{\tau_\mu}J\partial_{\xi^\mu}.$$

extend as smooth sections of ${}^bTE_A$ if and only if (3.5) holds.

For the rest of the proof of Theorem 3.2 we assume that (3.5) holds. In particular $J_a^b = 0$ if $b > a$ (since the λ_a increase with a). It follows that each of the real subbundles F_a is a complex vector bundle with complex structure J_a^a. We will write E_a for this complex vector bundle and κ_a for its rank; it may not be a complex subbundle of E since the range of the restriction of J to F_a may not be contained in F_a. But it also follows from (3.5) and the block structure of J that the subbundles

$$F_{\leq a} = \bigoplus_{b\leq a} F_b \tag{3.6}$$

are closed under J, so they are complex subbundles of E; we will denote them by $E_{\leq a}$.

Using the block structure of J it is easy to prove that if for each a we pick a frame $\eta_{a,\mu}$ of E_a, then these sections together give a frame for E.

Lemma 3.5. *Suppose* (3.5) *holds. Let*

$$\eta_{a,\mu}, \quad a = 1, \ldots, N,\ \mu = 1, \ldots, \kappa_a$$

be a frame of E over some open set $U \subset \mathcal{X}$, with $\eta_{a,\mu}$, $\mu = 1, \ldots, \kappa_a$ a frame of E_a. Let $\zeta^{a,\mu}$ denote the dual frame. Then $t^{-\lambda_a}\zeta^{a,\mu} \circ t^A$, initially defined for $t > 0$, extends smoothly to $t \geq 0$.

Proof. Since $\eta_{a,\mu}$ is a smooth complex frame of E_a, these sections together with the sections $J_a^a\eta_{a,\mu}$ form a smooth real frame for F_a, and

$$\eta_{a,\mu}, \quad \eta_{a,\mu+\kappa_a} = J_a^a\eta_{a,\mu}, \quad a = 1, \ldots, N,\ \mu = 1, \ldots, \kappa_a \tag{3.7}$$

is a (real) frame of F. Let $\xi^{a,\mu}$, $a=1,\dots,N$, $\mu=1,\dots,2\kappa_a$ be the dual frame of this frame. Using that the bundles $F_{\leq a}$ in (3.6) are closed under J and the duality relations between the $\zeta^{a,\mu}$ and the $\eta_{b,\nu}$ one proves that

$$\zeta^{a,\mu}=\xi^{a,\mu}+i\xi^{a,\mu+\kappa_\mu}+\sum_{b=a+1}^{N}Q^{a,\mu}_{b,\nu}\xi^{b,\nu+\kappa_b}$$

with some smooth functions $Q^{a,\mu}_{b,\nu}:U\to\mathbb{C}$. Thus

$$t^{-\lambda_a}\zeta^{a,\mu}\circ t^A=\xi^{a,\mu}+i\xi^{a,\mu+\kappa_a}+\sum_{b=a+1}^{N}\sum_{\nu=1}^{\kappa_b}t^{\lambda_b-\lambda_a}Q^{a,\mu}_{b,\nu}\xi^{b,\nu+\kappa_b}$$

is bounded at $t=0$, since the exponents $\lambda_b-\lambda_a$ are positive; smoothness at $t=0$ will follow if the exponents that actually appear, that is, those for which $Q^{a,\mu}_{b,\nu}\neq 0$ for some b and ν, are integers.

We will prove by induction in c that $t^{-\lambda_a}\zeta^{a,\mu}\circ t^A|_{F_{\leq c}}$ is smooth at $t=0$. If $c<a$, then $t^{-\lambda_a}\zeta^{a,\mu}\circ t^A|_{F_{\leq c}}=0$, and if $c=a$ then $t^{-\lambda_a}\zeta^{a,\mu}\circ t^A|_{F_{\leq c}}=\xi^{a,\mu}+i\xi^{a,\mu+\kappa_a}$. So suppose that $c>a$ and that $t^{-\lambda_a}\zeta^{a,\mu}\circ t^A|_{F_{\leq c-1}}$ is smooth. Thus if $Q^{a,\mu}_{b,\nu}\neq 0$ for some b with $a<b<c$ and some ν with $1\leq\nu\leq\kappa_b$, then $\lambda_b-\lambda_a\in\mathbb{N}$. We have $t^{-\lambda_a}\zeta^{a,\mu}(t^A\eta_{c,\vartheta})=t^{\lambda_c-\lambda_a}\zeta^{a,\mu}(\eta_{c,\vartheta})=0$ for $\vartheta=1,\dots,\kappa_c$. And for these indices ϑ,

$$\begin{aligned}
t^{-\lambda_a}\zeta^{a,\mu}(t^{\lambda_c}\eta_{c,\vartheta+\kappa_c})&=-it^{\lambda_c-\lambda_a}\zeta^{a,\mu}(J\eta_{c,\vartheta+\kappa_c})\\
&=-it^{\lambda_c-\lambda_a}\zeta^{a,\nu}(-\eta_{c,\vartheta}+\sum_{b'=a}^{c-1}J^{b'}_c\eta_{c,\vartheta+\kappa_c})\\
&=-it^{\lambda_c-\lambda_a}\zeta^{a,\nu}(\sum_{b'=a}^{c-1}J^{b'}_c\eta_{c,\vartheta+\kappa_c})\\
&=-it^{\lambda_c-\lambda_a}\zeta^{a,\nu}(J^a_c\eta_{c,\vartheta+\kappa_c})\\
&\qquad-it^{\lambda_c-\lambda_a}\sum_{b=a+1}^{c-1}\sum_{\nu=1}^{\kappa_b}Q^{a,\mu}_{b,\nu}\xi^{b,\nu+\kappa_b}(J^b_c\eta_{c,\vartheta+\kappa_c}).
\end{aligned}$$

Suppose that $\lambda_c-\lambda_a$ is not an integer. Then $J^a_c=0$. If we had

$$Q^{a,\mu}_{b,\nu}\xi^{b,\nu+\kappa_b}(J^b_c\eta_{c,\vartheta+\kappa_c})\neq 0$$

for some b with $a<b<c$ and some ν with $1\leq\nu\leq\kappa_b$, then $\lambda_c-\lambda_b$ and $\lambda_b-\lambda_a$ would both be integers, and so would be $\lambda_c-\lambda_a=\lambda_c-\lambda_b+\lambda_b-\lambda_a$, a contradiction. Thus, either $t^{-\lambda_a}\zeta^{a,\mu}(t^{\lambda_c}\eta_{c,\vartheta+\kappa_c})$ vanishes for $\vartheta=1,\dots,\kappa_c$, or $\lambda_c-\lambda_a$ is an integer. In either case, this function is smooth at $t=0$, so by linearity, $t^{-\lambda_a}\zeta^{a,\mu}\circ t^A|_{F_{\leq c}}$ is smooth at $t=0$. □

Fix a frame $\eta_{a,\mu}$ of E as in Lemma 3.5, with dual frame $\zeta^{a,\mu}$. Suppose $z^1,\dots,z^n$ are holomorphic coordinates for $\mathcal{X}$ in the domain of the sections $\eta_{a,\mu}$. The bundle of antiholomorphic vector fields of E is spanned by the $\partial_{\overline{\zeta}^{a,\mu}}$ and the

$$\partial_{\overline{z}^j}-\sum_{1\leq a,b\leq N}\sum_{\substack{1\leq\mu\leq\kappa_a\\1\leq\nu\leq\kappa_b}}\zeta^{a,\mu}\langle{\omega''}^{b,\nu}_{a,\mu},\partial_{\overline{z}^j}\rangle\partial_{\zeta^{b,\nu}}\tag{3.8}$$

with suitable $(0,1)$ forms ${\omega''}^{b,\nu}_{a,\mu}$ determined by $\overline{\partial}$ and the frame. By Lemma 3.4, the sections $d\wp^{-1}_{F,\rho}(\rho^{\lambda_a}\partial_{\overline{\zeta}^{a,\mu}})$ of $\mathbb{C}^bTF_A$ and their conjugates are smooth if and only if (3.5) holds. Assuming this already, we have, by Lemma 3.5,

$$\wp_{F,\rho}^{*}\zeta^{a,\nu} = \mathfrak{r}^{\lambda_a}\Theta^{a,\mu}$$

with some smooth function $\Theta^{a,\mu}$ on $F_{\rho,A}$. Thus the pull-back to F_A of the vector field in (3.8) is

$$d\wp_{F,\rho}^{-1}(\partial_{\overline{z}^j}) - \sum_{1\le a,b\le N}\ \sum_{\substack{1\le\mu\le\kappa_a\\ 1\le\nu\le\kappa_b}} \mathfrak{r}^{\lambda_a-\lambda_b}\Theta^{a,\mu}\langle {\omega''}^{b,\nu}_{a,\mu}, \partial_{\overline{z}^j}\rangle d\wp_{F,\rho}^{-1}(\rho^{\lambda_a}\partial_{\zeta^{b,\nu}}). \tag{3.9}$$

Since $d\wp_{F,\rho}^{-1}(\partial_{\overline{z}^j})$ is already a smooth section of $\mathbb{C}^bTF_{A,\rho}$, the above vector fields represent smooth sections of $\mathbb{C}^bTF_{A,\rho}$ if and only if for all a, b, ${\omega''}^{b,\nu}_{a,\mu} \neq 0 \implies \lambda_a - \lambda_b \in \mathbb{N}_0$, that is, in the presence of (3.5), the property that for each holomorphic local chart $z^1, \dots, z^n$ and frame $\eta_{a,\mu}$ as in Lemma 3.5 the vector fields (3.9) are smooth is equivalent to the statement that

$$\forall a,\ b:\ \overline{D}^b_a \neq 0 \implies \lambda_a - \lambda_b \in \mathbb{N}_0. \tag{3.10}$$

This completes the proof of Theorem 3.2, since (3.2) is equivalent to the conjunction of (3.5) and (3.10). □

The situation for the A-compactification of a holomorphic vector bundle $E \to \mathcal{X}$ is analogous to that of the blowup. Fix ρ as in (2.2). The map $\mathcal{N}_\rho \times [0,\varepsilon) \to E^A_\rho$ given by

$$(\theta, \mathfrak{r}) \mapsto (\frac{1}{\sqrt{1+\mathfrak{r}^2}}\cdot\theta, \frac{\mathfrak{r}}{\sqrt{1+\mathfrak{r}^2}})$$

is a smooth diffeomorphism for ε small enough (cf. [**5**]), and its composition with $\wp^E_\rho$ is $\mathfrak{r}^{-1}\cdot\theta$. Using this, the arguments of the proof of Theorem 3.2 are easily adapted to yield

THEOREM 3.6. *The complex structure of the complex manifold $E \subset E^A$ extends as a smooth b-complex structure on E^A if and only if*

$$\forall a,\ b:\ J^b_a \neq 0 \text{ or } \overline{D}^b_a \neq 0 \implies -(\lambda_a - \lambda_b) \in \mathbb{N}_0. \tag{3.11}$$

We will write ${}^bT^{0,1}E^A$ for the bundle of antiholomorphic elements in $\mathbb{C}^bTE^A$.

EXAMPLE 3.7. As an application of anisotropic compactifications consider the hyperquadric Σ in $\mathbb{C}^{\kappa+1}$ given by $h = 0$ where

$$h(\zeta) = \Im\zeta^{\kappa+1} - \sum_{j=1}^{\kappa}\varepsilon_j|\zeta^j|^2$$

with $\varepsilon_j = \pm 1$. The function h is homogeneous of degree 2 with respect to the action $t\cdot\zeta = (t\zeta^1, \dots, t\zeta^\kappa, t^2\zeta^{\kappa+1})$ induced by $A\zeta = (\zeta^1, \dots, \zeta^\kappa, 2\zeta^{\kappa+1})$. Let $\mathcal{M}$ be the A-compactification of $\mathbb{C}^{\kappa+1}$. The versions of Lemmas 3.3 and 3.4 for the compactification give that $\mathcal{M}$ is a b-complex manifold. Let $\rho : \mathbb{C}^{\kappa+1}\backslash 0 \to \mathbb{R}$ satisfy (2.2). It is not hard to prove that $\tilde{h} = (\wp^A)^*h/\rho^2$, a function on $\mathbb{C}^{\kappa+1}\backslash 0$, extends smoothly to $\mathcal{M}\backslash 0$ and that the restriction to $\partial\mathcal{M}$ of $d\tilde{h}$ is nonzero where $\tilde{h} = 0$. Thus we get a compactification Σ^A of Σ. The CR structure of Σ extends as a nondegenerate b-CR structure (cf. [**6**]) induced by the b-complex structure of $\mathcal{M}$.

4. Boundary structure

We continue to assume that $\mathcal{X}$ is a connected complex manifold, that $E \to \mathcal{X}$ is a holomorphic vector bundle of rank κ whose underlying real bundle is $F \to \mathcal{X}$, and that $A : F \to F$ is an isomorphism satisfying (2.5).

If (3.2) holds then the bundles (3.6) are closed under J; they are complex subbundles of E which we'll denote by $E_{\leq a}$. Since $\overline{D}_a^b = 0$ for $b > a$, each $E_{\leq a}$ is a holomorphic subbundle of E. Also, each of the vector bundles E_a (F_a with the complex structure given by J_a^a) is a holomorphic vector bundle with $\overline{D}_a^a$ as its $\overline{\partial}$ operator; as before we let κ_a denote its rank. If η is a local holomorphic section of $E_{\leq a}$, then $\pi_a \eta$ is a local holomorphic section of E_a. Both $J_t = t^{-A} J t^A$ and $\overline{\partial}_t = t^{-A} \overline{\partial} t^A$ are polynomial in t, and thus make sense for any value of t. Further, $J_t^2 = -\mathrm{Id}$ and $\overline{\partial}_t^2 = 0$ for any t. Let then E_t be the holomorphic vector bundle whose underlying real vector bundle is F with multiplication by i given by J_t and whose $\overline{\partial}$ operator is $\overline{\partial}_t$. If $t > 0$ then $t^A : E_t \to E$ is a biholomorphism. If η is a section of $E_{\leq a}$, then $\eta_t = t^{-A}\eta$ is a section of $E_{\leq a, t}$, and so is $t^{\lambda_a}\eta_t$. The latter, as a section of F, tends to $\pi_a \eta$ as $t \to 0^+$.

Similarly, if (3.11) holds, then each F_a is again a holomorphic vector bundle over $\mathcal{X}$ with complex structure J_a^a and $\overline{D}_a^a$ as $\overline{\partial}$ operator. The bundles

$$F_{\geq a} = \bigoplus_{b \geq a} F_a$$

are invariant under J; write $E_{\geq a}$ for $F_{\geq a}$ with the induced complex structure. The $E_{\geq a}$ are holomorphic subbundles of E. Let $E^{t^{-1}}$ be the vector bundle whose underlying real bundle is F, with $J_{t^{-1}} = t^A J t^{-A}$ and $\overline{\partial}_{t^{-1}} = t^A \overline{\partial} t^{-A}$ as its complex structure and $\overline{\partial}$ operator, respectively. These operators are again polynomial in t, and thus make sense for any value of t. We write E_∞ for the bundle F with complex structure and $\overline{\partial}$ operator corresponding to $t = 0$.

Lemma 4.1. *If (3.2) holds, then the pull-back of ${}^b T^{0,1} E_A$ to ∂E_A coincides with that of ${}^b T^{0,1} E_{0A}$. If (3.11) holds, then then the pull-back of ${}^b T^{0,1} E^A$ to ∂E^A coincides with that of ${}^b T^{0,1} E_\infty^A$.*

Note that in either statement, the structures are comparable, at least in principle, since $E_A = E_{0A}$ and $E^A = E^A_\infty$ as C^∞ manifolds. The first statement is an elementary computation in view of the work done in Section 3. The second follows from the first by an argument involving the remarks preceding Theorem 3.11

In this section we are interested only in describing the elliptic structure determined on the boundary by the b-complex structure. The lemma allows us to assume that J and A commute and that $\overline{\partial}$ acts diagonally on $E = \bigoplus_a E_a$. We will therefore assume that this is the case for the rest of this section. We can then simultaneously blow up the zero section and compactify. The resulting manifold, denoted $\mathcal{M}$ is, in terms of a function ρ satisfying (2.2), diffeomorphic to $\mathcal{N}_\rho \times [-1, 1]$. This can be seen by noting that the map $E \backslash 0 \to \mathcal{N}_\rho \times (-1, 1)$ defined by sending u to the point $(\pi_\rho(u), r(u))$ where $\pi_\rho(u)$ is the point on $\mathcal{N}_\rho$ in the same orbit as u and

$$r(u) = \frac{\rho(u) - 1}{\rho(u) + 1}$$

can de regarded as a map $\mathcal{M} \backslash \partial \mathcal{M} \to \mathcal{N}_\rho \times (-1, 1)$ and as such extends as a smooth diffeomorphism to all or $\mathcal{M}$. Under the diffeomorphism, the boundary of the blowup corresponds to $\mathcal{N}_\rho \times \{-1\}$, and the boundary of the compactification to $\mathcal{N}_\rho \times \{1\}$. We let

$$\dot{\wp}_\rho : \mathcal{N}_\rho \times (-1,1) \to E\backslash 0 \tag{4.1}$$

be the inverse of the map just defined and note that

$$d\dot{\wp}_\rho\Big(\frac{1-r^2}{2}\partial_r\Big) = R \tag{4.2}$$

where R is the infinitesimal generator of the action (2.1).

To determine the b-complex structure of $\mathcal{M}$ and consequently the structure induced on $\partial\mathcal{M}$, we will make a convenient choice of the function ρ. Pick a Hermitian metric h on E for which the vector bundles E_a are mutually orthogonal, let SE be the unit sphere bundle, and define ρ to be the solution of

$$R\rho = \rho, \quad \rho|_{SE} = 1.$$

Thus $\mathcal{N}_\rho = SE$, and we may, and will, view $\mathcal{M}$ as $SE \times [-1,1]$. We will write $\dot{\wp}$ for the map (4.1).

A special feature of this choice of ρ is that $\mathcal{T}' = JR$ is tangent to the level sets $\{\rho = \rho_0\}$ of ρ. To see this and other features of the b-complex structure, fix an orthonormal frame η_μ, $\mu = 1,\dots,\kappa$, for E over some open set $U \subset \mathcal{X}$ so that with $r_a = \operatorname{rank} E_{\le a}$,

$$\eta_\mu, \quad \mu = r_{a-1}+1,\dots,r_{a-1}+\kappa_a = r_a.$$

is a frame for E_a. In particular, $A\eta_\mu = \tau_\mu\eta_\mu$ where $\tau_\mu = \lambda_a$ if η_μ is a section of E_a. Let ζ^μ be the dual frame. Then

$$R = \sum_{\mu=1}^{\kappa} \tau_\nu(\zeta^\nu\partial_{\zeta^\nu} + \overline{\zeta}^\nu\partial_{\overline{\zeta}^\nu}). \tag{4.3}$$

Define

$$\mathcal{T}' = JR = i\sum_{\mu=1}^{\kappa} \tau_\nu(\zeta^\nu\partial_{\zeta^\nu} - \overline{\zeta}^\nu\partial_{\overline{\zeta}^\nu}). \tag{4.4}$$

Since $\mathcal{T}'\sum_\mu|\zeta^\mu|^2 = 0$, $\mathcal{T}'$ is in particular tangent to SE, and thus $\mathcal{T}'\rho = 0$ on SE. It is verified by direct calculation that $[\mathcal{T}', R] = 0$, so

$$0 = [\mathcal{T}', R]\rho = \mathcal{T}'R\rho - R\mathcal{T}'\rho = \mathcal{T}'\rho - R\mathcal{T}'\rho.$$

It follows that $\mathcal{T}'\rho = 0$.

In particular, since the hypersurfaces $\{r = r_0\}$, $r_0 \in (0,1)$, are mapped by $\dot{\wp}$ to the hypersurfaces $\{\rho = \rho_0\}$, the vector field $\mathcal{T}$ lifts to a vector field $\mathcal{T}$ on $\mathcal{M}$ which is tangent to $\{r = r_0\}$ and commutes with ∂_r. This means that $\mathcal{T}$ is the canonical lifting of the vector field $\mathcal{T}'$ on SE by the projection $SE \times [-1,1] \to SE$, a smooth vector field on all of $\mathcal{M}$. Since $R + iJR$ is an antiholomorphic vector field,

$$\frac{1-r^2}{2}\partial_r + i\mathcal{T} \tag{4.5}$$

is a global section of ${}^bT^{0,1}\mathcal{M}$.

In the coordinates we are using (on individual fibers) the action (2.1) is

$$t\cdot(\zeta^1,\dots,\zeta^\kappa) = (t^{\tau_1}\zeta^1,\dots,t^{\tau_\kappa}\zeta^\kappa) \tag{4.6}$$

One verifies by direct computation that the vector fields $u \mapsto \rho(u)^{\tau_\mu}\partial_{\overline{\zeta}^\mu}|_u$ are invariant under the action. Also the functions $\rho^{-\tau_\mu}\zeta^\mu$ are invariant under the action (2.1). Thus the vector fields

$$\overline{L}_{\mu\nu} = \rho^{-\tau_\nu}\zeta^\nu\rho^{\tau_\mu}\partial_{\overline{\zeta}^\mu} - \rho^{-\tau_\mu}\zeta^\mu\rho^{\tau_\nu}\partial_{\overline{\zeta}^\nu}, \tag{4.7}$$

which are CR vector fields of the standard CR structure of SE (in particular tangent to SE), are invariant under the action, $[R, \overline{L}_{\mu\nu}] = 0$. Using that $\rho(u)$ solves

$$\sum_\nu \rho(u)^{-2\tau_\nu}|\zeta^\nu(u)|^2 = 1$$

we get that

$$\frac{\partial\rho}{\partial\overline{\zeta}^\mu} = \frac{\zeta^\mu}{2\rho^{2\tau_\mu-1}f}$$

with $f(u) = \sum_{b,\nu}\lambda_b\rho(u)^{-2\lambda_b}|\zeta^{b,\nu}(u)|^2 = h(\rho(u)^{-A}u, \rho(u)^{-A}Au)$ and consequently, that the vector fields in (4.7) are in fact tangent to all the hypersurfaces $\{\rho = \rho_0\}$. Therefore their lifting by $\dot{\wp}$ to $\dot{\mathcal{M}}$ is the canonical lifting of these vector field on SE via the projection $SE \times [0, 1] \to SE$.

Now let ∇ be the Hermitian holomorphic connection. If $\overline{\partial}\eta_\mu = \sum_\nu \eta_\nu \otimes {\omega''}^\nu_\mu$, then

$$\nabla\eta_\mu = \sum_{\nu=1}^{\kappa}\eta_\nu \otimes \omega^\nu_\mu$$

with $\omega^\nu_\mu = -\overline{{\omega''}^\mu_\nu} + {\omega''}^\nu_\mu$. If η_μ is a section of E_a then ω^ν_μ vanishes unless η_ν is also a section of E_a. Let $z^1, \dots, z^n$ be a local holomorphic chart with domain U. Then the bundle of horizontal vectors of type $(0, 1)$ of E is spanned by

$$\overline{L}_j = \partial_{\overline{z}^j} - \sum_{\mu,\nu=1}^{\kappa}\Big[\zeta^\mu\langle\omega^\nu_\mu, \partial_{\overline{z}^j}\rangle\partial_{\zeta^\nu} + \overline{\zeta}^\mu\langle\overline{\omega}^\nu_\mu, \partial_{\overline{z}^j}\rangle\partial_{\overline{\zeta}^\nu}\Big].$$

These vector fields are tangent to SE because the connection is Hermitian. A direct computation using the block structure of the ω^ν_μ gives that $[\overline{L}_j, R] = 0$. Therefore, as above, $\overline{L}_j\rho = 0$ everywhere on the part of $E\backslash 0$ over U since this holds on SE. We conclude that $\overline{L}_j$ is also tangent to the level sets of ρ, and as before, that their liftings to $\dot{\mathcal{M}}$ via $\dot{\wp}$ coincide with the canonical liftings via the projection $SE \times (-1, 1) \to SE$ of these vector fields on SE.

Let $\overline{\mathcal{K}}$ be the standard CR structure of SE viewed as the tangential elements of $T^{0,1}E|_{SE}$. We have shown:

Proposition 4.2. *The b-complex structure of $\mathcal{M} = SE \times [-1, 1]$ is the direct sum of the canonical lifting of $\overline{\mathcal{K}}$ via the projection $\mathcal{M} \to SE$ and the span of the vector field* (4.5).

Both boundary components of $\mathcal{M}$ are diffeomorphic to SE, of course. The vector subbundle $\overline{\mathcal{V}}_b \subset \mathbb{C}T\partial M$ determined by ${}^bT^{0,1}\mathcal{M}$ is, on each component, the direct sum of $\overline{\mathcal{K}}$ and the span of the vector field $\mathcal{T}'$ in (4.4). However, note that if $\mathfrak{r}$ is a defining function for the boundary of $\mathcal{M}$, then $J(\mathfrak{r}\partial_\mathfrak{r}) = \mathcal{T}'$ on $SE \times \{-1\}$ while $J(\mathfrak{r}\partial_\mathfrak{r}) = -\mathcal{T}'$ on $SE \times \{1\}$ (cf. (1.2)).

The behavior near $\partial\mathcal{M}$ of tempered holomorphic functions on $\mathcal{M}$ (also of certain tempered representatives of the ${}^b\overline{\partial}$-cohomology) depends on the Levi form of the CR structure, cf. [**8**, Theorem 8.3]. We will compute the Levi form, focusing on the component $\mathcal{N} = SE \times \{-1\}$ of $\partial\mathcal{M}$ identified with SE; the canonical vector field is $\mathcal{T} = \mathcal{T}'$. We continue to use the orthonormal frame η_μ. Let θ_e be the (real) one-form on $\mathcal{N}$ which vanishes on $\mathcal{K} \oplus \overline{\mathcal{K}}$ and is equal to 1 when paired with

$$\mathcal{T}'_e = i \sum_{\mu=1}^{\kappa} (\zeta^\nu \partial_{\zeta^\nu} - \overline{\zeta}^\nu \partial_{\overline{\zeta}^\nu}). \tag{4.8}$$

The Levi form of $\mathcal{K}$ (the part of $T^{1,0}E|_{SE}$ tangential to SE) with respect to θ_e is

$$\mathrm{Levi}_{\theta_e}(v,w) = -id\theta_e(v,\overline{w}), \quad v,\ w \in \mathcal{K}_p,\ p \in SE.$$

By direct calculation it is verified that θ_e is the pull-back to SE of

$$\tilde{\theta}_e = \frac{1}{2i} \sum_{\nu=1}^{\kappa} (\overline{\zeta}^\nu \theta^\nu - \zeta^\nu \overline{\theta}^\nu) \tag{4.9}$$

with

$$\theta^\nu = d\zeta^\nu + \sum_{\mu=1}^{\kappa} \zeta^\mu \omega_\mu^\nu$$

and that $d\theta_e$ is (the pullback of)

$$d\tilde{\theta}_e = i \Big[\sum_{\nu=1}^{\kappa} \theta^\nu \wedge \overline{\theta}^\nu - \sum_{\mu,\nu=1}^{\kappa} \zeta^\mu \overline{\zeta}^\nu \Omega_\mu^\nu \Big]$$

The forms θ^ν together with their conjugates span the annihilator of the horizontal bundle of the connection. Hence

$$\mathrm{Levi}_{\theta_e}(L_j, \overline{L}_k) = - \sum_{\mu,\nu=1}^{\kappa} \zeta^\mu \overline{\zeta}^\nu \Omega_\mu^\nu (\partial_{z^j}, \partial_{\overline{z}^k})$$

$$\mathrm{Levi}_{\theta_e}(L_j, \overline{L}_{\mu\nu}) = 0$$

where $\Omega_\mu^\nu = d\omega_\mu^\nu + \sum_\gamma \omega_\gamma^\nu \wedge \omega_\mu^\gamma$ are the components of the curvature of ∇ with respect to the frame η_μ. On vertical CR vectors Levi_{θ_e} is positive definite, since those give the standard CR structure of the sphere.

As far as signature and/or degeneracy is concerned, the actual element of the characteristic set of the CR structure used to compute the Levi form is arbitrary. So rather than using θ_e, we will from now on use

$$\theta = \frac{1}{\sum_\nu \tau_\nu |\zeta^\nu|^2} \theta_e,$$

which has the property that $\langle \theta, \mathcal{T}' \rangle = 1$; this is a more natural normalization, since $\mathcal{T}$ is a vector field in $\overline{\mathcal{V}}$. Since $d(\sum_\nu \tau_\nu |\zeta^\nu|) \wedge \theta_e$ vanishes on $\mathcal{K} \oplus \overline{\mathcal{K}}$,

$$\mathrm{Levi}_\theta = \frac{1}{\sum_\nu \tau_\nu |\zeta^\nu|^2} \mathrm{Levi}_{\theta_e}$$

Normalizations aside, we see that the Levi form may behave very badly. Nevertheless, some general assertions can be made if $\mathcal{X}$ (therefore SE) is compact.

The Hermitian metric on E and a Hermitian metric on $\mathcal{X}$ give a Hermitian metric on the total space of E for which the horizontal bundle is orthogonal to the vertical one. The restriction of this metric to SE is $\mathcal{T}'$-invariant and the decomposition $\mathbb{C}TSE = \mathcal{K} \oplus \overline{\mathcal{K}} \oplus \operatorname{span} \mathcal{T}'$ is orthogonal. With it and the Riemannian measure it induces we construct the Kohn Laplacians,

$$\Box_{b,q} : C^\infty(SE; {\textstyle\bigwedge}^q \overline{\mathcal{K}}^*) \to C^\infty(SE; {\textstyle\bigwedge}^q \overline{\mathcal{K}}^*),$$

which are also $\mathcal{T}'$-invariant. Let

$$\mathcal{H}^q_{\overline{\partial}_b}(SE) \subset L^2(SE; {\textstyle\bigwedge}^q\overline{\mathcal{K}}^*)$$

denote the kernel of $\Box_{b,q}$ in $L^2(SE; \bigwedge^q\overline{\mathcal{K}}^*)$. Since $\Box_{b,q}$ is $\mathcal{T}'$-invariant, the Lie derivative $-i\mathrm{Lie}_{\mathcal{T}'}$ gives an operator

$$-i\mathrm{Lie}_{\mathcal{T}'} : \mathrm{Dom}(\mathrm{Lie}_{\mathcal{T}'}) \subset \mathcal{H}^q_{\overline{\partial}_b}(SE) \to \mathcal{H}^q_{\overline{\partial}_b}(SE) \tag{4.10}$$

where $\mathrm{Dom}(\mathrm{Lie}_{\mathcal{T}'}) = \{\phi \in \mathcal{H}^q_{\overline{\partial}_b}(SE) : -i\mathrm{Lie}_{\mathcal{T}'}\phi \in L^2(SE, \bigwedge^q\overline{\mathcal{K}}^*)\}$; (4.10) is a selfadjoint Fredholm operator, see [**8**, Theorem 7.14], so there is an orthogonal decomposition of $\mathcal{H}^q_{\overline{\partial}_b}(SE)$ into eigenspaces of $-i\mathrm{Lie}_{\mathcal{T}'}$,

$$\mathcal{H}^q_{\overline{\partial}_b}(\mathcal{N}) = \bigoplus_\tau \mathcal{H}^q_{\overline{\partial}_b,\tau}(SE).$$

The points in the spectrum of (4.10) are, in some cases, related to the coefficients τ_j in the definition of R or $\mathcal{T}'$.

With the principal symbol $\boldsymbol{\tau} = \boldsymbol{\sigma}(-i\mathcal{T})$ of $-i\mathcal{T}$ we separate $\mathrm{Char}(\Box_{b,q}) = \mathrm{Char}(\mathcal{K})$ (the span of θ) into the sets $\mathrm{Char}^\pm(\mathcal{K})$ according to the sign of $\boldsymbol{\tau}$. In [**8**] it was proved that if $\Box_{b,q}$ is hypoelliptic in, say, $\mathrm{Char}^+(\mathcal{K})$, then the number of positive eigenvalues of (4.10) is finite (this is true for a more general class of operators, cf [**8**, Section 6]). The proof of this is a combination of local and microlocal arguments that gives the following refinement:

Proposition 4.3. *Let $U \subset SE$ be an open set such that $\Box_{b,q}$ is hypoelliptic in* $\mathrm{Char}^+(\mathcal{K})$. *If ϕ_ℓ is an orthonormal sequence of eigenvectors of $-i\mathrm{Lie}_{\mathcal{T}'}$ in $\mathcal{H}^q_{\overline{\partial}_b}(\mathcal{N})$, $-i\mathrm{Lie}_{\mathcal{T}'}\phi_\ell = \tau_\ell\phi_\ell$, and if $\tau_\ell \to \infty$ as $\ell \to \infty$, then ϕ_ℓ converges to* 0 *uniformly on compact subsets of U as $\ell \to \infty$.*

We will only outline the proof. If the conclusion of the Proposition is false, there is a compact set $C \subset U$ on which ϕ_ℓ does not converge uniformly to 0 as $\ell \to \infty$. We may then pass to a subsequence of the $\{\phi_\ell\}$ for which the eigenvalues satisfy $\tau_{\ell+1} \geq 2\tau_\ell$. The key argument in the proof of Theorem 6.12 of [**8**] is that for such a sequence, if $\Box_{b,q}$ is hypoelliptic in $\mathrm{Char}^+(\mathcal{K})$ in the open set U, then every point in U has a neighborhood where $\{\phi_\ell\}$ tends to 0 uniformly. Thus we reach a contradiction, since $C \subset U$ is compact.

Example 4.4. Let $E_1 \to \mathcal{X}$ be a Hermitian holomorphic line bundle. Let m be a positive integer, and let $E_2 = E_1^*$ with the induced complex structure and Hermitian form. Let $E = E_1 \oplus E_2$, with the obvious Hermitian metric. Let η_1 be a local section of SE_1, let η_1^* be the dual section, and let $\eta_2 = \eta_1^* \otimes \cdots \otimes \eta_1^*$ (m times), a section of SE_2. Then η_1, η_2 is an orthonormal frame of E. If ω is the connection form of the Hermitian holomorphic connection of E_1 with respect to η_1, then the connection form associated with η_2 is $-m\omega$. The connection form of the hermitian holomorphic connection of E with respect to the frame η_1, η_2 is diagonal with $\Omega^1_1 = d\omega$ and $\Omega^2_2 = -m\Omega^1_1$. Thus

$$d\theta = i\Big[\theta^1 \wedge \overline{\theta}^1 + \theta^2 \wedge \overline{\theta}^2 - (|\zeta^1|^2 - m|\zeta^2|^2)\Omega^1_1\Big]$$

Suppose that E_1 is negative. Then the above formula gives that Levi_θ is positive definite in the region U_+ defined by $|\zeta^1|^2 > m|\zeta^2|^2$, and nondegenerate indefinite in the region U_{nd} defined by $|\zeta^1|^2 < m|\zeta^2|^2$. Thus $\Box_{b,0}$ (on functions) is microlocally hypoelliptic in $\mathrm{Char}^-(\mathcal{K})$ on U_+ and hypoelliptic on $\mathrm{Char}(\mathcal{K})$ in U_{nd}. The proposition implies that if ϕ_ℓ is a sequence of CR functions that satisfy $-i\mathrm{Lie}_{\mathcal{T}} = \tau_\ell \phi_\ell$ and $\tau_\ell \to \infty$, then ϕ_ℓ tends to 0 uniformly on compact subsets of U_{nd}, and if $\tau_\ell \to -\infty$, then $\phi_\ell \to 0$ on compact subsets of $U_+ \cup U_{\mathrm{nd}}$.

Suppose ψ is a nontrivial nontrivial holomorphic section of a sufficiently large power of E_1^*. Then ψ gives a holomorphic function

$$E_1 \oplus E_2 \ni p_1 \oplus p_2 \mapsto f(p_1 \oplus p_2) = \langle f, p_1 \otimes \cdots \otimes p_1 \rangle \in \mathbb{C},$$

and by restriction a CR function on SE. Suppose, for the sake of notational simplicity, that all the eigenvalues of the isomorphism A are equal to 1, so $\mathcal{T}' = T'_e$, the vector field in (4.8), and that f is a section of E_1^*. Using the frame η_1, η_2 we have $f(\zeta^1\eta_1 + \zeta^2\eta_2) = \zeta^1\psi(\eta_1)$ so $-i\mathrm{Lie}_{\mathcal{T}'_e} f = f$, and the powers f^ℓ satisfy $-i\mathrm{Lie}_{\mathcal{T}'_e} f^\ell = \ell f^\ell$. In particular $(f^{\ell_1}, f^{\ell_2}) = 0$ if $\ell_1 \neq \ell_2$, and by the proposition the functions $\phi_\ell = f^\ell/\|f^\ell\| \to 0$ on compact subsets of U_{nd} as $\ell - \to \infty$. It is not clear that there are CR eigenfunctions of $-i\mathrm{Lie}_{\mathcal{T}'}$ with negative eigenvalues.

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