Zeta functions of elliptic cone operators

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Abstract. This paper is an overview of aspects of the singularities of the zeta function, equivalently, of the small time asymptotics of the trace of the heat semigroup, of elliptic cone operators. It begins with a brief description of classical results for regular differential operators on smooth manifolds, and includes a concise introduction to the theory of cone differential operators. The later sections describe recent joint work of the author with J. Gil and T. Krainer on the existence of the resolvent of elliptic cone operators and the structure of its asymptotic behavior as the modulus of the spectral parameter tends to infinity within a sector in $\mathbb{C}$ on which natural ray conditions on the symbol of the operator are assumed. These ideas are illustrated with examples.

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1. Introduction

The principal aim of these notes is to give an overview of certain interesting structural properties of the zeta function of an elliptic cone operator on a compact manifold. We begin, in Section 2, with an account of the zeta function of elliptic operators in the classical settings, and continue in Section 3 with a description of a number of results mostly concerning the equivalent problem of the structure of the small time asymptotics of the trace of the heat semigroup of elliptic operators on manifolds with conical singularities (assuming of course some appropriate positivity conditions).

A reader wishing to go somewhat further into the details of the theory and the meaning of some terms used in the statements in Section 3 may benefit from the material in Sections 4 and 5, which go into some of the details of the theory of cone operators. Some aspects of the spectral theory of elliptic cone operators are

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presented in Section 6. Sections 4, 5, and 6 are needed for Sections 7 and 8. The first of these last two sections presents results concerning rays of minimal growth, while the last is intended to give an idea of the origin of the complicated structure of the singularities of the zeta function.

The expositions in Sections 4 to 8 are based on joint work with Juan B. Gil and Thomas Krainer contained in the papers [14, 15, 16, 17, 18].

2. Classical results

Leaving aside the *a posteriori* observation about the relation between the classical Riemann (or the Epstein) zeta function and the zeta function of the Laplacian on a circle (or torus), it is fair to say that the zeta function of a differential operator appeared first in a paper of T. Carleman [6]. There he proves Weyl's estimate for the eigenvalues of the Dirichlet Laplacian on a planar region $\mathcal{M}$ with piecewise C^2 boundary ("continuous curvature") using the Ikehara-Wiener Tauberian theorem (Ikehara [22], see Korevaar [27]). In Carleman's paper, the zeta function appears in the form

$$\frac{1}{2\pi i}\int_\gamma \sum_{k=1}^{\infty} \frac{\lambda\phi_k(p)^2}{\lambda_k(\lambda_k-\lambda)}\lambda^{-s}\,d\lambda$$

where the ϕ_k form a complete orthonormal system of eigenfunctions, $\Delta\phi_k = \lambda_k\phi_k$ with the λ_k forming a nondecreasing sequence, and γ is a line $\Im\lambda = a$, $0 < a < \lambda_1$ (throughout this note the convention will be to take the positive Laplacian). Of course this is the zeta function after integration over $\mathcal{M}$. The fact that $s = 1$ is a simple pole allows the use of Ikehara's theorem, and gives a first direct link between the residues of the zeta function and what one may term the classical spectral information.

Carleman's work not withstanding, the explicit study of zeta functions, and related objects, of elliptic differential operators, began with work of S. Minakshisundaram in the late 1940's, in particular his work with Å. Pleijel [33] in which one finds, among other things, the following:

Theorem 2.1 (Minakshisundaram and Pleijel [33]**).** *Let Δ the Laplacian on a compact orientable Riemannian manifold $\mathcal{M}$ without boundary, or with smooth boundary and either Dirichlet or Neumann boundary condition. Let λ_k be the eigenvalues repeated with multiplicity. Then the corresponding zeta function has a meromorphic extension to $\mathbb{C}$ with simple poles contained in*

$$\{(n-\ell)/m : \ell \in \mathbb{N}_0\}\backslash(-\mathbb{N}_0) \tag{2.2}$$

where $n = \dim\mathcal{M}$, $m = 2$ is the order of Δ, and $\mathbb{N}_0$ is the set of non-negative integers.

Minakshisundaram and Pleijel proved their theorems in [33] by first constructing the Schwartz kernel of a parametrix for the initial value problem for the heat equation. From this they constructed (using a Laplace transform) the

Schwartz kernel of a parametrix for the resolvent of Δ, which they then exploited using Cauchy's integral formula to write expressions for the zeta function from which the meromorphic continuation and other properties were read off.

The idea of going directly from the heat kernel to the zeta function via Mellin transform to establish the fundamental analytic properties of zeta functions appears in a paper of Minakshisundaram [31] (submitted only 4 days after the paper with Pleijel cited above) where he discusses the behavior of the zeta function associated with the flat Laplacian with Dirichlet or Neumann conditions on a domain with smooth boundary. A few years later Minakshisundaram [32] used this to give more direct proofs of his results with Pleijel. See [12] for a perspective on Minakshisundaram's conceptual contribution to spectral analysis.

Incidentally, the relation between the ζ function and the trace of the heat kernel is the following. Suppose that A is an unbounded selfadjoint operator on some Hilbert space with discrete spectrum $\{\lambda_k\}_{k=0}^\infty$ (assumed to lie in $(0,\infty)$ and to satisfy a Weyl-type estimate, $\lambda_k \sim ck^\alpha$ for some $c, \alpha > 0$). Then e^{-tA} is trace-class for $t > 0$,

$$\operatorname{Tr} e^{-tA} = \sum_{k=0}^\infty e^{-\lambda_k t}$$

and one has

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr} e^{-tA} t^s \,\frac{dt}{t} \tag{2.3}$$

for every s with sufficiently large real part.

The proofs of Minakshisundaram and Pleijel in [33] become at some point what amounts to an analysis of the Schwartz kernel of the complex powers of the Laplacian. This analysis was made explicit by R. T. Seeley, who gave far reaching extensions of these theorems, to general elliptic differential operators on compact manifolds without boundary acting on sections of a vector bundle [34], and to elliptic boundary value problems [35, 36], in both cases assuming that a ray condition if satisfied. Seeley showed, among other things, that for selfadjoint problems, the zeta function of a differential operator of order m on an n-dimensional manifold has a meromorphic continuation to all of $\mathbb{C}$ with simple poles contained in the set (2.2).

Greiner [21] obtained an expansion of the heat trace for small time for elliptic partial differential operators of even order acting on sections of a vector bundle (essentially constructing a parametrix for the heat operator via an anisotropic pseudodifferential calculus), again both for closed manifolds and for compact manifold with boundary and suitable boundary conditions. In Greiner's work, the principal symbol of the operator is assumed to have the property that the real part of its eigenvalues is bounded below uniformly by a positive number on the cosphere bundle of the manifold.

There is a direct, explicit relation by way of (2.3) between the coefficients of the small time asymptotics of $\operatorname{Tr} e^{-tA}$ (to the extent that such expansion exists) and the residues of the zeta function of A. Therefore the analysis of the residues of the zeta function and that of the coefficients of the expansion of $\operatorname{Tr} e^{-tA}$ at $t = 0$ are equivalent problems. There is by now a wealth of information gathered through, and about, the heat kernel, see [20] and [23] for instance, with many implications and applications in a number of areas beyond zeta functions; describing these would take us far away in a direction which is not the subject here, so in the rest of these notes we focus on manifolds with conical singularities.

3. Conical singularities

The simple meromorphic structure (location and order of the poles) of the zeta function, even its meromorphic extendability, begins to disappear when considering differential operators with singularities. This showed up first in the from of a logarithmic term in the short time expansion of the heat trace in J. Cheeger's analysis [8] of spectral properties of compact manifolds with conical singularities with straight cone metrics near the conical points. The latter means, in effect, that $\mathcal{M}$ is a compact manifold with boundary with a metric which is Riemannian in $\mathring{\mathcal{M}}$ and of the form

$$g_c = dx \otimes dx + x^2 \pi^* g \tag{3.1}$$

in a tubular neighborhood U of $\partial\mathcal{M}$; the map $\pi : U \to \partial\mathcal{M}$ is the projection, x is a defining function of $\partial\mathcal{M}$ (positive in the interior of $\mathcal{M}$), and g is a Riemannian metric on $\partial\mathcal{M}$. The structure of the metric (3.1) is that of the Euclidean metric g_e in polar coordinates. More generally, if $\mathcal{N}$ is a closed submanifold of S^{N-1} and

$$\mathcal{M} = [0,\infty) \times \mathcal{N},\ \wp : \mathcal{M} \to \mathbb{R}^N,\ \mathcal{M} \ni (x,y) \mapsto \wp(x,y) = xy \in \mathbb{R}^N,$$

then $\wp^* g_e$ has the structure of the metric in (3.1)

Cheeger's primary concern in [8] and various other of his papers at the time such as [9] (which considerably extends [8]) lies with various spectral invariants associated with the Laplacian for such metrics on forms of any degree. He observes that one can obtain a parametrix for the heat equation on $\mathcal{M}$ by gluing together parametrices for the problem in the interior of $\mathcal{M}$ and an exact parametrix near the boundary (this is the same general scheme as that in the first step of the paper of Minakshisundaram and Pleijel), from which the asymptotics of the trace of heat kernel can be obtained with arbitrary precision by a recursive process. The exact parametrix near the boundary is obtained using separation of variables. The end result is the validity of an expansion of the form

$$\operatorname{Tr} e^{-t\Delta_F} \sim \sum_{k=0}^{\infty} a_k t^{(k-n)/2} + a_0' \log t$$

where Δ_F means the Friedrichs extension, in which the novelty is the appearance of the logarithmic term; as before, $n = \dim \mathcal{M}$. Following Minakshisundaram [32] one obtains that the zeta function, computed using (2.3), extends as a meromorphic function to all of $\mathbb{C}$ with simple poles in the set (2.2) ($m = 2$ and $n = \dim \mathcal{M}$) and possibly also at 0.

Around the same time, Callias and Taubes [5] also found logarithmic singularities in the short time asymptotics of the heat trace of certain selfadjoint operators related to the Dirac operator with singular potential (where such a logarithmic term appears in a relative trace formula). An explicit calculation by Callias [4] shows that if A is the closure of

$$D_x^2 + \kappa/x^2 : C_c^\infty(\mathbb{R}_+) \subset L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \tag{3.2}$$

with Lebesgue measure and $\kappa \geq 3/4$ (which implies that (3.2) has exactly one selfadjoint extension, namely its Friedrichs extension) then (see *ibid.*, Theorem (5.κ))

$$\langle \operatorname{tr} e^{-tA}, \varphi\rangle \sim \sum_{k=0}^{\infty} \langle c_k, \varphi\rangle t^{(k-1)/2} + \sum_{k=1}^{\infty} \langle c_k', \varphi\rangle t^{k-1/2} \log t, \qquad \varphi \in C_c^\infty(\overline{\mathbb{R}}_+)$$

as $t \to 0^+$. Here $\operatorname{tr} e^{-tA}$ is the restriction of the Schwartz kernel of e^{-tA} for fixed $t > 0$ to the diagonal and the c_k and c_k' are certain distributions. As a consequence, the "zeta" function of A, computed via (2.3) has, at least in principle, double poles on the set (2.2) with $n = 1$ and $m = 2$.

Several questions on regions with conical singularities were studied by Kondrati'ev in the 1960's, see for example [26]. But it was the work of Cheeger cited above that generated the impetus for intense subsequent work by many authors, including Brüning, Melrose, Schulze, and Seeley, eventually also Lesch, Mazzeo, and many others, on various aspects of analysis on manifolds with conical singularities or cylindrical ends and other variants of the problem.

To go further we need some terminology (more will be given in subsequent sections). Analysis on a manifold with conical singularities or cylindrical ends really means analysis of a partial differential operator of a special type on a manifold $\mathcal{M}$ with boundary together with, explicitly or implicitly, a cone metric or a b-metric (see Section 4 for the general definitions; the metric g_c in (3.1) is an example of a cone metric, see (4.2), and $g_b = x^{-2} g_c$ is the prototype of a b-metric). The Laplacian with respect to a general cone metric serves as the model for elliptic cone operators, sometimes also called a Fuchs-type operator. For example, the Laplacian with respect to the product metric (3.1), easily computed, has the form

$$\frac{1}{x^2}\big((xD_x)^2 - i(n-2)xD_x + \Delta_g\big).$$

Here a neighborhood of $\partial\mathcal{M}$ in $\mathcal{M}$ is thought of as $\partial\mathcal{M} \times [0, \varepsilon)$, $\varepsilon > 0$, and Δ_g is the Laplacian on $\partial\mathcal{M}$ with respect to g. In general, a b-differential operator is a differential operator on $\mathcal{M}$ which near any point of the boundary has the form

$$P = \sum_{k+|\alpha| \leq m} a_{k,\alpha}(x, y)(xD_x)^k D_y^\alpha$$

with smooth coefficients $a_{k,\alpha}$, where (x, y) are coordinates near the point with x a defining function for $\partial\mathcal{M}$. The class of b-operators of order m mapping sections of a vector bundle E to sections of a vector bundle F is denoted $\operatorname{Diff}_b^m(\mathcal{M}; E, F)$, or just $\operatorname{Diff}^m_b(\mathcal{M};E)$ if $F=E$, or $\operatorname{Diff}^m_b(\mathcal{M})$ in the case of scalar operators. A general cone differential operator of order m is an operator A such that $x^m A$ is a b-operator. As indicted above, more details will be given in Section 4.

In the specific case of interest to us here, namely operators on manifolds with conical singularities, there were a number of results proved in fair generality in connection with the asymptotic of the trace of the heat kernel in the 1990's. Among these we single out the following two theorems, the first of which is a general result.

Theorem 3.3 (Lesch [28], Theorem 2.4.1). *Let $A\in x^{-\nu}\operatorname{Diff}^m_b(\mathcal{M};E)$ be a positive differential operator on a compact manifold with boundary, assume $x^\nu A$ is b-elliptic. Let $\mathcal{D}$ be the domain of a positive selfadjoint extension of A. Then $e^{-tA_{\mathcal{D}}}$ is a trace class operator and*

$$\operatorname{Tr}(e^{-tA_{\mathcal{D}}})\sim\sum_{k=0}^{n-1}a_k t^{(k-n)/m}+\mathcal{O}(\log t)\qquad \text{as } t\to 0^+.$$

It is interesting to pass along Lesch's observation that the coefficients a_k are independent of the extension, since they are local invariants determined by A. The effect of the domain is hidden in the $\mathcal{O}(\log t)$ term.

A complete expansion was obtained by Brüning and Seeley [2] for certain second order operators which near the boundary of $\mathcal{M}$ are of the form $D_x^2+x^{-1}A(x)$ where $A(x)$ is an unbounded selfadjoint operator satisfying certain lower bound estimates. The statement about $A(x)$ incorporates the choice of a domain for the operator. This is generally a necessary step since cone operators, initially defined on compactly supported smooth functions or sections in the interior of $\mathcal{M}$ may have many selfadjoint extensions.

Another complete expansion, this time for operators of arbitrary order but with special assumptions on its structure near the boundary (leading to the property essentially that separation of variables works) and on the domain is the following:

Theorem 3.4 (Lesch [28], Theorem 2.4.6). *Let $A=x^{-\nu}P$ with $P\in\operatorname{Diff}^m_b(\mathcal{M};E)$ b-elliptic. Suppose A is symmetric positive on its minimal domain. Suppose further that A has constant coefficients near $\partial\mathcal{M}$, and let $\mathcal{D}$ be the domain of a positive selfadjoint extension of A. Assume further that $\mathcal{D}$ is stationary (see* (5.9)*). Then $\operatorname{Tr}(e^{-tA_{\mathcal{D}}})$ has a full asymptotic expansion,*

$$\operatorname{Tr}(e^{-tA_{\mathcal{D}}})\sim\sum_{k=0}^{\infty}a_k t^{(k-n)/m}+b\log t\qquad \text{as } t\to 0^+. \tag{3.5}$$

Constant coefficients means that there is a tubular neighborhood $\pi:U\to\partial\mathcal{M}$, a defining function x, and a connection ∇ on E, such that with $P=x^\nu A$ (an element of $\operatorname{Diff}^m_b(\mathcal{M};E)$) one has that $[x\nabla_{D_x},A]=0$ near $\partial\mathcal{M}$. Here $D_x=-i\partial_x$ and ∂_x means the vector field tangent to the fibers of π such that $\partial_x x=1$.

A complete asymptotic expansion was also obtained by Gil [13] for a general elliptic cone operator, with the hypothesis of selfadjointness replaced by a sector property and with the assumption that the operator has only one closed extension:

Theorem 3.6 (Gil, [13], Theorem 4.9). *Let $\Lambda \subset \mathbb{C}$ be the complement of a closed sector in contained in $\Re\lambda > 0$. Let $A \in x^{-m}\operatorname{Diff}_b^m(\mathcal{M})$, $m > 0$, be such that $A - \lambda$ is parameter-elliptic with respect to some $\gamma \in \mathbb{R}$ on Λ (see* [13, Definition 3.1]*). Suppose further that*

$$A : C_c^\infty(\mathring{\mathcal{M}}) \subset x^{\gamma - n/2 - m} L_b^2(\mathcal{M}) \to x^{\gamma - n/2 - m} L_b^2(\mathcal{M})$$

has only one closed extension. Then the heat trace admits the asymptotic expansion

$$\operatorname{Tr} e^{-tA} \sim \sum_{k=0}^{\infty} a_k t^{(k-n)/m} + \sum_{k=0}^{\infty} a'_k t^{k/m} \log t \qquad \text{as } t \to 0^+,$$

where a_k and a'_k are constants depending on the symbolic structure of A.

Gil proves his theorem by first constructing the resolvent of A on the sector Λ and then using a Dunford integral to obtain the heat semigroup. A similar result can be deduced from the work of Loya [29] on the structure of the resolvent of a cone operator (in which the underlying assumptions are similar to those of Gil, *op. cit.*). Incidentally, the asymptotic structure of the resolvent for elliptic pseudodifferential operators on a closed manifold was obtained by Agranovich [1].

Returning to the specific topic of the structure of the zeta function itself, Falomir, Pisani, and Wipf [10] discovered an example (an ordinary differential operator) showing that the location of the poles of the zeta function need not be the set (2.2). This work was followed by investigations of a similar nature by Falomir, Muschietti, Pisani, and Seeley [11] and then by work by Kirsten, Loya, and Park [24] for second order Laplace-like cone operators with constant coefficients near the boundary (see the definition of this concept above after Theorem 3.4) showing that the zeta function may not have a meromorphic extension at all due to the presence of logarithmic terms. These same authors showed in a more extensive analysis [25] (still in the very important case of Laplace-like operators, with constant coefficients near the boundary) that the poles of the zeta function may occur at arbitrary places in $\mathbb{C}$, and that the singularities may be logarithmic.

The most general result on the asymptotic expansion of the resolvent of a general cone operator as the modulus of spectral parameter tends to infinity within a sector, with no other assumption than the correct ellipticity and ray (or sector) conditions was obtained by Gil, Krainer, and the author of the present note in [18], see Theorem 8.2 below. The implications of the complicated asymptotics on the zeta function are similar to those obtained by Kirsten, Loya, and Park in [24].

4. Cone differential operators

Cone differential operators are a generalization of the kind of operators one obtains when writing regular differential operators with smooth coefficients in spherical coordinates. The underlying manifold is a manifold with boundary (interpreted as the spherical blowup of a manifold with conical [isolated] singularities).

Explicitly, let $\mathcal{M}$ be a manifold with boundary and $E, F \to \mathcal{M}$ be complex vector bundles over $\mathcal{M}$, then a cone differential operator of order m is an element of $x^{-m}\operatorname{Diff}^m_b(\mathcal{M}; E, F)$, where x is a defining function for $\partial\mathcal{M}$, $x > 0$ in $\overset{\circ}{\mathcal{M}}$ and $\operatorname{Diff}^m_b(\mathcal{M}; E, F)$ is the class of totally characteristic operators, or b-differential operators, of Melrose (see [30]); this is a subspace of the space $\operatorname{Diff}^m(\mathcal{M}; E, F)$ of linear differential operators

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F).$$

Thus the elements of $\operatorname{Diff}^m_b(\mathcal{M}; E, F)$ are linear differential operators with smooth coefficients up to the boundary that can be represented locally near the boundary as matrices of linear combinations with smooth coefficients of products of up to m vector fields which are tangential to the boundary. Equivalently, using conjugation with the multiplication operator x^k,

$$\begin{aligned}\operatorname{Diff}^m_b&(\mathcal{M}; E, F)\\ &= \{P \in \operatorname{Diff}^m(\mathcal{M}; E, F) : x^{-k} P x^k \in \operatorname{Diff}^m(\mathcal{M}; E, F)\ \forall k \in \mathbb{N}_0\}.\end{aligned} \tag{4.1}$$

Because of their definition, the natural primary structure bundle when dealing with b-differential operators is the b-tangent bundle, ${}^b\pi : {}^bT\mathcal{M} \to \mathcal{M}$, the vector bundle over $\mathcal{M}$ whose smooth sections are in one-to-one correspondence with the submodule $C^\infty_{\tan}(\mathcal{M}; T\mathcal{M})$ of the $C^\infty(\mathcal{M}; \mathbb{R})$-module $C^\infty(\mathcal{M}; T\mathcal{M})$ whose elements are vector fields which are tangential to the boundary. Since $C^\infty_{\tan}(\mathcal{M}; T\mathcal{M})$ is locally free finitely generated over $C^\infty(\mathcal{M}; \mathbb{R})$, there are indeed a vector bundle ${}^bT\mathcal{M}$ and bundle homomorphism ${}^b\mathrm{ev} : {}^bT\mathcal{M} \to T\mathcal{M}$ inducing an isomorphism ${}^b\mathrm{ev}_* : C^\infty(\mathcal{M}; {}^bT\mathcal{M}) \to C^\infty_{\tan}(\mathcal{M}; T\mathcal{M})$.

A b-differential operator $P \in \operatorname{Diff}^m_b(\mathcal{M}; E, F)$ has a well defined principal symbol, a smooth section ${}^b\boldsymbol{\sigma}(P)$ of ${}^b\pi^* \operatorname{Hom}(E, F)$ over ${}^bT^*\mathcal{M}\backslash 0$, related to the standard principal symbol of P through the commutative diagram

$$\begin{array}{ccc} \pi^*\operatorname{Hom}(E,F) & \xrightarrow{({}^b\mathrm{ev}^*)^*} & {}^b\pi^*\operatorname{Hom}(E,F) \\ {\scriptstyle \boldsymbol{\sigma}(P)}\uparrow & & \uparrow{\scriptstyle {}^b\boldsymbol{\sigma}(P)} \\ T^*\mathcal{M}\backslash 0 & \xrightarrow[{}^b\mathrm{ev}^*]{} & {}^bT^*\mathcal{M}\backslash 0 \end{array}$$

in which the bottom map is the dual of ${}^b\mathrm{ev}$ (off of the zero section) and the top map is the natural map.

There is a vector bundle over $\mathcal{M}$ whose smooth sections (up to the boundary) are in one-to one correspondence with the elements of $x^{-1}C^\infty_{\tan}(\mathcal{M}; T\mathcal{M})$, and which we may denote by $x^{-1}\,{}^bT\mathcal{M}$. One can make the case that the principal symbols of elements of cone operators live on the dual of this bundle, denoted $x\,{}^bT^*\mathcal{M}$. But the definition appears to depend on the defining function x. Following a more natural path, define first (see [15]) $C^\infty_{\mathrm{cn}}(\mathcal{M}, T^*\mathcal{M})$ as the $C^\infty(\mathcal{M}; \mathbb{R})$-submodule of $C^\infty(\mathcal{M}; T^*\mathcal{M})$ whose sections are conormal to $\partial\mathcal{M}$, that is, the elements $\alpha \in C^\infty(\mathcal{M}; T^*\mathcal{M})$ such that $\iota^*\alpha = 0$ (where $\iota : \partial\mathcal{M} \to \mathcal{M}$) is the inclusion map). Then again $C^\infty_{\mathrm{cn}}(\mathcal{M}, T^*\mathcal{M})$ is a locally free finitely generated module over $C^\infty(\mathcal{M}; \mathbb{R})$, so it is $C^\infty(\mathcal{M}; \mathbb{R})$-isomorphic to the space of smooth sections of a vector bundle ${}^c\pi : {}^cT^*\mathcal{M} \to \mathcal{M}$. It is not hard to see that ${}^cT^*\mathcal{M}$ is isomorphic to $x\, {}^bT^*\mathcal{M}$. We can now make a precise definition:

A cone metric is a smooth metric on ${}^cT^*\mathcal{M}$, that is, a smooth section of the symmetric tensor product $S^2\, {}^cT^*\mathcal{M}$ which is pointwise strictly positive. (4.2)

The map $C^\infty(\mathcal{M}; {}^cT^*\mathcal{M}) \to C^\infty(\mathcal{M}; T^*\mathcal{M})$ induces a vector bundle homomorphism

$$ {}^c\mathrm{ev} : {}^cT^*\mathcal{M} \to T^*\mathcal{M} $$

The bundle ${}^cT^*\mathcal{M}$ together with the map ${}^c\mathrm{ev}$ is the natural structure bundle in the context of cone operators. A cone operator $A \in x^{-m}\operatorname{Diff}(\mathcal{M}; E, F)$ has as principal symbol a smooth section ${}^c\boldsymbol{\sigma}(A)$ of ${}^c\pi^*\operatorname{Hom}(E, F)$. This principal symbol is related to the standard principal symbol of A over $\mathring{\mathcal{M}}$ (there A is a standard differential operator) by

$$ {}^c\boldsymbol{\sigma}(A)(\eta) = \boldsymbol{\sigma}(A)({}^c\mathrm{ev}(\eta)). $$

Ellipticity of $A \in x^{-m}\operatorname{Diff}^m_b(\mathcal{M}; E, F)$ (c-ellipticity) is defined as invertibility of ${}^c\boldsymbol{\sigma}(A)$. Writing $P = x^m A$ one verifies that c-ellipticity of A is equivalent to b-ellipticity of P.

Associated with A there is another symbol, the "wedge" symbol. This is a differential operator on the inward pointing normal bundle $N_+\partial\mathcal{M}$ of $\partial\mathcal{M}$ in $\mathcal{M}$ (with the zero section, its boundary, included), which we define in the next few paragraphs.

Any $P \in \operatorname{Diff}^m_b(\mathcal{M}; E, F)$ determines an operator

$$ P_b \in \operatorname{Diff}^m_b(\partial\mathcal{M}; E_{\partial\mathcal{M}}, F_{\partial\mathcal{M}}) $$

(where $E_{\partial\mathcal{M}}$ means the part of E over $\partial\mathcal{M}$, as a vector bundle over $\partial\mathcal{M}$) by way of the following procedure. Let ϕ be a smooth section of E along $\partial\mathcal{M}$, let $\tilde\phi$ be a smooth extension to a neighborhood of $\partial\mathcal{M}$. The characterization (4.1) implies that $(P\tilde\phi)|_{\partial\mathcal{M}}$ is independent of the extension $\tilde\phi$. Define

$$ P_b\phi = (P\tilde\phi)|_{\partial\mathcal{M}}. $$

Noting that $x^{-i\sigma}Px^{i\sigma} \in \operatorname{Diff}^m_b(\mathcal{M}; E, F)$ one may define the indicial family of P as the family

$$ \mathbb{C} \ni \sigma \mapsto \widehat{P}(\sigma) = (x^{-i\sigma}Px^{i\sigma})_b \in \operatorname{Diff}^m(\partial\mathcal{M}, E_{\partial\mathcal{M}}, F_{\partial\mathcal{M}}) $$

and the indicial operator

$$ P_\wedge \in \operatorname{Diff}^m(N_+\partial\mathcal{M}; E_\wedge, F_\wedge) $$

with the aid of the Mellin inversion formula, as

$$ P_\wedge u = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^\infty (x_\wedge/x'_\wedge)^{i\sigma}\widehat{P}(\sigma)u(x'_\wedge, y)\,\frac{dx'_\wedge}{x'_\wedge}\,d\sigma, \quad u \in C^\infty_c(\mathring{N}_+\partial\mathcal{M}; E_\wedge). $$

Here $E_\wedge$ is the pullback of E to $N_+\partial\mathcal{M}$ by the projection map and $x_\wedge$ is the function $dx : N_+\partial\mathcal{M} \to \mathbb{R}$ determined by x; this function is linear on the fibers. The definition of $P_\wedge$ appears to depend on the defining function x, but in fact it does not.

Finally if $A \in x^{-m}\operatorname{Diff}^m_b(N_+\partial\mathcal{M}; E, F)$, define

$$A_\wedge = x_\wedge^{-m} P_\wedge$$

using $P = x^m A$. Again $A_\wedge$ is independent of the defining function x.

If A is c-elliptic, then so is $A_\wedge$, and if P is b-elliptic, then $\widehat{P}(\sigma)$ is elliptic for every σ, $P_\wedge$ is b-elliptic, and the set

$$\operatorname{spec}_b(P) = \{\sigma \in \mathbb{C} : \widehat{P}(\sigma) \text{ is not invertible}\},$$

the boundary spectrum of P (or $A = x^{-m}P$), is a discrete set with the property that $\operatorname{spec}_b(P) \cap \{\sigma \in \mathbb{C} : a < \Im\sigma < b\}$ is finite for every a and $b \in \mathbb{R}$ (see Melrose [30]).

The operator $A_\wedge$ has an important homogeneity property inherited from that of $P_\wedge$. For $\varrho > 0$ let

$$\chi_\varrho : E_\wedge \to E_\wedge$$

denote parallel transport from $p \in N_+\partial\mathcal{M}$ to ϱp. In terms of the definition of $E_\wedge$ as the pull back of the bundle $\pi_E : E \to \mathcal{M}$ by the projection map $\pi : N_+\partial\mathcal{M} \to \partial\mathcal{M}$, we have

$$E_\wedge = \{(p, \eta) : p \in N_+\partial\mathcal{M},\ \eta \in E,\ \pi_{\partial\mathcal{M}}(p) = \pi_E(\eta)\}$$

and $\chi_\varrho(p, \eta) = (\varrho p, \eta)$. This parallel transport was implicitly used above, in the formula defining $P_\wedge$. Define

$$\kappa_\varrho \phi = \varrho^{-\mu}\chi_\varrho^* \phi \quad \text{for sections } \phi : N_{\partial\mathcal{M}} \to E_\wedge \tag{4.3}$$

Using this it is very easy to see that

$$P_\wedge \kappa_\varrho = \kappa_\varrho P_\wedge$$

and from this, that

$$\kappa_\varrho^{-1} A_\wedge \kappa_\varrho = \varrho^m A_\wedge. \tag{4.4}$$

The factor $\varrho^{-\mu}$ in (4.3), correctly chosen, will end up giving that κ_ϱ is unitary. This property is not too important because the formulas in which κ_ϱ appears it does so either as a conjugating operator, as it already did, or as the image by it of some space.

Example 4.5. Let $\mathcal{M}_0$ be a smooth closed orientable Riemannian 2-manifold, let Δ be the positive Laplacian, let $p_0 \in \mathcal{M}_0$. Let $\mathcal{M}$ be the spherical blowup of $\mathcal{M}_0$ at p_0 and $\wp : \mathcal{M} \to \mathcal{M}_0$ the blowdown map. Thus (i) $\mathcal{M}$ is diffeomorphic to $\mathcal{M}_0 \backslash D$ where $D \subset \mathcal{M}_0$ is a (small) metric open disc centered at p_0 in which we have normal coordinates (y_1, y_2) and (ii) $\wp$ gives a diffeomorphism from $\mathring{\mathcal{M}} = \mathcal{M}\backslash\partial\mathcal{M}$ to $\mathcal{M}_0 \backslash \{p_0\}$ and sends a suitable tubular neighborhood U of the circle $\partial\mathcal{M}$ to D by way of the map $(x, \theta) \mapsto (y_1, y_2) = (x\cos\theta, x\sin\theta)$.

Let A be the operator determined by Δ on $\mathring{\mathcal{M}}$. Then $A_\wedge$ is just the Euclidean Laplacian of $\mathbb{R}^2$ in polar coordinates (an operator on $\mathring{N}_+(\partial\mathcal{M}) = S^1 \times (0,\infty)$).

5. Domains

The L^2 theory of elliptic cone operators on a compact manifold with boundary is very much like that of elliptic differential operators on a closed manifold. Indeed, suppose given a smooth b-measure on $\mathcal{M}$ (a Borel measure $\mathfrak{m}_b$ such that $x\mathfrak{m}_b$ is a smooth positive measure on $\mathcal{M}$) and hermitian metrics on E and F with which the spaces $x^\mu L^2_b(\mathcal{M};E) = L^2(\mathcal{M};E;x^{-2\mu}\mathfrak{m}_b)$, likewise $x^\mu L^2_b(\mathcal{M};F)$ are defined. Basing the analysis on these Hilbert spaces, any c-elliptic operator $A \in x^{-m}\operatorname{Diff}^m_b(\mathcal{M};E,F)$ is, on its natural maximal domain,

$$\mathcal{D}_{\max}(A) = \{u \in x^\mu L^2_b(\mathcal{M};E) : Au \in x^\mu L^2_b(\mathcal{M};F)\},$$

a Fredholm operator (Lesch [28, Proposition 1.3.16]).

But the theory is also like that of regular elliptic operators on manifolds with boundary, with boundary conditions. This is because if $A \in x^{-m}\operatorname{Diff}^m_b(\mathcal{M};E,F)$ is c-elliptic, then the closure of

$$A : C^\infty_c(\mathring{\mathcal{M}};E) \subset x^\mu L^2_b(\mathcal{M};E) \to x^\mu L^2_b(\mathcal{M};F) \tag{5.1}$$

is again a Fredholm operator (Lesch *op. cit.*). The domain of the closure, $\mathcal{D}_{\min}(A)$, is often a proper subspace of $\mathcal{D}_{\max}(A)$. In this case a condition needs to be imposed to determine which domain is being used to study the operator.

For a proof of the Fredholm properties of A on its minimal and maximal domains (and many other facts concerning elliptic come operators) the reader may consult, as already indicated, Lesch [28]. Alternatively, the reader may fill in the details of the following outline in which parts of Chapters 4 through 6 of Melrose [30] are assumed.

Let $P = x^m A$. Since P is b-elliptic, the set

$$\{\Im\sigma : \sigma \in \operatorname{spec}_b(P)\}$$

is a discrete subset of $\mathbb{R}$ without points of accumulation. The closure of

$$P : C^\infty_c(\mathcal{M};E) \subset x^{\mu'} L^2_b(\mathcal{M};E) \to x^{\mu'} L^2_b(\mathcal{M};F)$$

is Fredholm if and only if $\mu' \notin -\Im\operatorname{spec}_b(P)$ ([30, Theorem 5.40]), in fact in this case there are operators

$$\begin{aligned} Q &: x^{\mu'} L^2_b(\mathcal{M};F) \to x^{\mu'} H^m_b(\mathcal{M};E),\\ R_{\mathfrak{l}} &: x^{\mu'} L^2_b(\mathcal{M};E) \to x^{\mu'+\varepsilon} H^\infty_b(\mathcal{M};E),\\ R_{\mathfrak{r}} &: x^{\mu'} L^2_b(\mathcal{M};F) \to x^{\mu'+1} H^\infty_b(\mathcal{M};F) \end{aligned} \tag{5.2}$$

(ε is positive and smaller than $\min\{-\mu' - \rho : \rho \in \Im\operatorname{spec}_b(P),\ \rho < -\mu'\}$) such that

$$QP = I - R_{\mathfrak{l}}, \qquad PQ = I - R_{\mathfrak{r}}. \tag{5.3}$$

For nonnegative integers m the Sobolev spaces H_b^m are defined inductively as follows. First $H_b^0 = L_b^2$. Next if $m > 0$ is an integer, then $u \in H_b^m$ iff $Yu \in H_b^{m-1}$ for all $Y \in C^\infty_{\tan}(\mathcal{M}; T\mathcal{M})$. The space H_b^{-m} is the dual of H_b^m, and for noninteger s, H_b^s is defined by interpolation; see [30] for a more detailed description. A fundamental property of these Sobolev spaces is that the inclusion $x^{\mu''} H_b^{s''} \hookrightarrow x^{\mu'} H_b^{s'}$ is compact if $\mu'' > \mu'$ and $s'' > s'$; this uses that $\mathcal{M}$ is compact. Thus $R_{\mathfrak{l}}$ and $R_{\mathfrak{r}}$ are compact as operators $x^{\mu'} L_b^2 \to x^{\mu'} L_b^2$.

Suppose $\delta > 0$ is such that $(-\mu - m, -\mu - m + \delta] \cap \Im \operatorname{spec}_b(P) = \emptyset$, pick $\mu' = \mu + m - \delta$ above and let Q and $R_{\mathfrak{r}}$ be as stated. So

$$Q : x^{\mu+m-\delta} L_b^2(\mathcal{M}; F) \to x^{\mu+m-\delta} H_b^m(\mathcal{M}; E)$$

and

$$R_{\mathfrak{r}} : x^{\mu+m-\delta} L_b^2(\mathcal{M}; F) \to x^{\mu+m-\delta+1} H_b^\infty(\mathcal{M}; F)$$

If $f \in x^\mu L_b^2$, then $x^m f \in x^{\mu+m} L_b^2$. Since the latter space is a subspace of $x^{\mu+m-\delta} L_b^2$, $Qx^m f \in x^{\mu+m-\delta} H_b^m$. The inclusion $x^{\mu+m-\delta} H_b^m \subset x^\mu H_b^m$ gives in particular $Qx^m f \in x^\mu L_b^2$. Note that $Qx^m : x^\mu L_b^2 \to x^\mu L_b^2$ is compact. Further,

$$AQx^m f = x^{-m}(PQx^m f) = f - x^{-m} R_{\mathfrak{r}} x^m f.$$

Since

$$x^{-m} R_{\mathfrak{r}} x^m f \in x^{\mu+m-\delta+1} H_b^\infty(\mathcal{M}; F) \subset x^\mu L_b^2,$$

Qx^m maps $x^\mu L_b^2$ into the maximal domain of A. The inclusion in the last displayed equation is compact, so $B = Qx^m$ is a compact parametrix for A with compact error. One can show that B maps into the minimal domain, and with some more work (see [19]), that

$$\mathcal{D}_{\min}(A) = \mathcal{D}_{\max}(A) \cap \bigcap_{\delta > 0} x^{\mu+m-\delta} H_b^m(\mathcal{M}; E). \tag{5.4}$$

It is easy to see that $x^{\mu+m} H_b^m(\mathcal{M}; E) \subset \mathcal{D}_{\min}(A)$. If $-\mu - m \in \Im \operatorname{spec}_b(P)$, then (5.4) is the best statement one can make about the minimal domain. On the other hand, if $-\mu - m \notin \Im \operatorname{spec}_b(P)$, then we may repeat the argument above with $\delta = 0$ and conclude that $\mathcal{D}_{\min}(A) = x^{\mu+m} H_b^m(\mathcal{M}; E)$. In either case we see that A as an operator on its minimal domain is Fredholm. The argument also gives that $A^\star$, the formal adjoint of A, is Fredholm on its minimal domain. But the Hilbert space adjoint of $A^\star$ with its minimal domain is A with its maximal domain, so A with its maximal domain is also Fredholm.

Since A with either the minimal or the maximal domain is Fredholm, every closed extension of (5.1) is Fredholm. The domain $\mathcal{D}$ of any such extension contains $\mathcal{D}_{\min}$ and is contained in $\mathcal{D}_{\max}$. It also follows that $\mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$ is finite dimensional. So we may parametrize the closed extensions of (5.1) by the set of subspaces of $\mathcal{D}_{\max}/\mathcal{D}_{\min}$.

Define

$$(u, v)_A = (Au, Av) + (u, v), \quad u, v \in \mathcal{D}_{\max}(A)$$

where the inner products on the right are those of $x^\mu L^2(\mathcal{M};F)$ and $x^\mu L^2(\mathcal{M};E)$, respectively. This defines an inner product on $\mathcal{D}_{\max}(A)$, and the latter space is complete with respect to the induced norm, which of course is equivalent to the graph norm. Note that the space $\mathcal{D}_{\min}(A)$ is the closure of $C_c^\infty(\mathcal{M};E)$ in $\mathcal{D}_{\max}(A)$ with respect to the graph norm. We always view $\mathcal{D}_{\max}(A)$ as a Hilbert space with the inner product just defined. The arguments just presented show that the inclusion

$$\mathcal{D}_{\max}(A) \hookrightarrow x^\mu L^2_b(\mathcal{M};E) \tag{5.5}$$

is compact.

The space $\mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$ is isomorphic to the orthogonal $\mathcal{E}(A) = \mathcal{D}_{\min}(A)^\perp$ of $\mathcal{D}_{\min}(A)$ in $\mathcal{D}_{\max}(A)$. One can prove (see [15]) that

$$\mathcal{E}(A) = \ker(A^\star A + I) \cap \mathcal{D}_{\max}(A).$$

Here $\ker(A^\star A + I)$ is the kernel of $A^\star A + I$ acting on the the space of E-valued distributions on $\mathring{\mathcal{M}}$. We may now write the domain of any closed extension of (5.1) as $\mathcal{D} = D + \mathcal{D}_{\min}$ where D is a subspace of $\mathcal{E}(A)$. This is a particularly useful description when discussing selfadjoint extensions of symmetric elliptic cone operators. Define

$$\mathrm{Gr}(\mathcal{E}(A)) = \{D : D \text{ is a subspace of } \mathcal{E}(A)\}.$$

Thus $\mathrm{Gr}(\mathcal{E}(A)) = \bigcup_k \mathrm{Gr}_k(\mathcal{E}(A))$ is the disjoint union of the various Grassmannian varieties associated with $\mathcal{E}(A)$.

All aspects of this section have counterparts associated with the operator $A_\wedge$, except for the Fredholm property of the extensions of

$$A_\wedge : C_c^\infty(N_+\partial\mathcal{M};E_\wedge) \subset x_\wedge^\mu L^2_b(N_+\partial\mathcal{M};E_\wedge) \to x_\wedge^\mu L^2_b(N_+\partial\mathcal{M};F_\wedge).$$

It remains true that $\mathcal{D}_{\wedge,\min}$ has finite codimension in $\mathcal{D}_{\wedge,\max}$, hence the domain of any closed extension is of the form $D_\wedge + \mathcal{D}_{\wedge,\min}$. Additionally, because of (4.4), we have that κ_ϱ acts on $\mathcal{D}_{\wedge,\max}$ albeit not as a unitary map, only as an isomorphism of Banach spaces. It is easy to see, again using (4.4), that κ_ϱ preserves $\mathcal{D}_{\wedge,\min}$. It follows that $\pi_{\wedge,\max}\kappa_\varrho = \pi_{\wedge,\max}\kappa_\varrho\pi_{\wedge,\max}$ and that we also have an action

$$\boldsymbol{\kappa}_\varrho : \mathcal{E}_\wedge \to \mathcal{E}_\wedge, \quad \boldsymbol{\kappa}_\varrho v = \pi_{\wedge,\max}\kappa_\varrho v. \tag{5.6}$$

Letting $\pi_{\wedge,\max} : \mathcal{D}_{\wedge,\max} \to \mathcal{D}_{\wedge,\max}$ be the orthogonal projection, we get diffeomorphisms

$$\mathrm{Gr}_k(\mathcal{E}(A_\wedge)) \ni D_\wedge \mapsto \boldsymbol{\kappa}_\varrho D_\wedge \in \mathrm{Gr}_k(\mathcal{E}(A_\wedge)). \tag{5.7}$$

The infinitesimal generator of the one-parameter family of diffeomorphisms $\boldsymbol{\kappa}_{e^t}$,

$$\mathcal{T}(D_\wedge) = \frac{d}{dt}\bigg|_{t=0} \boldsymbol{\kappa}_{e^t} D_\wedge$$

is a smooth (in fact real-analytic) vector field.

There is a natural isomorphism (see [15])

$$\theta : \mathcal{E}(A) \to \mathcal{E}(A_\wedge) \tag{5.8}$$

that allows passage from domains for closed extensions of A to domains of closed extensions of $A_\wedge$. This map and the action (5.7) are fundamental in the analysis of the resolvent of A with a given domain.

$$\text{An element } D_\wedge \in \mathrm{Gr}_k(\mathcal{E}(A_\wedge)) \text{ is said to be stationary if it is a fixed point of the action } \boldsymbol{\kappa}\text{. A domain } \mathcal{D} = D + \mathcal{D}_{\min} \text{ (or } \mathcal{D}_\wedge = D_\wedge + \mathcal{D}_{\min}) \text{ is said to be stationary if } \theta(D) \text{ (or } D_\wedge) \text{ is so.} \tag{5.9}$$

Stationary domains always exist because the Euler characteristic of $\mathrm{Gr}_k(\mathcal{E}_\wedge)$ is not zero, so the vector field $\mathcal{T}$ must vanish somewhere.

In the following sections we drop the argument A from objects whenever there is no ambiguity ($\mathcal{D}_{\min}$, $\mathcal{D}_{\max}$, $\mathcal{E}$, etc.), and add $\wedge$ as a subscript for objects associated with $A_\wedge$ ($\mathcal{D}_{\wedge,\min}$, $\mathcal{D}_{\wedge,\max}$, $\mathcal{E}_\wedge$, etc.).

Example 5.10. Continuing with the setup of Example 4.5, the $\overset{\circ}{L}{}^2$ spaces are those defined by the measure associated with the metric. Using that $\overset{\circ}{\mathcal{M}}$ is diffeomorphic to $\mathcal{M}_0\backslash\{p_0\}$ we see that $\mathcal{D}_{\min}$ is naturally isomorphic to the domain of the closure of

$$\Delta : C_c^\infty(\mathcal{M}_0\backslash\{p_0\}) \subset L^2(\mathcal{M}_0) \to L^2(\mathcal{M}_0).$$

It is immediate that $\{u \in H^2(\mathcal{M}_0) : u(p_0) = 0\}$ contains $\mathcal{D}_{\min}(A)$; in fact equality holds,

$$\mathcal{D}_{\min} = \{u \in H^2(\mathcal{M}_0) : u(p_0) = 0\},$$

as the reader may easily verify.

The maximal domain contains $H^2(\mathcal{M}_0)$ (more properly, $\wp^* H^2(\mathcal{M}_0)$). But it contains more elements. Namely, let $g^2 : \mathcal{M}_0 \to \mathbb{R}$ be a smooth function such that $g^2(p) = \mathrm{dist}(p, p_0)^2$ if p is near p_0. Then $\log g \in L^2(\mathcal{M})$. Since

$$\Delta \log g = \delta_{p_0} + h = \text{the Dirac } \delta \text{ at } p_0 \text{ plus a smooth function on } \mathcal{M}_0,$$

$A \log g = h$ (because A only "sees" what happens in $\mathcal{M}_0\backslash\{p_0\}$). Thus $A \log g \in L^2$, hence $\log g \in \mathcal{D}_{\max}(A)$. One can show that

$$\mathcal{D}_{\max}(A) = H^2(\mathcal{M}_0) \oplus \mathrm{span} \log g.$$

Similarly, for $A_\wedge$, which we may treat as the Euclidean Laplacian restricted to $\mathbb{R}^2\backslash 0$, we have that $\mathcal{D}_{\wedge,\min} = \{u \in H^2(\mathbb{R}^2) : u(0) = 0\}$ and

$$\mathcal{D}_{\wedge,\max} = H^2(R^2) \oplus \mathrm{span} \log g_\wedge.$$

where $g_\wedge$ is any compactly supported function on $\mathbb{R}^2$ which is smooth outside 0 and satisfies $g_\wedge(p) = \|p\|$ (the Euclidean norm of $p \in \mathbb{R}^2$) near 0. Thus if χ is an arbitrary compactly supported function on $\mathbb{R}^2$ with $\chi(0) \neq 0$, then

$$\mathcal{D}_{\wedge,\max} = \mathcal{D}_{\wedge,\min} \oplus \mathrm{span}\{\chi, \log g_\wedge\}.$$

The space $\mathcal{E}_\wedge$ is 2-dimensional, spanned by the functions $\pi_{\wedge,\max}\chi$ and $\pi_{\wedge,\max} \log g_\wedge$. It is also equal to $\ker(A_\wedge^2 + I) \cap \mathcal{D}_{\wedge,\max}$ (we are using that $A_\wedge$ is symmetric on $C_c^\infty(\mathbb{R}^2\backslash 0)$). The statement that $u \in L^2(\mathbb{R}^2)$ satisfies $A_\wedge^2 u + u = 0$ is equivalent to the statement that $\Delta^2 u + u$ is supported at 0. Passing to Fourier transform we see that $\widehat{u}(\xi) = p(\xi)/(1 + \|\xi\|^4)$ where p is a polynomial. Using that $\widehat{u}(\xi)$ is also in

$L^2(\mathbb{R}^2)$ and that $A_\wedge u \in L^2(\mathbb{R}^2\backslash 0)$ (that is, $\Delta u = c\delta_0 + f$, $c \in \mathbb{C}$, $f \in L^2(\mathbb{R}^2)$) we get conditions on p from which one concludes that u must be a linear combination of the functions

$$u_1(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\frac{1}{1+\|\xi\|^4}\,d\xi, \qquad u_2(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\frac{\|\xi\|^2}{1+\|\xi\|^4}\,d\xi.$$

Clearly $u_1 \in H^2(\mathbb{R}^2)$. As for u_2, note that

$$\Delta u_2(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\frac{\|\xi\|^4}{1+\|\xi\|^4}\,d\xi = \delta_0 - u_1(x).$$

Restricting to $\mathbb{R}^2\backslash 0$ we have $\Delta u_2 = -u_1$, that is, $A_\wedge u_2 = -u_1$. Also $A_\wedge u_1 = u_2$, so these two functions do belong to $\ker(A_\wedge^2 + I)$. Note that $(u_1, u_2)_{A_\wedge} = 0$, so $\{u_1, u_2\}$ is an $A_\wedge$- orthogonal basis of $\mathcal{E}_\wedge$. Finally, note that $\|u_1\|^2_{A_\wedge} = \|u_2\|^2_{A_\wedge}$; let μ denote this number ($\mu = 1/8$). The formulas

$$\pi_{\wedge,\max}\kappa_\varrho u_j = \frac{1}{\mu}\big((\kappa_\varrho u_j, u_1)_{A_\wedge} u_1 + (\kappa_\varrho u_j, u_2)_{A_\wedge} u_2\big), \quad j = 1, 2$$

give

$$\pi_{\wedge,\max}\kappa_\varrho u_1 = u_1, \quad \pi_{\wedge,\max}\kappa_\varrho u_2 = -\frac{2}{\pi}\log\varrho^2\, u_1 + u_2.$$

The integrals leading to these formulas can be evaluated by elementary means using polar coordinates.

Since $\mathcal{E}_\wedge$ is 2-dimensional, the only interesting Grassmannian variety based on it is $\mathrm{Gr}_1(\mathcal{E}_\wedge)$, the 1-dimensional complex projective space, in other words, the Riemann sphere. The action of $\boldsymbol{\kappa}_\varrho$ on elements of $\mathrm{Gr}_1(\mathcal{E}_\wedge)$ is easily described using the formulas for $\pi_{\wedge,\max}\kappa_\varrho u_j$. If $D \in \mathrm{Gr}_1(\mathcal{E}_\wedge)$ is spanned by $\alpha u_1 + \beta u_2$, $(\alpha,\beta) \neq 0$, then of course

$$\boldsymbol{\kappa}_\varrho D = \mathrm{span}\{(\alpha - \frac{2\beta}{\pi}\log\varrho^2)u_1 + \beta u_2\}. \tag{5.11}$$

The curve $\varrho \mapsto \boldsymbol{\kappa}_\varrho D$ has a limit as $\varrho \to 0$ or ∞. Namely, if $\beta = 0$, then $\boldsymbol{\kappa}_\varrho D = D_{\wedge,F} = \mathrm{span}\{u_1\}$ and if $\beta \neq 0$, then (once $\log\varrho \neq \alpha\pi/4\beta$)

$$\mathrm{span}\{(\alpha - \frac{2\beta}{\pi}\log\varrho^2)u_1 + \beta u_2\} = \mathrm{span}\{u_1 + \frac{\pi\beta}{\pi\alpha - 2\beta\log\varrho^2}u_2\}$$

also tends to $D_{\wedge,F}$, regardless of whether ϱ tends to 0 or ∞.

Also the infinitesimal generator of the one-parameter group $t \mapsto \boldsymbol{\kappa}_{e^t}$ can easily be described in terms of the homogenous coordinates on S^2. Writing either $u_1 + \zeta u_2$ or $zu_1 + u_2$ as basis for elements of $\mathrm{Gr}_1(\mathcal{E}_\wedge)$, the formulas above give, if $D = \mathrm{span}\{z_0 u_1 + u_2\}$, that the curve $\varrho \mapsto \boldsymbol{\kappa}_\varrho D$ is

$$\varrho \mapsto z(\varrho) = z_0 - \frac{2}{\pi}\log\varrho^2$$

in terms of the ζ coordinate. The derivative of $\zeta(\varrho)$ is

$$\frac{dz}{d\varrho}\frac{\partial}{\partial z}\Big|_{\zeta(\varrho)} + \frac{d\overline{z}}{d\varrho}\frac{\partial}{\partial\overline{z}}\Big|_{z(\varrho)} = -\frac{4}{\varrho\pi}\bigg(\frac{\partial}{\partial z}\Big|_{z(\varrho)} + \frac{\partial}{\partial\overline{z}}\Big|_{z(\varrho)}\bigg).$$

Evaluating at $\varrho = 1$ gives $\mathcal{T}$ at D. Thus if $z = x + iy$, then

$$\mathcal{T} = -\frac{4}{\pi}\frac{\partial}{\partial x}.$$

In terms of the coordinate $\zeta = \xi + i\eta$ we have

$$\mathcal{T} = -\frac{4}{\pi}\left((\eta^2 - \xi^2)\frac{\partial}{\partial \xi} + 2\xi\eta\frac{\partial}{\partial y}\right)$$

which has a zero at $\zeta = 0$ (which corresponds to $D_{\wedge,F}$).

6. Spectra

We assume now that $F = E$. Write $A_{\mathcal{D}}$ for the operator $A \in x^{-m}\operatorname{Diff}^m_b(\mathcal{M}; E)$ with domain $\mathcal{D}$; we continue to assume that A is c-elliptic and $\mathcal{M}$ is compact. Since $\operatorname{Ind}(A_{\mathcal{D}}) \neq 0$ implies $\operatorname{spec}(A_{\mathcal{D}}) = \mathbb{C}$, having index 0 is necessary in order for $A_{\mathcal{D}}$ to have nonempty resolvent set. It was pointed out by Lesch, *op. cit.* that the index of A with domain $\mathcal{D}$ ($A_{\mathcal{D}}$ for short) is given by the formula

$$\operatorname{Ind}(A_{\mathcal{D}}) = \operatorname{Ind}(A_{\mathcal{D}_{\min}}) + \dim D.$$

Since $\dim \mathcal{D} \geq 0$, a necessary condition for A to admit a closed extension with nonempty resolvent set is that $\operatorname{Ind}(A_{\mathcal{D}_{\min}}) \leq 0$. Since $\dim D \leq \dim \mathcal{E}$, also the condition $\operatorname{Ind}(A_{\mathcal{D}_{\max}}) \geq 0$ is necessary. Of course these two conditions together imply that there is a subspace $D \subset \mathcal{E}$ such that with $\mathcal{D} = D + \mathcal{D}_{\min}$ we have $\operatorname{Ind}(A_{\mathcal{D}}) = 0$. For this reason we assume henceforth that

$$\operatorname{Ind}(A_{\mathcal{D}_{\min}}) \leq 0 \leq \operatorname{Ind}(A_{\mathcal{D}_{\max}}). \tag{6.1}$$

Thus generally we will be interested in the extensions of (5.1) with domain $\mathcal{D} = D + \mathcal{D}_{\min}$ where $D \in \operatorname{Gr}_{d''}(\mathcal{E})$, $d'' = -\operatorname{Ind}(A_{\mathcal{D}_{\min}})$.

Suppose $A_{\mathcal{D}} - \lambda_0$ is invertible. Then, since the inclusion $\mathcal{D} \hookrightarrow x^{\mu}L^2_b(\mathcal{M}; E)$ is compact (because of (5.5)), the spectrum of A is a discrete subset of $\mathbb{C}$. It is convenient to classify the spectrum as follows (see [15]). Let

$$\text{bg-spec}(A) = \bigcap_{\mathcal{D} = D + \mathcal{D}_{\min}} \operatorname{spec}(A_{\mathcal{D}})$$

where D runs over all elements of $\operatorname{Gr}(\mathcal{E})$. The set bg-spec$(A)$ is the background spectrum of A; it is the subset of $\mathbb{C}$ present in all closed extensions of (5.1). It is easy to verify that $\lambda \in \text{bg-spec}(A)$ if and only if $A_{\mathcal{D}_{\min}} - \lambda$ is not injective or $A_{\mathcal{D}_{\max}} - \lambda$ is not surjective. We also define

$$\text{bg-res}(A) = \mathbb{C} \backslash \text{bg-spec}(A).$$

With this we can split spec$(A_{\mathcal{D}})$ as

$$\operatorname{spec}(A_{\mathcal{D}}) = \text{bg-spec}(A) \cup \big(\text{bg-res}(A) \cap \operatorname{spec}(A_{\mathcal{D}})\big).$$

The part of the spectrum of $A_{\mathcal{D}}$ in bg-res(A) can be analyzed further.

For $\lambda \in \text{bg-res}(A)$ define

$$\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda).$$

The dimension of $\mathcal{K}_\lambda$ is independent of λ, equal to $d' = \operatorname{Ind}(A_{\mathcal{D}_{\max}})$. These vector spaces form a complex vector bundle over bg-res(A).

Let $\mathcal{D} = D + \mathcal{D}_{\min}$ be some domain. If $\lambda \in$ bg-res(A), then $\lambda \in \operatorname{spec}(A_{\mathcal{D}})$ if and only if $A_{\mathcal{D}} - \lambda$ has nontrivial kernel K. Of course $K = \mathcal{K}_\lambda \cap \mathcal{D}$, so

$$\text{bg-res}(A) \cap \operatorname{spec}(A_{\mathcal{D}}) = \{\lambda \in \text{bg-res}(A) : \mathcal{D} \cap \mathcal{K}_\lambda \neq 0\}.$$

Let $\pi_{\max} : \mathcal{D}_{\max} \to \mathcal{D}_{\max}$ be the orthogonal projection on $\mathcal{E}$. The restriction of $\pi_{\max}$ to $\mathcal{K}_\lambda$ is injective. This is elementary: if $\phi \in \mathcal{K}_\lambda$ and $\pi_{\max}(\phi) = 0$, then $\phi \in \mathcal{D}_{\min}$; since $A_{\mathcal{D}_{\min}} - \lambda$ is injective (because $\lambda \in$ bg-res(A)), $\phi = 0$. Letting $K_\lambda = \pi_{\max}\mathcal{K}_\lambda$ one gets from this that

$$\lambda \in \text{bg-res}(A) \cap \operatorname{spec}(A_{\mathcal{D}}) \iff K_\lambda \cap D \neq 0. \tag{6.2}$$

Note that $K_\lambda \cap D = 0$ implies $K_\lambda \oplus D = \mathcal{E}$ because $d' + d'' = \dim \mathcal{E}$.

Given $D \in \operatorname{Gr}_{d''}(\mathcal{E})$, let

$$\mathscr{V}_D = \{K \in \operatorname{Gr}_{d'}(\mathcal{E}) : D \cap K \neq 0\}. \tag{6.3}$$

The set $\operatorname{Gr}_{d'}(\mathcal{E})$ is a compact complex manifold and $\mathscr{V}_D$ is a complex subvariety of complex codimension 1 (locally given as the set of zeros of a determinant). Of course we also have the reverse variety: if $K \in \operatorname{Gr}_{d'}(\mathcal{E})$, then there is an associated variety $\mathscr{V}_K \subset \operatorname{Gr}_{d''}(\mathcal{E})$. Using this terminology we may rephrase (6.2) as

$$\text{bg-res}(A) \cap \operatorname{spec}(A_{\mathcal{D}}) = \{\lambda : K_\lambda \in \mathscr{V}_D\}. \tag{6.4}$$

In other words, in terms of the map

$$\text{bg-res}(A) \ni \lambda \xrightarrow{K} K_\lambda \in \operatorname{Gr}_{d'}(\mathcal{E}),$$

a holomorphic map, we have bg-res$(A) \cap \operatorname{spec}(A_{\mathcal{D}}) = K^{-1}(\mathscr{V}_D)$.

Again all objects described so far have their counterparts in the case of $A_\wedge$, for example $\mathcal{K}_{\wedge,\lambda} = \ker(A_{\mathcal{D}_{\wedge,\max}} - \lambda)$ when $\lambda \in$ bg-res$(A_\wedge)$. The homogeneity property (4.4) implies that bg-spec$(A_\wedge)$ consists of a union of closed rays and sectors issuing from the origin. Namely, if $\lambda_0 \in \mathbb{C}\backslash 0$ and $A_\wedge - \lambda_0$ is not injective on $\mathcal{D}_{\wedge,\min}$ or not surjective on $\mathcal{D}_{\wedge,\max}$, then the same is true for $\kappa_\varrho^{-1}(A - \lambda_0)\kappa_\varrho$ since

$$\kappa_\varrho^{-1}(A - \lambda_0)\kappa_\varrho = \varrho^m(A_\wedge - \varrho^{-m}\lambda_0). \tag{6.5}$$

Thus bg-res$(A_\wedge)$ is a union of open sectors with vertex at the origin. It is immediate that

$$\begin{gathered}\forall D_\wedge \in \mathcal{E}_\wedge\text{: } A_\wedge - \lambda \text{ with domain } \mathcal{D}_\wedge = D_\wedge + \mathcal{D}_{\wedge,\min} \text{ is Fredholm and} \\ \operatorname{Ind}(A_{\wedge,\mathcal{D}_\wedge} - \lambda) \text{ is constant on each component of bg-res}(A_\wedge).\end{gathered} \tag{6.6}$$

Note also that (6.5) implies

$$\kappa_\varrho \mathcal{K}_{\wedge,\lambda} = \mathcal{K}_{\wedge,\varrho^m\lambda} \tag{6.7}$$

Example 6.8. Continuing with Example 5.10, we have

$$\text{bg-spec}(A) = \{\lambda \in \mathbb{C} : \exists u \in C^\infty(\mathcal{M}_0),\ u \neq 0, u(p_0) = 0,\ \Delta u = \lambda u\}.$$

This is a subset of spectrum of Δ on $\mathcal{M}_0$. An interesting description of this set is as the set of eigenvalues of Δ for which there is a mode with p_0 in its nodal set.

And since $A_\wedge$ is the positive Euclidean Laplacian on $\mathbb{R}^2\backslash 0$,

$$\text{bg-spec}(A_\wedge) = [0,\infty) \subset \mathbb{C}$$

Namely, if λ is real and nonnegative, then $A_\wedge - \lambda$, while injective on $\mathcal{D}_{\wedge,\min}$, is not surjective on $\mathcal{D}_{\wedge,\max}$, in fact its range is dense but not closed.

7. Rays of minimal growth for elliptic cone operators

Following Seeley's program [34], the first step in determining the meromorphic structure of the zeta function of an elliptic cone operator is to determine the existence of rays of minimal growth.

Theorem 7.1 ([14, Theorem 6.36]). *Let $A \in x^{-m}\operatorname{Diff}^m_b(\mathcal{M},E)$, let $\Lambda \subset \mathbb{C}$ be a closed sector, and let $\mathcal{D} = D + \mathcal{D}_{\min}$ be the domain of a closed extension of* (5.1)*. Suppose that ${}^c\boldsymbol{\sigma}(A) - \lambda$ is invertible on $({}^cT^*\mathcal{M}\backslash 0)\times\Lambda$ and that Λ is a sector of minimal growth for $A_\wedge$ with domain $\theta(D)+\mathcal{D}_{\wedge,\min}$. Then Λ is a sector of minimal growth for $A_{\mathcal{D}}$.*

The proof in [14], relies on constructing first a left inverse for $A-\lambda$ on the minimal domain of A (always $\lambda \in \Lambda$, $|\lambda|$ large), and then correcting additively to get an inverse for $A_{\mathcal{D}} - \lambda$. We will describe some aspects of this in the case of $A_{\wedge,\mathcal{D}_\wedge} - \lambda$. In particular, we will discuss the issue of exactly when is Λ a sector of minimal growth for $A_\wedge$ with domain $\mathcal{D}_\wedge = \theta(D) + \mathcal{D}_{\wedge,\min}$.

Suppose for the time being that Λ is a closed sector such that $\Lambda\backslash 0$ is contained in bg-res$(A_\wedge)$. For example, if Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$, then there is $R \geq 0$ such that $A_{\wedge,\mathcal{D}_\wedge} - \lambda$ is invertible for $\lambda \in \Lambda_R = \{\lambda \in \Lambda : |\lambda| > R\}$. This implies that $\Lambda_R \subset$ bg-res$(A_\wedge)$, therefore that

$$\Lambda\backslash 0 \subset \text{bg-res}(A_\wedge), \tag{7.2}$$

because the latter set is a union of open sectors.

Because of (6.6), $A_\wedge - \lambda$ has a left inverse on $\mathcal{D}_{\wedge,\min}$ for every $\lambda \in \Lambda\backslash 0$. In fact there is a left inverse with range $\mathcal{D}_{\wedge,\min}$ and kernel equal to the orthogonal space (in L^2_b) of $(A_\wedge - \lambda)(\mathcal{D}_{\wedge,\min})$ which we will denote $B_{\wedge,\min}(\lambda)$. Note that

$$B_{\wedge,\min}(\lambda)(A_\wedge - \lambda) : \mathcal{D}_{\wedge,\max} \to \mathcal{D}_{\wedge,\max} \text{ is a projection onto } \mathcal{D}_{\wedge,\min}. \tag{7.3}$$

Also due to (6.6), we have that $A_\wedge - \lambda$ with domain $\mathcal{D}_{\wedge,\max}$ has a right inverse for each $\lambda \in \Lambda_0$ with range equal to $\mathcal{K}^\perp_{\wedge,\lambda} \cap \mathcal{D}_{\wedge,\max}$. Here $\mathcal{K}^\perp_{\wedge,\lambda}$ is the orthogonal space, also in L^2_b of $\mathcal{K}_{\wedge,\lambda}$. We will denote this specific right inverse by $B_{\wedge,\max}(\lambda)$.

The homogeneity property (4.4) of $A_\wedge$ and the fact that κ_ϱ preserves L^2 orthogonality of spaces (unitarity is not the relevant reason) imply that

$$\kappa_\varrho^{-1} B_{\wedge,\min}(\varrho^m\lambda)\kappa_\varrho = \varrho^{-m} B_{\wedge,\min}(\lambda),\ \kappa_\varrho^{-1} B_{\wedge,\max}(\varrho^m\lambda)\kappa_\varrho = \varrho^{-m} B_{\wedge,\max}(\lambda) \tag{7.4}$$

It is automatic that

$$B_{\wedge,\min}(\lambda), B_{\wedge,\max}(\lambda) : x^\mu L^2_b(\mathcal{M};E) \to \mathcal{D}_{\wedge,\max}$$

are continuous maps depending smoothly on λ. Further, their homogeneity implies decay of their norm as operators $x^\mu L^2_b(\mathcal{M};E) \to x^\mu L^2_b(\mathcal{M};E)$. For instance,

$$\begin{aligned}\|B_{\wedge,\min}(\varrho^m\lambda)u\| &= \|\varrho^{-m}\kappa_\varrho B_{\wedge,\min}(\lambda)\kappa_\varrho^{-1}u\| = \varrho^{-m}\|B_{\wedge,\min}(\lambda)\kappa_\varrho^{-1}u\| \\ &\le \varrho^{-m}\|B_{\wedge,\min}(\lambda)\|\,\|\kappa_\varrho^{-1}(\lambda)u\| = \varrho^{-m}\|B_{\wedge,\min}(\lambda)\|\,\|u\|,\end{aligned}$$

so $\|B_{\wedge,\min}(\lambda)\| \le |\lambda|^{-1}\|B_{\wedge,\min}(\lambda/|\lambda|)\|$ when $\lambda \in \Lambda\backslash 0$.

Now pick a domain $\mathcal{D}_\wedge = D_\wedge + \mathcal{D}_{\wedge,\min}$. Assume that for $\lambda \in \Lambda\backslash 0$ we have $\operatorname{Ind}(A_{\mathcal{D}_\wedge} - \lambda) = 0$; this is the case if we already know that Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$. Then, as discussed in the previous section in the case of A, and keeping in mind (7.2), the part of the resolvent set of $A_{\wedge,\mathcal{D}_\wedge}$ in $\Lambda\backslash 0$ is

$$\mathscr{R} = \operatorname{res}(A_{\wedge,\mathcal{D}_\wedge}) \cap (\Lambda\backslash 0) = \{\lambda \in \Lambda\backslash 0 : \mathcal{K}_{\wedge,\lambda} \cap \mathcal{D}_\wedge = 0\}. \tag{7.5}$$

Recall that $\mathcal{K}_{\wedge,\lambda} = \ker(A_{\wedge,\mathcal{D}_{\wedge,\max}} - \lambda)$. Because of the analyticity in the parameter λ of $(A_{\wedge,\mathcal{D}_{\wedge,\max}} - \lambda)$, the set $\mathscr{R}$ is the complement of a closed discrete set in $\Lambda\backslash 0$.

If $\lambda \in \mathscr{R}$, then

$$\mathcal{K}_{\wedge,\lambda} + \mathcal{D}_\wedge = \mathcal{D}_{\wedge,\max}$$

as a direct sum. For such λ define

$$\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge} = \text{projection on } \mathcal{K}_{\wedge,\lambda} \text{ along } \mathcal{D}_\wedge.$$

Now,

$$(A_\wedge - \lambda)B_{\wedge,\max}(\lambda) = I$$

and of course

$$(A_\wedge - \lambda)\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge} B_{\wedge,\max}(\lambda) = 0$$

so

$$(A_\wedge - \lambda)(I - \pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge})B_{\wedge,\max}(\lambda) = I.$$

The operator

$$B_{\mathcal{D}_\wedge}(\lambda) = (I - \pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge})B_{\wedge,\max}(\lambda) \tag{7.6}$$

obviously maps into $\mathcal{D}_\wedge$ an thus

$$B_{\mathcal{D}_\wedge}(\lambda) : x^\mu L^2_b(N_+\partial\mathcal{M};E_\wedge) \to \mathcal{D}_\wedge$$

is a right inverse for

$$(A_\wedge - \lambda) : \mathcal{D}_\wedge \to x^\mu L^2_b(N_+\partial\mathcal{M};E_\wedge).$$

We will let the reader show that $B_{\mathcal{D}_\wedge}(\lambda)$ is also surjective, so that we may conclude that $B_{\mathcal{D}_\wedge}(\lambda)$ is the resolvent (in $\mathscr{R}$) of the unbounded operator

$$A_\wedge : \mathcal{D}_\wedge \subset x^\mu L^2_b(N_+\partial\mathcal{M};E_\wedge) \to x^\mu L^2_b(N_+\partial\mathcal{M};E_\wedge).$$

Another tautological formula for $B_{\mathcal{D}_\wedge}(\lambda)$ is

$$B_{\mathcal{D}_\wedge}(\lambda) = B_{\wedge,\min}(\lambda) + \big(I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\big)B_{\mathcal{D}_\wedge}(\lambda)$$

(just expand the right hand side and use that $(A_\wedge - \lambda)B_{\mathcal{D}_\wedge}(\lambda)$ is the identity on $x^\mu L^2_b(N_+\partial\mathcal{M}; E_\wedge)$). Replacing (7.6) in the right hand side of this formula and some more elementary algebraic manipulations (this time exploiting the fact that $(A_\wedge - \lambda)B_{\wedge,\max}(\lambda)$ is the identity on $x^\mu L^2_b(N_+\partial\mathcal{M}; E_\wedge)$) leads to

$$B_{\mathcal{D}_\wedge}(\lambda) = B_{\wedge,\max}(\lambda) - \big(I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\big)\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge} B_{\wedge,\max}(\lambda).$$

Because of (7.3), the operator $I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)$ is a projection with kernel $\mathcal{D}_{\wedge,\min}$. Using that $I - \pi_{\wedge,\max}$ is a projection on $\mathcal{D}_{\wedge,\min}$ we thus get

$$I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda) = \big(I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\big)\pi_{\wedge,\max}$$

The projection $\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge}$ also vanishes on $\mathcal{D}_{\wedge,\min}$, so

$$\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge} = \pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge}\pi_{\wedge,\max}$$

Therefore

$$B_{\mathcal{D}_\wedge}(\lambda) = B_{\wedge,\max}(\lambda) - \big(I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\big)\pi_{\wedge,\max}\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge}\pi_{\wedge,\max}B_{\wedge,\max}(\lambda).$$

We will now rewrite $\pi_{\wedge,\max}\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge}\pi_{\wedge,\max}$.

The subspace $K_{\wedge,\lambda} = \pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda}$ of $\mathcal{E}_\wedge$ is isomorphic to $\mathcal{K}_{\wedge,\lambda}$. The characterization (7.5) of $\mathscr{R}$ can also be given in terms of $K_{\wedge,\lambda}$ and $D_\wedge$, as

$$\mathscr{R} = \{\lambda \in \Lambda\backslash 0 : K_{\wedge,\lambda} \cap D_\wedge = 0\}. \tag{7.7}$$

Thus whenever $\lambda \in \mathscr{R}$, $\mathcal{E}_\wedge = K_{\wedge,\lambda} \oplus D_\wedge$. Let then $\pi_{K_\lambda,D_\lambda} : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$ be the projection on K_λ along $D_\wedge$. Then

$$\pi_{\wedge,\max}\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge}\pi_{\wedge,\max} = \pi_{K_{\wedge,\lambda},D_\wedge}\pi_{\wedge,\max}.$$

Indeed, suppose $u \in \mathcal{E}_\wedge$. Then

$$u = \phi + v, \quad \phi = \pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge}u \in \mathcal{K}_{\wedge,\lambda}, \quad v \in \mathcal{D}_\wedge.$$

Let $\phi_0 = \pi_{\wedge,\max}\phi$, $v_0 = \pi_{\wedge,\max}v$. both $\phi_1 = \phi - \phi_0$ and $v_1 = v - v_0$ belong to $\mathcal{D}_{\wedge,\min}$. Since $u \in \mathcal{E}_\wedge$, the formula $u = (\phi_0 + v_0) + (\phi_1 + v_1)$ gives $\phi_1 + v_1 = 0$. So

$$\pi_{\wedge\max}\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}_\wedge}u = \phi_0 = \pi_{K_{\wedge,\lambda},D_\wedge}u.$$

Thus

$$B_{\mathcal{D}_\wedge}(\lambda) = B_{\wedge,\max}(\lambda) - \big(I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\big)\pi_{K_{\wedge,\lambda},D_\wedge}\pi_{\wedge,\max}B_{\wedge,\max}(\lambda). \tag{7.8}$$

We now discuss necessary and sufficient conditions for Λ to be sector of minimal growth for $A_\wedge$ with domain $D_\wedge$. This pertains to two issues: existence of the resolvent for all sufficiently large $\lambda \in \Lambda$, and decay estimates for the norm of the resolvent. To get a hold on these issues, we fix $\lambda_0 \in \Lambda\backslash 0$ and analyze $B_{\mathcal{D}_\wedge}(\lambda)$ as λ moves to ∞ along the ray through λ_0. We do this by setting $\lambda = \varrho^m\lambda_0$ in (7.8) and analyzing the expressions that result from using (7.4) as $\varrho \to \infty$.

The issue of existence of the inverse of $A_{\wedge,\mathcal{D}_\wedge} - \lambda$ for $\lambda = \varrho^m\lambda_0$ for ϱ large is by now easily understood. The condition $K_{\wedge,\varrho^m\lambda_0} \cap D = 0$ is both necessary and sufficient in order for $A_{\wedge,\mathcal{D}_\wedge} - \varrho^m\lambda_0$ to be invertible. Since $\boldsymbol{\kappa}_\varrho : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$ is an isomorphism, this condition is equivalent to $\boldsymbol{\kappa}_\varrho^{-1}K_{\wedge,\varrho^m\lambda_0}\cap\boldsymbol{\kappa}_\varrho^{-1}D=0$, that is, $K_{\wedge,\lambda_0}\cap\boldsymbol{\kappa}_\varrho^{-1}D=0$ (see (6.7)). Therefore, the requirement is

$$\boldsymbol{\kappa}_\varrho^{-1}D_\wedge\notin\mathscr{V}_{K_{\wedge,\lambda_0}}\text{ for all sufficiently large }\varrho.$$

We assume this henceforth.

As for the issue of decay, straight from (7.4) we get

$$B_{\wedge,\max}(\varrho^m\lambda_0)=\varrho^{-m}\kappa_\varrho B_{\wedge,\max}(\lambda_0)\kappa_\varrho^{-1}.$$

Further,

$$\begin{aligned}I-B_{\wedge,\min}(\varrho^m\lambda_0)(A_\wedge-\varrho^m\lambda_0)&=I-\varrho^{-m}\kappa_\varrho B_{\wedge,\min}(\lambda_0)\kappa_\varrho^{-1}(A_\wedge-\varrho^m\lambda_0)\\&=I-\varrho^{-m}\kappa_\varrho B_{\wedge,\min}(\lambda_0)(A_\wedge-\lambda_0)\kappa_\varrho^{-1}\\&=\kappa_\varrho\big(I-B_{\wedge,\min}(\lambda_0)(A_\wedge-\lambda_0)\big)\kappa_\varrho^{-1}\end{aligned}$$

in which (6.5) was used in the second equality. Altogether this gives

$$\begin{aligned}B_{\mathcal{D}_\wedge}(\varrho^m\lambda_0)=\varrho^{-m}\kappa_\varrho\big\{&B_{\wedge,\max}(\lambda_0)\\&-\big(I-B_{\wedge,\min}(\lambda_0)(A_\wedge-\lambda_0)\big)\kappa_\varrho^{-1}\pi_{K_{\wedge,\varrho^m\lambda_0},D_\wedge}\pi_{\wedge,\max}\kappa_\varrho B_{\wedge,\max}(\lambda_0)\big\}\kappa_\varrho^{-1}.\end{aligned}$$

In the second term we replace the factor

$$\kappa_\varrho^{-1}\pi_{K_{\wedge,\varrho^m\lambda_0},D_\wedge}\pi_{\wedge,\max}\kappa_\varrho$$

by

$$\pi_{\wedge,\max}\kappa_\varrho^{-1}\pi_{K_{\wedge,\varrho^m\lambda_0},D_\wedge}\pi_{\wedge,\max}\kappa_\varrho\pi_{\wedge,\max}$$

taking advantage again of the fact that $I-B_{\wedge,\min}(\lambda_0)(A_\wedge-\lambda_0)$ vanishes on $\mathcal{D}_{\wedge,\min}$, and also that $\pi_{\wedge,\max}\kappa_\varrho=\pi_{\wedge,\max}\kappa_\varrho\pi_{\wedge,\max}$ because κ_ϱ preserves $\mathcal{D}_{\wedge,\min}$. With the notation of (5.6),

$$\pi_{\wedge,\max}\kappa_\varrho^{-1}\pi_{K_{\wedge,\varrho^m\lambda_0},D_\wedge}\pi_{\wedge,\max}\kappa_\varrho|_{\mathcal{E}_\wedge}=\boldsymbol{\kappa}_\varrho^{-1}\pi_{K_{\wedge,\varrho^m\lambda_0},D_\wedge}\boldsymbol{\kappa}_\varrho$$

Using (6.7) one easily obtains $\boldsymbol{\kappa}_\varrho^{-1}\pi_{K_{\wedge,\varrho^m\lambda_0},D_\wedge}\boldsymbol{\kappa}_\varrho=\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}$. So, finally we arrive at

$$\begin{aligned}B_{\mathcal{D}_\wedge}(\varrho^m\lambda_0)=\varrho^{-m}\kappa_\varrho\big\{&B_{\wedge,\max}(\lambda_0)\\&-\big(I-B_{\wedge,\min}(\lambda_0)(A_\wedge-\lambda_0)\big)\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}\pi_{\wedge,\max}B_{\wedge,\max}(\lambda_0)\big\}\kappa_\varrho^{-1}.\end{aligned}\tag{7.9}$$

Note that the norm of $B_{\mathcal{D}_\wedge}(\varrho^m\lambda_0):x^\mu L^2(N_+\partial\mathcal{M},E_\wedge)\to L^2(N_+\partial\mathcal{M},E_\wedge)$ is bounded by ϱ^m times the norm of

$$B_{\wedge,\max}(\lambda_0)-\big(I-B_{\wedge,\min}(\lambda_0)(A_\wedge-\lambda_0)\big)\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}\pi_{\wedge,\max}B_{\wedge,\max}(\lambda_0)$$

The fact that the only dependence of ϱ is in $\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}$ lends credence to what we showed in [15, Theorem 8.3], namely that the ray through λ_0 is a ray of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$ if and only if the norm of $\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}$ is bounded as $\varrho\to\infty$, and that this norm is bounded if

> there is a neighborhood U of $\mathscr{V}_{K_{\lambda_0}}$ and $\varrho_0>0$ such that $\varrho>\varrho_0\implies\boldsymbol{\kappa}_\varrho^{-1}D_\wedge\notin U$.

Completing this, we showed in [16, Theorem 4.3] that this last displayed condition is also sufficient for the boundedness of $\|\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}\|$ as $\varrho\to\infty$. This condition is equivalent to the statement that the limit set

$$\Omega^-(D_\wedge)=\{D'_\wedge\in \mathrm{Gr}_{d''}(\mathcal{E}_\wedge):\exists\{\varrho_k\}_{k=1}^\infty,\ \lim_{k\to\infty}\varrho_k=\infty,\ \lim_{k\to\infty}\boldsymbol{\kappa}_{\varrho_k}^{-1}D_\wedge=D'_\wedge\}$$

is disjoint from $\mathscr{V}_{K_{\wedge,\lambda_0}}$; this is how the condition is stated in the theorem just cited. The number d'' is the negative of the index of $A_{\wedge,\min}-\lambda$ for $\lambda\in\Lambda_0$ (see (6.6)).

If one insists on having a condition on the sector Λ, one can take the arc $C=\{\lambda\in\Lambda:|\lambda|=|\lambda_0|\}$ and define

$$\mathscr{V}_{K_{\wedge,C}}=\bigcup_{\lambda\in C}\mathscr{V}_{K_{\wedge,\lambda}}.$$

Then $\mathscr{V}_{K_{\wedge,C}}$ is a closed subset of $\mathrm{Gr}_{d''}(\mathcal{E}_\wedge)$, and Λ is a sector of minimal growth if and only if

$$\Omega^-(D_\wedge)\cap\mathscr{V}_{K_{\wedge,C}}=\emptyset. \tag{7.10}$$

The set $\Omega^-(D_\wedge)$ is in some sense the principal symbol of $\mathcal{D}_\wedge$ (or of $\mathcal{D}=D+\mathcal{D}_{\min}$ if $D_\wedge=\theta(D)$), and the condition (7.10) is like an ellipticity condition.

Obviously:

$$\text{if } \mathcal{D}_\wedge \text{ is stationary (see (5.9)) then } \pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge} \text{ is independent of } \varrho. \tag{7.11}$$

This property results in a considerable simplification of the analysis of the asymptotics of the resolvent of $A_{\mathcal{D}}-\lambda$.

Example 7.12. Continuing with Example 5.10 (see also Example 6.8), assume in all formulas that follow that $\lambda\notin[0,\infty)$. Recall that $\text{bg-spec}(A_\wedge)=[0,\infty)\subset\mathbb{C}$.

Let

$$\phi(\lambda)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{e^{ix\cdot\xi}}{|\xi|^2-\lambda}\,d\xi,\quad \lambda\notin[0,\infty).$$

Then $\phi(\lambda)\in L^2(\mathbb{R}^2)$. Also

$$(\Delta\phi(\lambda))\widehat{\ }(\xi)=1+\frac{\lambda}{|\xi|^2-\lambda},$$

which means that

$$\Delta\phi(\lambda)=\delta_0+\lambda\phi(\lambda).$$

so by restriction to $\mathbb{R}^2\backslash 0$,

$$A_\wedge\phi(\lambda)=\lambda\phi(\lambda)$$

Thus $\phi(\lambda)\in\mathcal{D}_{\wedge,\max}$ and spans $\mathcal{K}_{\wedge,\lambda}$.

Let $\mathfrak{p}_\lambda:L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ be the orthogonal projection on $\mathcal{K}_{\wedge,\lambda}$:

$$\mathfrak{p}(f)=4\pi|\lambda|(f,\phi(\lambda))\phi(\lambda).$$

Let $B_F(\lambda)$ be the inverse of $(\Delta-\lambda):H^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$. The reason for the subindex F is that the space $\mathcal{D}_{\wedge,F}=H^2(\mathbb{R}^2)$ is the Friedrichs extension of

$$\Delta:C_c^\infty(\mathbb{R}^2\backslash 0)\subset L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2),$$

so $B_F(\lambda)$ is actually the resolvent of $A_{\wedge,\mathcal{D}_{\wedge,F}}$. We have

$$B_{\wedge,\max}(\lambda) = (1-\mathfrak{p}_\lambda)B_F(\lambda).$$

Indeed, this $B_{\wedge,\max}(\lambda)$ is the right inverse of $(A_\wedge - \lambda)$ with range orthogonal (in L^2) to $\mathcal{K}_{\wedge,\lambda}$.

The Hilbert space adjoint of $A_{\wedge,\mathcal{D}_{\wedge,\min}}$ is $A_{\wedge,\mathcal{D}_{\wedge,\max}}$ so the range of $A_{\wedge,\mathcal{D}_{\wedge,\min}} - \lambda$ is the orthogonal space (in L^2) of $\ker(A_{\wedge,\mathcal{D}_{\wedge,\max}} - \lambda)$ which of course is $\mathcal{K}_{\wedge,\lambda}$. So

$$B_{\wedge,\min}(\lambda) = B_F(\lambda)(I-\mathfrak{p}_\lambda).$$

Let $D_\wedge = \operatorname{span}\{\alpha u_1 + \beta u_2\}$ with fixed $(\alpha,\beta)\neq 0$, and let $\mathcal{D}_\wedge = D_\wedge + \mathcal{D}_{\wedge,\min}$. To compute the spectrum of $A_{\wedge,\mathcal{D}_\wedge}$ we need to compute the spaces $K_{\wedge,\lambda}$, which here just means to compute $\pi_{\wedge,\max}\phi(\lambda)$. In terms of the basis u_1, u_2 of $\mathcal{E}_\wedge$ discussed in Example 5.10,

$$\pi_{\wedge,\max}\phi(\lambda) = -\frac{2}{\pi}\log(-\lambda)u_1 + u_2. \tag{7.13}$$

The log is the principal branch of the logarithm with cut $(-\infty,0]$. The part of the spectrum of $A_{\wedge,\mathcal{D}_\wedge}$ in bg-res$(A_\wedge)$ is

$$\{\lambda\in \text{bg-res}(A_\wedge) : \alpha u_1 + \beta u_2 \text{ and } -\frac{2}{\pi}\log(-\lambda)u_1 + u_2 \text{ are linearly dependent}\}.$$

In other words, $\lambda\in\operatorname{spec}(A_{\wedge,\mathcal{D}_\wedge})\cap\text{bg-res}(A_\wedge)$ if and only if the determinant, $f(\lambda,\alpha,\beta) = \alpha + (2\beta/\pi)\log(-\lambda)$, of

$$\begin{bmatrix} \alpha & -\frac{2}{\pi}\log(-\lambda) \\ \beta & 1 \end{bmatrix}$$

is zero. If $\beta = 0$ then $f(\lambda,\alpha,\beta)$ has no zeros, corresponding to the fact that the spectrum of the Friedrichs extension is exactly $[0,\infty)$. If $\beta\neq 0$ and $\Im(\alpha/\beta)\notin 2+4\mathbb{Z}$, then there is exactly one zero, at

$$\lambda = -e^{-\pi\alpha/2\beta}.$$

It follows that in any closed sector Λ with $\Lambda\backslash 0\subset\text{bg-res}(A_\wedge)$ there is at most one eigenvalue of $A_{\wedge,\mathcal{D}_\wedge}$.

We have already shown that $\boldsymbol{\kappa}_\varrho D_\wedge \to D_{\wedge,F}$ as ϱ tends to 0, equivalently,

$$\boldsymbol{\kappa}_\varrho^{-1} D_\wedge \to D_\wedge \quad \text{as } \varrho\to\infty.$$

So $\Omega^-(D_\wedge) = D_{\wedge,F}$ and thus, for any $\lambda_0\in\text{bg-res}(A_\wedge)$,

$$\Omega^-(D_\wedge)\cap\mathcal{K}_{\wedge,\lambda_0} = \emptyset$$

since $\pi_{\wedge,\max}\phi(\lambda)$ is never an element of $D_{\wedge,F}$.

8. Asymptotics

Suppose that it has been determined by way of [14, Theorem 6.36] (quoted above as Theorem 7.1) that the closed $\Lambda \subset \mathbb{C}$ is a sector of minimal growth for a given extension $A_{\mathcal{D}}$ of our elliptic cone operator. In [17] we discussed the asymptotics in the case of a stationary domain, and in [18] we were able to complete our results to general general domains. We will discuss some aspects of the latter result below.

In the case of stationary domains (see (5.9)) we have:

Theorem 8.1. [17, Theorem 1.1] *Suppose $\mathcal{D}$ is stationary in the sense of* (5.9). *Then, for any $\varphi \in C^\infty(\mathcal{M};\mathrm{End}(E))$ and $\ell \in \mathbb{N}$ with $m\ell > n = \dim \mathcal{M}$,*

$$\mathrm{Tr}\big(\varphi(A_{\mathcal{D}}-\lambda)^{-\ell}\big) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{m_j} \alpha_{jk}\lambda^{\frac{n-j}{m}-\ell}\log^k\lambda \quad \text{as } |\lambda|\to\infty,$$

with a suitable branch of the logarithm, with constants $\alpha_{jk} \in \mathbb{C}$. The numbers m_j vanish for $j < n$, and $m_n \le 1$. In general, the α_{jk} depend on φ, A, $\mathcal{D}$, and ℓ, but the coefficients α_{jk} for $j < n$ and $\alpha_{n,1}$ do not depend on $\mathcal{D}$. If both A and φ have coefficients independent of x near $\partial\mathcal{M}$, then $m_j = 0$ for all $j > n$.

The asymptotics of the trace of the resolvent, which ultimately determines the behavior of the ζ function, depends fundamentally on the asymptotics of the resolvent of $A_{\wedge,\mathcal{D}_\wedge}$, which by virtue of (7.9) depends in an essential manner on the asymptotics of $\pi_{\mathcal{K}_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}$. If the domain is stationary then $\pi_{\mathcal{K}_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}$ has a simple asymptotics (indeed, it is homogeneous of degree 0 in ϱ, see (7.11)). On the other hand, if $\mathcal{D}$ is not stationary, its asymptotics, therefore that of the resolvent, can be rather complicated:

Theorem 8.2. [18, Theorem 1.4] *For any $\varphi \in C^\infty(\mathcal{M};\mathrm{End}(E))$ and $\ell \in \mathbb{N}$ with $m\ell > n$,*

$$\mathrm{Tr}\big(\varphi(A_{\mathcal{D}}-\lambda)^{-\ell}\big) \sim \sum_{j=0}^{\infty} r_j(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)\lambda^{\nu_j/m} \quad \text{as } |\lambda|\to\infty,$$

where each r_j is a rational function in $N+1$ variables, $N \in \mathbb{N}_0$, with real numbers μ_k, $k = 1,\dots,N$, and $\nu_j > \nu_{j+1} \to -\infty$ as $j \to \infty$. We have $r_j = p_j/q_j$ with $p_j,\ q_j \in \mathbb{C}[z_1,\dots,z_{N+1}]$ such that $q_j(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)$ is uniformly bounded away from zero for large λ.

The asymptotic behavior of $\pi_{\mathcal{K}_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}$ is rooted in the behavior of the curve $\varrho \mapsto \boldsymbol{\kappa}_\varrho^{-1}D_\wedge$ in $\mathrm{Gr}_{d''}(\mathcal{E}_\wedge)$. We gain an understanding of this by analyzing the infinitesimal generator of the action

$$\varrho \mapsto \boldsymbol{\kappa}_\varrho : \mathcal{E}_\wedge \to \mathcal{E}_\wedge.$$

Since this is a one-parameter group of isomorphisms, there is a linear operator $\mathfrak{a} : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$ such that $\boldsymbol{\kappa}_\varrho = e^{-\log\varrho\,\mathfrak{a}}$ (the choice of sign for $\mathfrak{a}$ is a matter of convenience; we are interested in letting ϱ tend to ∞ in $\boldsymbol{\kappa}_\varrho^{-1}D_\wedge$, so in fact we are looking at $e^{\log \varrho \mathfrak{a}}$). The precise determination of $\mathfrak{a}$ is best done using the Mellin transform.

Fix some defining function for $\partial\mathcal{M}$, let $P_\wedge = x_\wedge^m A_\wedge$ (an elliptic b-operator with respect to the boundary of $N_+\partial\mathcal{M}$; the latter is trivialized by the choice of x as $\partial\mathcal{M} \times [0,\infty)$). Let $\widehat{P}_\wedge(\sigma)$ be the indicial family of $P_\wedge$. Let

$$\Sigma = \{\sigma \in \operatorname{spec}_b(P_\wedge) : -\mu - m < \Im\sigma < -\mu\},$$

let $\mathfrak{Mero}_\Sigma$ be the space of meromorphic functions on $\mathbb{C}$ with poles in Σ to the space of smooth sections of $E_{\partial\mathcal{M}} \to \partial\mathcal{M}$ and let $\mathfrak{Holo}$ be the subspace consisting of entire functions. Then $\widehat{P}$ induces maps

$$\widehat{P} : \mathfrak{Mero}_\Sigma \to \mathfrak{Mero}_\Sigma, \quad \widehat{P} : \mathfrak{Holo} \to \mathfrak{Holo}$$

which in turn give a map

$$\widehat{P} : \mathfrak{Mero}_\Sigma/\mathfrak{Holo} \to \mathfrak{Mero}_\Sigma/\mathfrak{Holo}.$$

Then $\mathcal{E}_\wedge$ is canonically isomorphic to the kernel, $\widehat{\mathcal{E}}_\wedge$, of this map. Namely, if $u \in \mathcal{E}_\wedge$ and $\omega : N_+\partial\mathcal{M} \to \mathbb{R}$ is smooth compactly supported, equal 1 in a neighborhood of the zero section, then

$$\mathscr{M}(u)(y,\sigma) = \int_0^\infty x_\wedge^{-i\sigma} u(x_\wedge, y)\omega(x_\wedge, y)\frac{dx_\wedge}{x_\wedge}$$

is holomorphic for $\Im\sigma \geq -\mu$, meromorphic in $\Im\sigma > -\mu - m$ with poles in Σ, and $\widehat{P}\mathscr{M}(u)$ is holomorphic in $\Im\sigma > -m-\mu$. Taking the singular parts of $\mathscr{M}(u)$ at the points of Σ gives an element $\mathrm{s}_\Sigma\mathscr{M}(u) \in \mathfrak{Mero}_\Sigma$ such that $\widehat{P}\mathrm{s}_\Sigma\mathscr{M}(u)$ is entire. This gives a map

$$\mathcal{E}_\wedge \ni u \mapsto [\mathrm{s}_\Sigma\mathscr{M}(u)] \in \widehat{\mathcal{E}}_\wedge,$$

where [ ] means class in $\mathfrak{Mero}_\Sigma$ modulo $\mathfrak{Holo}$. This map is the isomorphism mentioned above.

Now, $u \mapsto \mathscr{M}(u)$ conjugates κ_ϱ with multiplication by $\varrho^{i\sigma}$:

$$\begin{aligned}\mathscr{M}(\kappa_\varrho u)(y,\sigma) &= \int_0^\infty x_\wedge^{-i\sigma}\varrho^{-\mu}u(\varrho x_\wedge, y)\omega(x_\wedge, y)\frac{dx_\wedge}{x_\wedge}\\ &= \varrho^{i\sigma-\mu}\int_0^\infty x_\wedge^{-i\sigma}u(x_\wedge, y)\omega(x_\wedge/\varrho, y)\frac{dx_\wedge}{x_\wedge}\\ &\equiv \varrho^{i\sigma-\mu}\mathscr{M}(u) \mod \mathfrak{Holo}.\end{aligned}$$

Associated with each $\sigma_j \in \Sigma$, $j = 1,\dots,N$, there is the subspace $\widehat{\mathcal{E}}_{\wedge,\sigma_j} \subset \widehat{\mathcal{E}}_\wedge$ whose elements have representatives in $\mathfrak{Mero}_\Sigma$ with pole only at σ_j. We may view $\widehat{\mathcal{E}}_{\wedge,\sigma_j}$ directly as a space of singular parts of elements of $\mathfrak{Mero}_\Sigma$ with pole only at σ_j. If

$$\sum_{k=1}^{\nu}\frac{\phi_j}{(\sigma-\sigma_j)^k}$$

is an element of $\widehat{\mathcal{E}}_{\wedge,\sigma_j}$, then (using $\varrho^{i\sigma} = \varrho^{i\sigma_j}\varrho^{i(\sigma-\sigma_j)}$)

$$\varrho^{i\sigma-\mu}\sum_{k=1}^{\nu}\frac{\phi_k}{(\sigma-\sigma_j)^k} \equiv \varrho^{i\sigma_j-\mu}\sum_{\vartheta=1}^{\nu}\frac{1}{(\sigma-\sigma_j)^{\vartheta}}\sum_{k-\ell=\vartheta}\frac{i^\ell\log^\ell\varrho}{\ell!}\phi_k \mod \mathfrak{Holo}$$

Thus $\mathfrak{a}$, viewed on the Mellin transform side, has eigenvalues $-i\sigma_j+\mu$, $\sigma_j\in\Sigma$, and the generalized eigenspace corresponding to $-i\sigma_j+\mu$ is $\widehat{\mathcal{E}}_{\wedge,\sigma_j}$.

Each space $\widehat{\mathcal{E}}_{\wedge,\sigma_j}$ corresponds to a subspace $\mathcal{E}_{\wedge,\sigma_j}\subset\mathcal{E}_\wedge$. These spaces are, as we saw, the generalized eigenspaces of $\mathfrak{a}$. We may write $\mathfrak{a}=\mathfrak{a}_0+N$ where $\mathfrak{a}_0$ is diagonal and N is nilpotent. Let $\mathfrak{a}':\mathcal{E}_\wedge\to\mathcal{E}_\wedge$ be the operator which acts on each $\mathcal{E}_{\wedge,\sigma_j}$ by multiplication by $-i\Re\sigma_j$. The eigenvalues of $\mathfrak{a}_0-\mathfrak{a}'$ are the numbers $\Im\sigma_j+\mu$. Order the set of these numbers as $\mu_0>\mu_1>\dots$ (i.e. no repetitions). Since $\sigma_j\in\Sigma$, $-m<\mu_k<0$. Let

$$\tilde{\mathcal{E}}_{\wedge,\mu_k} = \bigoplus_{\substack{\sigma_j\in\Sigma\\ \Im\sigma_j+\mu=\mu_k}}\mathcal{E}_{\wedge,\sigma_j}.$$

Also let π_{σ_j} be the projection on $\mathcal{E}_{\wedge,\sigma_j}$ and N_{σ_j} the restriction of N to this space.

In [18, Sections 3 and 4] we showed the following. Given any subspace $D_\wedge\subset\mathcal{E}_\wedge$, there are functions $v_k:\mathbb{R}\to\mathcal{E}_\wedge$, $k=1,\dots,d''=\dim D_\wedge$, (perhaps not defined at $t=0$) such that

$$e^{t\mathfrak{a}}D_\wedge=\operatorname{span}\{v_k(t):k=1,\dots,d''\},\quad t\gg 0,$$

is of the form

$$v_k(t)=e^{t\mathfrak{a}'}g_k(t)+\sum_{\substack{\sigma\in\Sigma\\ \Im\sigma+\mu<\mu_k}}e^{t(-i\sigma+\mu-\mu_k)}\hat{p}_{k,\sigma}(t). \tag{8.3}$$

The $g_k(t)$ are polynomials in $1/t$ with values in $\tilde{\mathcal{E}}_{\wedge,\mu_k}$ and the collection of vectors

$$g_{\infty,k}=\lim_{t\to\infty}g_k(t)$$

is an independent set spanning a subspace $D_{\wedge,\infty}$, and

$$\hat{p}_{k,\sigma}(t)=e^{tN_\sigma}\pi_\sigma p_{k,\sigma}(t),\quad \sigma\in\Sigma,$$

where the $p_{k,\sigma}(t)$ are polynomials in t and $1/t$ with values in $\mathcal{E}_\wedge$.

The numbers $-i\sigma+\mu-\mu_k$ appearing in the exponents in the sum in (8.3) all have negative real part. It follows that $\|v_k(t)-e^{t\mathfrak{a}'}g_k(t)\|\to 0$ as $t\to\infty$. From this one concludes that the distance in $\mathrm{Gr}_{d''}(\mathcal{E}_\wedge)$ between $\boldsymbol{\kappa}_\varrho^{-1}D_\wedge=e^{\log\varrho\mathfrak{a}}D_\wedge$ and $e^{\log\varrho\mathfrak{a}'}D_{\wedge,\infty}$ tends to 0 as $t\to\infty$. Separately one can show (consult the details in [18]) that

$$\{e^{t\mathfrak{a}'}D_{\wedge,\infty}:t\in\mathbb{R}\}$$

is an embedded torus, which immediately proves that $\Omega^-(D_\wedge)$ is a subset of this torus, and in fact is equal to it. The dimension of the torus may be zero (a point) in which case the statement is that the limit $\lim_{\varrho\to\infty}e^{\log\varrho\mathfrak{a}}D_\wedge$ exists. This will be the case for any $D_\wedge$ if no two distinct elements of Σ have the same imaginary part.

The term $e^{t\mathfrak{a}'}g_k(t)$ in (8.3) is

$$\sum_{\substack{\sigma\in\Sigma\\ \Im\sigma+\mu=\mu_k}} e^{t(-i\Im\sigma+\mu-\mu_k)}\hat{p}_{k,\sigma}(t).$$

So all the numbers $-i\sigma+\mu-\mu_k$ appearing in the exponents of the right hand side in (8.3) are of the form $-i\sigma+\mu-\Re(-i\sigma_j+\mu)$ with $\sigma,\sigma'\in\Sigma$ and $\Re(-i\sigma+\mu)<\Re(-i\sigma_j+\mu)$. The collection of these numbers is thus

$$\{-i\sigma-\Im\sigma' : \sigma,\sigma'\in\Sigma,\ \Re(-i\sigma)<\Re(-i\sigma')\},$$

The additive semigroup $\mathfrak{S}\subset\mathbb{C}$ generated by this set is a subset of $\{\vartheta\in\mathbb{C} : \Re\vartheta\le 0\}$ with the property that $\{\vartheta\in\mathfrak{S} : \Re\vartheta>\mu\}$ is finite for every $\mu\in\mathbb{R}$.

All this information comes together to produce, after some more work, the following theorem slightly adapted from Theorem 7.4 of [18] (see the proof there):

Theorem 8.4. *If Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$, $\mathcal{D}_\wedge = D_\wedge + \mathcal{D}_{\wedge,\min}$, and $\lambda_0\in\Lambda\backslash 0$, then there are polynomials $p_\vartheta(z^1,\dots,z^N,t)$ with values in* $\mathrm{End}(\mathcal{E}_\wedge)$ *and $\mathbb{C}$-valued polynomials $q_\vartheta(z^1,\dots,z^N,t)$ such that*

$$\exists\, C,\ R>0 \text{ such that } |q_\vartheta(\varrho^{i\Re\sigma_1},\dots,\varrho^{i\Re\sigma_N},t)|>C \text{ if } \ \varrho>R \tag{8.5}$$

and such that

$$\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge} = \sum_{\vartheta\in\mathfrak{S}} \frac{\varrho^\vartheta p_\vartheta(\varrho^{i\Re\sigma_1},\dots,\varrho^{i\Re\sigma_N},\log\varrho)}{q_\vartheta(\varrho^{i\Re\sigma_1},\dots,\varrho^{i\Re\sigma_N},\log\varrho)},\ \ \varrho>R \tag{8.6}$$

with uniform convergence in norm in $\varrho>R$. The σ_j are an enumeration of Σ.

Note that the exponents ϑ of the factors ϱ^ϑ have real parts tending to $-\infty$. The parameter ϱ can be complexified (while keeping it in a sector around the positive real axis) and then replaced by $\zeta^{1/m}$ where $\zeta=\lambda/\lambda_0$ using the principal branch of m-th root). This is how Theorem 7.4 of [18] is stated.

Replacing $\lambda=\varrho^m e^{i\theta}\lambda_0$ in (7.8) and following through to (7.9) one obtains

$$\begin{aligned} &B_{\mathcal{D}_\wedge}(\lambda) = (\lambda_0/\lambda)\kappa_{(\lambda/\lambda_0)^{1/m}}\big\{B_{\wedge,\max}(\lambda_0)\\ &-\big(I-B_{\wedge,\min}(\lambda_0)(A_\wedge-\lambda_0)\big)\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}^{-1}_{(\lambda/\lambda_0)^{1/m}}D_\wedge}\pi_{\wedge,\max}B_{\wedge,\max}(\lambda_0)\big\}\kappa^{-1}_{(\lambda/\lambda_0)^{1/m}}. \end{aligned} \tag{8.7}$$

Of course we need to verify that (7.9) remains true after complexifying ϱ, but that is indeed the case since all elements of (7.9) depend real-analytically on ϱ. The fact that $B_{\wedge\min}(\lambda)$ and $B_{\wedge,\max}(\lambda)$ as they appear in (7.8) do not depend analytically on λ is immaterial because the formula we are extending analytically is (7.9).

The final step in obtaining the asymptotics of $B_{\mathcal{D}_\wedge}(\lambda)$ in $\lambda\in\Lambda$ as $|\lambda|\to\infty$ is to replace $\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}$ in the formula for $B_{\mathcal{D}_\wedge}(\lambda)$.

It is the complicated structure of the expansion of $\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}$ in (8.6) that is responsible for the unusual behavior of the zeta function of a cone operator. Indeed, if the domain is stationary, then (obviously) $\pi_{K_{\wedge,\lambda_0},\boldsymbol{\kappa}_\varrho^{-1}D_\wedge}$ is independent of ϱ.

Example 8.8. The indicial family of $A_\wedge$ of Example 5.10, the Laplacian on the blowup of $\mathbb{R}^2$ at 0, is

$$\sigma^2 + D_\theta^2.$$

Thus the boundary spectrum of A (or $A_\wedge$) is $i\mathbb{Z}$. We have been viewing $A_\wedge$ as an unbounded operator on subspaces of

$$L^2(\mathbb{R}^2) = r^{-1}L^2_b(S^1 \times [0,\infty); \frac{dr}{r}\,d\theta)$$

so $\Sigma = \{\sigma \in \operatorname{spec}_b(A_\wedge) : -1 < \Im\sigma < -2\} = \{0\}$. So we expect a rather simple structure for the asymptotics of $\pi_{K_{\wedge,\lambda_0},D_\wedge}$ for any one dimensional subspace $D_\wedge \subset \mathcal{E}_\wedge$.

We know from Example 7.12 that for any $\lambda_0 \notin [0,\infty)$, the ray through λ_0 is a ray of minimal growth for $A_{\mathcal{D}_\wedge}$, for any $\mathcal{D}_\wedge = D_\wedge + \mathcal{D}_{\wedge,\min}$. The space $K_{\wedge,\lambda_0}$ is spanned by

$$\pi_{\wedge,\max}\phi(\lambda_0) = -\frac{2}{\pi}\log(-\lambda_0)u_1 + u_2,$$

see (7.13). If $D_\wedge$ is the space spanned by $\psi = \alpha u_1 + \beta u_2$, $(\alpha,\beta) \neq 0$, then $\boldsymbol{\kappa}_\varrho^{-1}D_\wedge$ is spanned by

$$\boldsymbol{\kappa}_\varrho^{-1}\psi = (\alpha + \frac{2\beta}{\pi}\log\varrho^2)u_1 + \beta u_2,$$

see (5.11). The pair $\{\pi_{\wedge,\max}\phi(\lambda_0), \boldsymbol{\kappa}_\varrho^{-1}\psi\}$ is a basis of $\mathcal{E}_\wedge$ when $\varrho^2\lambda_0$ is not in the spectrum of $A_{\wedge,\mathcal{D}_\wedge}$, in which case we may express an arbitrary element $v = v_1u_1 + v_2u_2 \in \mathcal{E}_\wedge$ in terms of $\pi_{\wedge,\max}\phi(\lambda_0)$ and $\boldsymbol{\kappa}_\varrho^{-1}\psi$:

$$v = \frac{-\pi\beta\,v_1 + \pi(\alpha + 2\beta\log\varrho^2)\,v_2}{\pi\alpha + 2\beta\log(-\varrho^2\lambda_0)}\pi_{\wedge,\max}\phi(\lambda_0) + \frac{\pi\,v_1 + 2\log(-\lambda_0)\,v_2}{\pi\alpha + 2\beta\log(-\varrho^2\lambda_0)}\boldsymbol{\kappa}_\varrho^{-1}\psi$$

Consequently,

$$\pi_{K_{\wedge,\lambda_0},D_\wedge}v = \frac{-\pi\beta\,v_1 + \pi(\alpha + 4\beta\log\varrho)\,v_2}{\pi\alpha + 4\beta\log\varrho\,\log(-\lambda_0)}\,\pi_{\wedge,\max}\phi(\lambda_0).$$

Note that $D_\wedge$ is stationary if and only if $\beta = 0$, which corresponds to the Friedrichs extension.

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