Topological implications of global hypoellipticity

Gerardo A. Mendoza

I collect here some results concerning implications of a topological nature of conditions such as ellipticity or hypoellipticity of a differential or pseudodifferential operator. Results of this kind help to better understand the scope of hypotheses of such essentially analytic conditions. This is of interest, in particular, in the case of complexes of differential operators, whether elliptic or not (for example, CR complexes). Such complexes have been the subject of extensive investigation by many authors, one of the most remarkable results in the theory being that of P. Cordaro and J. Hounie [**1**] on local solvability (the validity of the Poincaré Lemma) for a certain class of non-elliptic complexes.

The following theorem is part of joint work (research partially supported by FAPESP, contract nr. 2008/56767-0) with A. P. Bergamasco and S. L. Zani [**2**] on topological restrictions imposed by the assumption of global C^∞-hypoellipticity.

THEOREM 0.1. *Suppose $\mathcal{M}$ is a closed orientable connected surface, $E, F \to \mathcal{M}$ are line bundles, and*

$$P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F) \tag{0.1}$$

is a first order differential operator of principal type. If P is globally hypoelliptic, then

$$\mathrm{c}(F) - \mathrm{c}(E) = \pm \mathbf{e}(\mathcal{M}).$$

Here $\mathrm{c}(E)$ is the total Chern class of E and $\mathbf{e}(\mathcal{M})$ is the Euler class of $\mathcal{M}$. Global hypoellipticity of course means that P has the property that if $u \in C^{-\infty}(\mathcal{M}; E)$ and $Pu \in C^\infty(\mathcal{M}; F)$ then in fact $u \in C^\infty(\mathcal{M}; E)$. Finally, principal type is meant here in the classical sense: the restriction of the principal symbol of P to any fiber $T^*_x\mathcal{M}$ of $T^*\mathcal{M}$ is nonzero as a (linear) function $T_x\mathcal{M} \to \mathrm{Hom}(E_x, F_x)$. The proof of Theorem 0.1 uses a microlocal argument concerning solvability of the transpose of P which ends up allowing us to deform P to an elliptic differential operator of order 1 using an idea from [**5**]. Once this is accomplished, the result is a consequence of Theorem 0.4 discussed below.

Theorem 0.1 generalizes the following theorem of J. Hounie [**3**]:

Theorem 0.2. *Suppose $\mathcal{M}$ is a closed orientable smooth surface and L is a vector field on $\mathcal{M}$ of principal type. If L is, as a differential operator, globally hypoelliptic, then $\mathcal{M}$ is a torus.*

The assumption that L is of principal type is equivalent to the statement that L is nowhere zero. Thus if L is a real vector field, then this implies immediately that $\mathcal{M}$ is a torus. However L may be a complex vector field, so Hounie's theorem is not immediate since the complexification of the tangent bundle of any manifold admits a globally defined nowhere zero vector field; a hypothesis such as global hypoellipticity is needed. Theorem 0.1 reduces to Hounie's theorem when E and F are the trivial vector bundle and P is has no zeroth order term (invariantly, P annihilates the constants).

Next is a theorem in which one reaches the same conclusion as in Theorem 0.1, starting with a different hypothesis. First some concepts. Let $\mathcal{M}$ be an arbitrary smooth paracompact manifold and $\iota : \mathcal{V} \hookrightarrow \mathbb{C}T\mathcal{M}$ a subbundle. Then there is an associated differential operator $\mathbb{D} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}; \mathcal{V}^*)$, namely if f is a smooth function, let $\mathbb{D}f = \iota^* df$. In other words, $\mathbb{D}f$ is the restriction of df to $\mathcal{V}$. When $\mathcal{V}$ is involutive, $\mathbb{D}$ is the first operator in a complex of first order differential operators on the exterior powers of $\mathcal{V}^*$ (see Treves [**6**]). Many natural differential complexes arise in this manner. Let $m = \dim \mathcal{M} - \operatorname{rk} \mathcal{V}$. The subbundle $\mathcal{V}$ is said to be a hypo-complex structure (see Treves, op cit.) if for each $x \in \mathcal{M}$ there exists some open neighborhood $\mathcal{U}$ of x and a C^∞ function $Z : U \to \mathbb{C}^m$ whose components Z_i satisfy $\mathbb{D}Z_i = 0$ over $\mathcal{U}$ and have independent differentials at x, with the property that for any u such that $\mathbb{D}u = 0$ near x there is h holomorphic near $Z(x)$ such that $u = h \circ Z$ near x. The following result [**5**, Theorem 7.3] was part of a general analysis of subbundles of $\mathbb{C}T\mathcal{M}$ carried out in joint work with H. Jacobowitz also aimed at getting a better sense of the analytical conditions that can be placed on complexes of the kind just described.

Theorem 0.3. *Suppose $\mathcal{M}$ is an orientable two-manifold and $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ is a hypo-complex subbundle (with $\operatorname{rk} \mathcal{V} = 1$). Then there exists a smooth family of subbundles $\mathcal{V}_t \subset \mathbb{C}T\mathcal{M}$, $0 \le t \le 1$ in which $\mathcal{V}_0 = \mathcal{V}$ and $\mathcal{V}_t$ is a holomorphic structure on $\mathcal{M}$ for each $t > 0$.*

This theorem can be viewed as intermediate between Theorems 0.1 and 0.2. Indeed, in the terminology of the first theorem, we have again that E is the trivial line bundle as in Theorem 0.2 but now $F = \mathcal{V}$. The conclusion of Theorem 0.1, namely that $\mathbf{c}_1(\mathcal{V}) = \pm\mathbf{e}(\mathcal{M})$, holds here because the Euler class of $\mathcal{M}$ is, except for sign (corresponding to choice a of orientation), the first Chern class of a holomorphic or antiholomorphic structure on $\mathcal{M}$.

With the stronger assumption of ellipticity we have the following correspondingly stronger result obtained in collaboration with H. Jacobowitz [**4**]

Theorem 0.4. *Let $\mathcal{M}$ be a connected compact manifold and $E, F \to \mathcal{M}$ complex vector bundles. If there is an elliptic classical pseudodifferential operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ then $\mathrm{c}(F) - \mathrm{c}(E) = k\mathbf{e}(\mathcal{M})$ for some $k \in \mathbb{Z}$.*

This is shown using the Gysin sequence associated with the cosphere bundle of $\mathcal{M}$, see [**4**]. A somewhat weaker result can be obtained if $\mathcal{M}$ is orientable using a Mayer-Vietoris sequence, as follows. Let p be the principal symbol of an elliptic pseudodifferential operator as in the statement of the theorem. Thus $p : \dot\pi^* E \to \dot\pi^* F$ is an isomorphism (where $\dot\pi : T^*\mathcal{M} \setminus 0 \to \mathcal{M}$ is the projection). Suppose for a moment that $\mathcal{M}$ admits a global nonvanishing (continuous) differential one-form α. The image of $\mathcal{M}$ by α is of course diffeomorphic to $\mathcal{M}$ and the isomorphism $p|_{\alpha(\mathcal{M})} : \dot\pi^* E|_{\alpha(\mathcal{M})} \to \dot\pi^* F|_{\alpha(\mathcal{M})}$ therefore descends to an isomorphism $E \to F$. That is, under the hypotheses of the theorem, if the Euler class of $\mathcal{M}$ vanishes, then $E = F$. Now, for general $\mathcal{M}$, pick $x \in \mathcal{M}$ arbitrarily, and let $\mathcal{U}$ be a neighborhood of x diffeomorphic to a ball. Then E is isomorphic to F over $\mathcal{M} \setminus \{x\}$ as well as over $\mathcal{U}$, since both these manifolds have vanishing Euler characteristic. The maps

$$H^{2q}(\mathcal{M}) \to H^{2q}(\mathcal{M} \setminus \{x\}) \oplus H^{2q}(\mathcal{U})$$

in the Mayer-Vietoris sequence in integral cohomology for the pair $\mathcal{M} \setminus \{x\}, \mathcal{U}$ are injective when $0 < 2q < \dim \mathcal{M}$, so since the image of $\mathrm{c}_q(F) - \mathrm{c}_q(E)$ in $H^{2q}(\mathcal{M} \setminus \{x\}) \oplus H^{2q}(\mathcal{U})$ vanishes, $\mathrm{c}_q(E) - \mathrm{c}_q(F) = 0$ when $2q < \dim \mathcal{M}$ (here $\mathrm{c}_q(E)$ is the q-th Chern class of E). So $\mathrm{c}(F) - \mathrm{c}(E)$ is either 0 (when $\dim \mathcal{M}$ is odd) or a homogeneous class of top degree (when $\dim \mathcal{M}$ is even). The theorem asserts that the latter is proportional to the Euler class of $\mathcal{M}$.

Suppose now that $\mathcal{M}$ is a closed connected orientable surface, let $\mathbf{e}$ be the Euler class of $\mathcal{M}$ and let E and F be line bundles over $\mathcal{M}$. If the operator (0.1) is an elliptic differential differential operator, then $\mathrm{c}_1(F) - \mathrm{c}_1(E) = k\mathbf{e}$, $k \in \mathbb{Z}$. The number $|k|$ is the order of P (see [**4**, Corollary 2.5]). Thus Theorem 0.1 includes Theorem 0.4 in the restricted context (in dimension and rank) of the former.

References

[1] P. Cordaro, J. Hounie, *Local solvability for a class of differential complexes*, Acta Math. **187** (2001), no. 2, 191–212.

[2] A. P. Bergamasco, G. A. Mendoza, S. L. Zani, *On Global Hypoellipticity*, to appear in Comm. in Partial Differential Equations.

[3] J. Hounie, *Globally hypoelliptic vector fields on compact surfaces*, Comm. in Partial Differential Equations, **7** (1982), no. 4, 343–370.

[4] H. Jacobowitz and G. A. Mendoza, *Elliptic equivalence of vector bundles.* Indiana Univ. Math. J. **51** (2002), 705–725.

[5] _______, *Subbundles of the complexified tangent bundle*, Trans. Amer. Math. Soc. **355** (2003), 4201–4222.

[6] F. Treves, *Hypo-analytic structures. Local theory*, Princeton Mathematical Series, **40**, Princeton University Press, Princeton, NJ, 1992.

Department of Mathematics, Temple University, Philadelphia, PA 19122, USA
E-mail address: gmendoza@temple.edu