

SPECTRALLY UNSTABLE DOMAINS

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ABSTRACT. Let H be a separable Hilbert space, $A_c : \mathcal{D}_c \subset H \rightarrow H$ a densely defined unbounded operator, bounded from below, let \mathcal{D}_{\min} be the domain of the closure of A_c and \mathcal{D}_{\max} that of the adjoint. Assume that \mathcal{D}_{\max} with the graph norm is compactly contained in H and that \mathcal{D}_{\min} has finite positive codimension in \mathcal{D}_{\max} . Then the set of domains of selfadjoint extensions of A_c has the structure of a finite-dimensional manifold \mathfrak{SA} and the spectrum of each of its selfadjoint extensions is bounded from below. If ζ is strictly below the spectrum of A with a given domain $\mathcal{D}_0 \in \mathfrak{SA}$, then ζ is not in the spectrum of A with domain $\mathcal{D} \in \mathfrak{SA}$ near \mathcal{D}_0 . But \mathfrak{SA} contains elements \mathcal{D}_0 with the property that for every neighborhood U of \mathcal{D}_0 and every $\zeta \in \mathbb{R}$ there is $\mathcal{D} \in U$ such that $\text{spec}(A_{\mathcal{D}}) \cap (-\infty, \zeta) \neq \emptyset$. We characterize these “spectrally unstable” domains as being those satisfying a nontrivial relation with the domain of the Friedrichs extension of A_c .

1. INTRODUCTION

Throughout the paper, H is a separable Hilbert space,

$$(1.1) \quad A_c : \mathcal{D}_c \subset H \rightarrow H$$

is a densely defined unbounded operator which is semibounded from below, and

$$A : \mathcal{D}_{\max} \subset H \rightarrow H$$

is the adjoint operator, automatically an extension of the symmetric operator (1.1).

The space \mathcal{D}_{\max} is a Hilbert space with the inner product

$$(1.2) \quad (u, v)_A = (Au, Av) + (u, v), \quad u, v \in \mathcal{D}_{\max}$$

where the inner product on the right is that of H . It is further assumed that the inclusion $\mathcal{D}_{\max} \hookrightarrow H$ is compact and that \mathcal{D}_{\min} , the domain of the closure of (1.1) (the closure of \mathcal{D}_c in \mathcal{D}_{\max}) has finite positive codimension in \mathcal{D}_{\max} .

With these assumptions, all closed extensions of (1.1) are Fredholm and the set of domains of extensions with index 0 can be parametrized by the

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elements of a compact manifold (a Grassmannian) in which the domains of the selfadjoint extensions form a real analytic compact submanifold $\mathfrak{S}\mathfrak{A}$. It is a fact that all these selfadjoint extensions have discrete spectrum bounded from below. (See Section 2 for details.) Write $A_{\mathcal{D}}$ for the operator with domain \mathcal{D} . The assertion that

every $\mathcal{D}_0 \in \mathfrak{S}\mathfrak{A}$ has a neighborhood U_0 for which there is $C_0 \in \mathbb{R}$ such that $\mathcal{D} \in U_0 \implies \text{spec}(A_{\mathcal{D}}) \subset \{\lambda : \Re\lambda > C_0\}$

is false. Namely, if it were to hold, then $\mathfrak{S}\mathfrak{A}$, being compact, would admit a finite cover by open sets U_j such that the spectrum of $A_{\mathcal{D}}$ is bounded from below by the same constant in each set U_j . Hence there would be an absolute lower bound for the spectra of all selfadjoint extensions, which is not true (see Lemma 2.10 below). So in fact there is $\mathcal{D}_0 \in \mathfrak{S}\mathfrak{A}$ such that

(1.3) for every neighborhood U of \mathcal{D}_0 and every $\zeta \in \mathbb{R}$ there is $\mathcal{D} \in U$ such that $\text{spec}(A_{\mathcal{D}}) \cap (-\infty, \zeta) \neq \emptyset$.

Such domains will be called spectrally unstable. The main purpose of this paper is to establish the following characterization of these domains (proof in Section 7):

Theorem 1.4. *Let $\mathcal{D}_F \in \mathfrak{S}\mathfrak{A}$ be the domain of the Friedrichs extension of (1.1). The element $\mathcal{D} \in \mathfrak{S}\mathfrak{A}$ is spectrally unstable if and only if*

$$(\mathcal{D} \cap \mathcal{D}_F) / \mathcal{D}_{\min} \neq 0.$$

The set of elements in $\mathfrak{S}\mathfrak{A}$ for which $(\mathcal{D} \cap \mathcal{D}_F) / \mathcal{D}_{\min} \neq 0$ is a real analytic subvariety of codimension 1.

Viewing the problem from the perspective of the von Neumann theory [8] (see [9, Theorem X.2]), let $\mathcal{K}_{\pm i} = \ker(A_{\mathcal{D}_{\max}} \mp i)$. With the assumptions of the first two paragraphs above, these subspaces of H have the same finite dimension. Let $\mathcal{D}_0 \in \mathfrak{S}\mathfrak{A}$. The spectrum of $U_{\mathcal{D}_0} = (A_{\mathcal{D}_0} - i)(A_{\mathcal{D}_0} + i)^{-1}$, the Cayley transform of $A_{\mathcal{D}_0}$, consists of 1 and a discrete subset of the circle $S^1 \subset \mathbb{C}$. The part of the spectrum of $U_{\mathcal{D}_0}$ in $\Im\lambda < 0$ accumulates at 1, and so the fact that arbitrarily small perturbations of \mathcal{D}_0 to $\mathcal{D} \in \mathfrak{S}\mathfrak{A}$ can lead to an apparently spontaneous generation of spectrum of $A_{\mathcal{D}}$ arbitrarily close to $-\infty$ is not surprising. What Theorem 1.4 does, is characterize those domains \mathcal{D}_0 for which arbitrarily small perturbations lead to spectrum of the Cayley transform spilling over from $\Im\lambda \leq 0$ to $\Im\lambda > 0$ across 1.

Note in passing that for no $\mathcal{D} \in \mathfrak{S}\mathfrak{A}$ can the part of the spectrum of $U_{\mathcal{D}}$ on the semicircle in $\Im\lambda > 0$ accumulate at 1, since the spectrum of any $A_{\mathcal{D}}$ is bounded below by [1, Theorem 7, pg. 217], quoted here as Theorem 2.11.

The key technical results are a very simple ‘‘regularity’’ result, Proposition 4.1, and Theorem 6.9, a statement concerning recovering the essential part of the domain of the Friedrichs extension as a limit of spaces associated with $\ker(A_{\mathcal{D}_{\max}} - \lambda)$. To describe these more precisely let \mathcal{E} be the orthogonal complement of \mathcal{D}_{\min} in \mathcal{D}_{\max} and π_{\max} the orthogonal projection on \mathcal{E} , all

with the inner product (1.2). Domains of closed extensions of (1.1) correspond to the various subspaces $D \subset \mathcal{E}$ via $\mathcal{D} = D + \mathcal{D}_{\min}$, with selfadjoint extensions corresponding to the points of a submanifold $\mathfrak{S}\mathfrak{A}$ of the Grassmannian of subspaces of \mathcal{E} of a certain dimension (so it is not \mathcal{D}_F that belongs to $\mathfrak{S}\mathfrak{A}$ in Theorem 1.4, but a certain subspace $D_F \subset \mathcal{E}$). Let $\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda)$ and $K_\lambda = \pi_{\max}\mathcal{K}_\lambda$. Then $\lambda \mapsto K_\lambda$ is a smooth curve in $\mathfrak{S}\mathfrak{A}$ if λ is sufficiently negative, and $\lim_{\lambda \rightarrow -\infty} K_\lambda = D_F$. This is a consequence of the following. For any domain $\mathcal{D} = D + \mathcal{D}_{\min}$ with $D \in \mathfrak{S}\mathfrak{A}$ and any $s \geq 0$ we define Hilbert spaces $H_{\mathcal{D}}^s$ using $A_{\mathcal{D}}$; these Sobolev-like spaces give $H_{\mathcal{D}}^0 = H$ and $H_{\mathcal{D}}^1 = \mathcal{D}$. For $u \in D^\perp$, the linear functional δ_u defined by $\mathcal{D} \ni v \mapsto (Av, u) - (v, Au) \in \mathbb{C}$ is an element of the dual space of $H_{\mathcal{D}}^1$, and may also be in $H_{\mathcal{D}^\dagger}^{-s}$ for $0 < s < 1$, the dual of $H_{\mathcal{D}}^s$. We show that $\delta_u \notin H_{\mathcal{D}^\dagger}^{-1/2}$ for $\mathcal{D}_F = D_F + \mathcal{D}_{\min}$ if $u \neq 0$.

Elliptic semibounded cone operators on compact manifolds \mathcal{M} with boundary acting on weighted L^2 -spaces of sections of a Hermitian vector bundle $E \rightarrow \mathcal{M}$,

$$A : C_c^\infty(\overset{\circ}{\mathcal{M}}; E) \subset x^{-\nu}L_b^2(\mathcal{M}; E) \rightarrow x^{-\nu}L_b^2(\mathcal{M}; E),$$

have the properties stated in the first two paragraphs, see Lesch [7, Proposition 1.3.16 and its proof]. The fine structure of the domain of the Friedrichs extension for these differential operators was given in [4, Theorem 8.12]; the interested reader may consult these references for detailed information about such operators. The research leading to the papers [5, 6] was the motivation for looking into the instability issue. Friedrichs defined his extension in [3]. The nature of the domain in the abstract context was elucidated by Freudenthal in [2].

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2. DOMAINS, SELFADJOINTNESS

All closed extensions of (1.1) considered here will have as domain a subspace of \mathcal{D}_{\max} containing \mathcal{D}_{\min} . Thus the domain of every closed extension of (1.1) is of the form

$$\mathcal{D} = D + \mathcal{D}_{\min}$$

with D a subspace of the orthogonal complement, \mathcal{E} , of \mathcal{D}_{\min} in \mathcal{D}_{\max} with respect to the inner product (1.2); \mathcal{E} is finite-dimensional by hypothesis. In particular, the domain of the Friedrichs extension of (1.1) has the form $\mathcal{D}_F = D_F + \mathcal{D}_{\min}$ for some subspace $D_F \subset \mathcal{E}$.

The resolvent family of

$$A : \mathcal{D}_F \subset H \rightarrow H$$

consists of compact operators $B_F(\lambda) : H \rightarrow H$, since they are also continuous as operators $H \rightarrow \mathcal{D}_F$ and the inclusion $\mathcal{D}_F \hookrightarrow H$ is compact. It follows that A with domain \mathcal{D}_{\min} or \mathcal{D}_{\max} is Fredholm, and from this and the finiteness

of $\dim \mathcal{E}$, that every closed extension of (1.1) is Fredholm (with compact resolvent when it exists). It is easily verified that the index of A with domain $\mathcal{D} = D + \mathcal{D}_{\min}$ is

$$(2.1) \quad \text{ind } A_{\mathcal{D}} = \text{ind } A_{\mathcal{D}_{\min}} + \dim D.$$

Since $A_{\mathcal{D}_{\min}} - \lambda I$ is injective for large negative λ , $\text{ind } A_{\mathcal{D}_{\min}} \leq 0$. And since $A_{\mathcal{D}_{\max}} - \lambda I$ is surjective for such λ , $\text{ind } A_{\mathcal{D}_{\max}} \geq 0$. From $\text{ind } A_{\mathcal{D}_{\max}} = \text{ind } A_{\mathcal{D}_{\min}} + \dim \mathcal{E}$ and $\text{ind } A_{\mathcal{D}_{\max}} = -\text{ind } A_{\mathcal{D}_{\min}}$ (because $A_{\mathcal{D}_{\max}}$ and $A_{\mathcal{D}_{\min}}$ are adjoints of each other) one derives that $\dim \mathcal{E} = 2d$ with $d = -\text{ind } A_{\mathcal{D}_{\min}}$; this is a positive number since $\dim \mathcal{E} > 0$. One can then view the set of domains of selfadjoint extensions of (1.1) as

$$\mathfrak{S}\mathfrak{A} = \{D \subset \mathcal{E} : A \text{ with domain } D + \mathcal{D}_{\min} \text{ is selfadjoint}\},$$

a subset of $\text{Gr}_d(\mathcal{E})$, the Grassmannian of d -dimensional subspaces of \mathcal{E} . As such, $\mathfrak{S}\mathfrak{A}$ is a compact real analytic submanifold of dimension d^2 (see Proposition 2.9).

Let

$$[\cdot, \cdot]_A : \mathcal{D}_{\max} \times \mathcal{D}_{\max} \rightarrow \mathbb{C}$$

denote the skew-Hermitian form

$$[u, v]_A = (Au, v) - (u, Av).$$

Then $[u, v]_A = 0$ if either u or v belongs to \mathcal{D}_{\min} , so

$$[u, v]_A = [\pi_{\max}u, \pi_{\max}v]_A$$

where

$$\pi_{\max} : \mathcal{D}_{\max} \rightarrow \mathcal{D}_{\max}$$

is the orthogonal projection on \mathcal{E} . The restriction of the Green form $[\cdot, \cdot]_A$ to \mathcal{E} is non-degenerate because the Hilbert space adjoint of A with domain \mathcal{D}_{\max} is A with domain \mathcal{D}_{\min} .

The facts collected in the following lemma can be verified directly, or following the arguments in [6, Section 6].

Lemma 2.2. *We have*

$$(2.3) \quad \mathcal{E} = \{u \in \mathcal{D}_{\max} : Au \in \mathcal{D}_{\max} \text{ and } A^2u = -u\}.$$

If $u \in \mathcal{E}$, then $Au \in \mathcal{E}$, and the map

$$(2.4) \quad A|_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E}$$

is an isometry with inverse $-A|_{\mathcal{E}}$. If $u, v \in \mathcal{E}$, then

$$(2.5) \quad [u, Av]_A = (u, v)_A.$$

Consequently, for any subspace $D \subset \mathcal{E}$, the adjoint of

$$A : D + \mathcal{D}_{\min} \subset H \rightarrow H$$

is

$$(2.6) \quad A : A(D^\perp) + \mathcal{D}_{\min} \subset H \rightarrow H$$

where D^\perp is the orthogonal complement of D in \mathcal{E} . Consequently

$$(2.7) \quad D \in \mathfrak{S}\mathfrak{A} \iff A(D^\perp) = D \iff A(D) = D^\perp.$$

and in particular, $D \in \mathfrak{S}\mathfrak{A} \implies D^\perp \in \mathfrak{S}\mathfrak{A}$.

We discuss the claim about the adjoint. The combination of (2.3) and (2.4) gives $A^2|_{\mathcal{E}} = -I$, so (2.5) can also be written as

$$[u, v]_A = -(u, Av)_A.$$

Suppose $\mathcal{D} = D + \mathcal{D}_{\min}$ with $D \subset \mathcal{E}$. The domain of the adjoint of $A_{\mathcal{D}}$ is $\mathcal{D}^* = D^* + \mathcal{D}_{\min}$ for some subspace $D^* \subset \mathcal{E}$. Since $A_{\mathcal{D}_{\min}}$ is symmetric, the condition that $v \in \mathcal{D}^*$ reduces to the statement that $[u, v]_A = 0$ for all $u \in D$, equivalently,

$$v \in D^* \iff (u, Av)_A = 0 \text{ for all } u \in D.$$

Thus $v \in D^* \iff Av \in D^\perp$, and so $D^* = (AD)^\perp$. Also $D = (AD^*)^\perp$, so $D^\perp = AD^*$, and using $A^2 = -I$ again we get $D^* = A(D^\perp)$, which gives the assertion in (2.6).

If $D \in \text{Gr}_d(\mathcal{E})$ and $T : D \rightarrow D^\perp$ is a linear map, then

$$\text{graph } T = \{u + Tu : u \in D\} \subset \mathcal{E}$$

is again an element of $\text{Gr}_d(\mathcal{E})$. The set U_D of all such elements is a neighborhood of D in $\text{Gr}_d(\mathcal{E})$.

Lemma 2.8. *Suppose $D \in \mathfrak{S}\mathfrak{A}$. Then*

$$U_D \cap \mathfrak{S}\mathfrak{A} = \{\text{graph } T : \text{the map } AT : D \rightarrow D \text{ is selfadjoint}\}.$$

Here *selfadjoint* means with respect to the A -inner product.

Since $A|_{\mathcal{E}}$ is unitary, if $T : D \rightarrow D^\perp$ is such that $AT : D \rightarrow D$ is selfadjoint, then also $TA : D^\perp \rightarrow D^\perp$ is selfadjoint.

Proof. Let $D \in \mathfrak{S}\mathfrak{A}$, let $T : D \rightarrow D^\perp$ be a linear map. In view of (2.7), the condition that $\text{graph } T \in \mathfrak{S}\mathfrak{A}$ is that

$$(u + Tu, A(v + Tv))_A = 0 \text{ for all } u, v \in D$$

For a general $T : D \rightarrow D^\perp$ and $u, v \in D$ we have

$$(u + Tu, A(v + Tv))_A = (u, Av)_A + (u, ATv)_A + (Tu, Av)_A + (Tu, ATv)_A.$$

Since $D \in \mathfrak{S}\mathfrak{A}$ and $u, v \in D$, $(u, Av)_A = 0$, and since $Tu, Tv \in D^\perp$ and $D^\perp \in \mathfrak{S}\mathfrak{A}$, also $(Tu, ATv)_A = 0$. Further, since A is an isometry on \mathcal{E} and $A^2 = -I$, $(Tu, Av)_A = -(ATu, v)$. Thus

$$(u + Tu, A(v + Tv))_A = (u, ATv)_A - (ATu, v)_A$$

so graph $T \in \mathfrak{S}\mathfrak{A}$ iff $AT : D \rightarrow D$ is selfadjoint with respect to the A -inner product. \square

Thus $\mathfrak{S}\mathfrak{A}$, as a subset of $\text{Gr}_d(\mathcal{E})$, is structurally simple:

Proposition 2.9 ([6] Proposition 6.3). *The set $\mathfrak{S}\mathfrak{A}$ is a smooth real-algebraic subvariety of $\text{Gr}_d(\mathcal{E})$.*

The dimension of the vector space of selfadjoint operators $D \rightarrow D$ (a real vector space) is d^2 , so $\mathfrak{S}\mathfrak{A}$ is a real submanifold of $\text{Gr}_d(\mathcal{E})$ of dimension d^2 .

Lemma 2.10 ([6] Proposition 6.4). *Every $\lambda \in \mathbb{R}$ appears as eigenvalue of some selfadjoint extension of A .*

Proof. Let $\lambda \in \mathbb{R}$. If $\ker(A_{\mathcal{D}_{\min}} - \lambda) \neq 0$, then $\lambda \in \text{spec}(A_{D+\mathcal{D}_{\min}})$ for every $D \in \mathfrak{S}\mathfrak{A}$, so the lemma holds in this case. Suppose now that $A_{\mathcal{D}_{\min}} - \lambda$ is injective and let $\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda)$. Then $\mathcal{K}_\lambda \cap \mathcal{D}_{\min} = 0$, so $K_\lambda = \pi_{\max} \mathcal{K}_\lambda$ has the same dimension as \mathcal{K}_λ . The injectivity of $A_{\mathcal{D}_{\min}} - \lambda$ implies the surjectivity of its adjoint, $A_{\mathcal{D}_{\max}} - \lambda$, so the index of the latter, namely d , is equal to the dimension of its kernel. So $K_\lambda \in \text{Gr}_d(\mathcal{E})$. Let $\mathcal{D} = K_\lambda + \mathcal{D}_{\min}$. To verify that $K_\lambda \in \mathfrak{S}\mathfrak{A}$ let $u, v \in \mathcal{K}_\lambda$ and $u_0, v_0 \in \mathcal{D}_{\min}$ (note that $\mathcal{D} = \mathcal{K}_\lambda + \mathcal{D}_{\min}$). Then $[u + u_0, v + v_0]_A = [u, v]_A$ using that the Hilbert space adjoint of $A_{\mathcal{D}_{\min}}$ is $A_{\mathcal{D}_{\max}}$ and that $A_{\mathcal{D}_{\min}}$ is symmetric. So

$$[u + u_0, v + v_0]_A = (u, Av) - (Au, v) = (u, \lambda v) - (\lambda u, v) = 0$$

since $\lambda \in \mathbb{R}$. It follows that $A_{\mathcal{D}}$ is symmetric, and from this and $\text{ind } A_{\mathcal{D}} = 0$, that A is selfadjoint. \square

We end with the following fundamental fact:

Theorem 2.11. *Let m be a lower bound of A_c . Every selfadjoint extension of A_c is semibounded from below and the part of its spectrum in $(-\infty, m)$ is discrete with at most d eigenvalues counting multiplicity.*

This is [1, Theorem 7, pg. 217]. Indeed, in view of the semiboundedness of (1.1), all we need to verify is that the deficiency indices of A_c are finite and equal. Since A_c is semibounded from below, $A_{\mathcal{D}_{\min}} - \lambda$ is injective if $\Im \lambda \neq 0$ or $\lambda \in \mathbb{R}$ is sufficiently negative. For such λ , $\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda)$ has constant dimension d , because of (2.1) and the definition of d as $-\text{ind } A_{\mathcal{D}_{\min}}$. In particular, the spaces \mathcal{K}_i and \mathcal{K}_{-i} have the same dimension. But these spaces are the orthogonal complements in H of the ranges of $A_{\mathcal{D}_{\min}} + i$ and $A_{\mathcal{D}_{\min}} - i$. We note in passing that both \mathcal{K}_i and \mathcal{K}_{-i} are subspaces of \mathcal{E} , with $\mathcal{E} = \mathcal{K}_i \oplus \mathcal{K}_{-i}$. This is the decomposition of \mathcal{E} into the eigenspaces of the almost complex structure of \mathcal{E} determined by A .

3. \mathcal{D} -SOBOLEV SPACES

Let $A : \mathcal{D} \subset H \rightarrow H$ be a selfadjoint extension of (1.1), let

$$\Pi_{\mathcal{D},\lambda} : H \rightarrow H$$

be the orthogonal projection on $\ker(A_{\mathcal{D}} - \lambda)$. Define, for arbitrary $s \geq 0$,

$$H_{\mathcal{D}}^s = \{u \in H : \sum_{\lambda \in \text{spec}(A_{\mathcal{D}})} (1 + |\lambda|)^{2s} \|\Pi_{\mathcal{D},\lambda} u\|^2 < \infty\}.$$

This is a Hilbert space with inner product

$$(u, v)_s = \sum_{\lambda \in \text{spec}(A_{\mathcal{D}})} (1 + |\lambda|)^{2s} (\Pi_{\mathcal{D},\lambda} u, \Pi_{\mathcal{D},\lambda} v).$$

We will write $\|\cdot\|_s$ for the norm of $H_{\mathcal{D}}^s$. We shall not make explicit the dependence on \mathcal{D} of the norm or the inner product, and omit s altogether when $s = 0$.

Clearly $H_{\mathcal{D}}^{s'}$ is densely and continuously contained in $H_{\mathcal{D}}^s$ if $s' > s \geq 0$.

Lemma 3.1. *The spaces $H_{\mathcal{D}}^1$ and \mathcal{D} are equal and the A -norm on \mathcal{D} and the norm of $H_{\mathcal{D}}^1$ are equivalent. The space \mathcal{D}_c is contained in $H_{\mathcal{D}}^s$ for every $0 \leq s \leq 1$, and its closure in $H_{\mathcal{D}}^1$ is \mathcal{D}_{\min} .*

In particular, $H_{\mathcal{D}}^1 \neq \mathcal{D}_{\max}$ since $\mathcal{D} \neq \mathcal{D}_{\max}$. We will write $\dot{H}_{\mathcal{D}}^s$ for the closure of \mathcal{D}_c in $H_{\mathcal{D}}^s$ ($0 \leq s \leq 1$). Evidently $\dot{H}_{\mathcal{D}}^s$ is independent of \mathcal{D} (despite the notation), but $\dot{H}_{\mathcal{D}}^s$ may depend on \mathcal{D} if $s < 1$.

Proof. Suppose $v \in H_{\mathcal{D}}^1$, let

$$v_n = \sum_{\lambda < n} \Pi_{\mathcal{D},\lambda}(v)$$

and note that

$$Av_n = \sum_{\lambda < n} \lambda \Pi_{\mathcal{D},\lambda}(v)$$

Since $v \in H$, $v_n \rightarrow v$ in H , but since in fact $v \in H_{\mathcal{D}}^1$, Av_n also converges in H . Since $A_{\mathcal{D}}$ is closed, $v \in \mathcal{D}$. Thus $H_{\mathcal{D}}^1 \subset \mathcal{D}$. The opposite inclusion follows from an application of the Spectral Theorem. An explicit calculation gives

$$\frac{1}{4} \|u\|_1^2 \leq \|u\|_A^2 \leq \|u\|_1^2, \quad u \in \mathcal{D}.$$

That the closure of \mathcal{D}_c in $H_{\mathcal{D}}^1$ is \mathcal{D}_{\min} follows from this and that $\mathcal{D}_c \subset H_{\mathcal{D}}^s$ for $0 \leq s \leq 1$ follows from $H_{\mathcal{D}}^1 \subset H_{\mathcal{D}}^s$ for such s . \square

Let $H_{\mathcal{D}^\dagger}^{-s}$ be the dual of $H_{\mathcal{D}}^s$ with the norm topology. Denote the pairing of $\psi \in H_{\mathcal{D}^\dagger}^{-s}$ and $u \in H_{\mathcal{D}}^s$ by $\langle \psi, u \rangle_s$. Define $h_s^\sharp : H_{\mathcal{D}}^s \rightarrow H_{\mathcal{D}^\dagger}^{-s}$ by setting

$$(3.2) \quad \langle h_s^\sharp v, u \rangle_s = (u, v)_s.$$

The Riesz representation theorem gives that the map h_s^\sharp is surjective, so invertible since it is also injective, and an antilinear isometry. The inverse will be denoted h_s^\flat .

The space $H_{\mathcal{D}^\dagger}^{-s}$ is again a Hilbert space with inner product

$$(\psi, \eta)_{-s} = (h_s^\flat \eta, h_s^\flat \psi)_s, \quad \psi, \eta \in H_{\mathcal{D}^\dagger}^{-s}.$$

The Hilbert space norm of an element of $H_{\mathcal{D}^\dagger}^{-s}$ is equal its norm as linear functional $H_{\mathcal{D}}^s \rightarrow \mathbb{C}$.

Suppose $0 \leq s \leq 1$, let $\dot{H}_{\mathcal{D}^\dagger}^{-s}$ be the dual of $\dot{H}_{\mathcal{D}}^s$. The inclusion map

$$\iota_s : \dot{H}_{\mathcal{D}}^1 \rightarrow H_{\mathcal{D}}^s$$

gives the dual map

$$\iota_s^\dagger : H_{\mathcal{D}^\dagger}^{-s} \rightarrow \dot{H}_{\mathcal{D}^\dagger}^{-1}.$$

We are interested in the elements of the kernel of these maps.

The kernel of ι_s^\dagger , the annihilator in $H_{\mathcal{D}^\dagger}^{-s}$ of the closure of $\dot{H}_{\mathcal{D}}^1$ in $H_{\mathcal{D}}^s$, is isomorphic via h_s^\flat to the orthogonal complement of $\dot{H}_{\mathcal{D}}^s$ in $H_{\mathcal{D}}^s$, so $\dim \ker \iota_s^\dagger = \dim H_{\mathcal{D}}^s / \dot{H}_{\mathcal{D}}^s$. In particular, $\dim \ker \iota_1^\dagger = d$, since by Lemma 3.1, $\dot{H}_{\mathcal{D}}^1 = \mathcal{D}_{\min}$ and $H_{\mathcal{D}}^1 = D + \mathcal{D}_{\min}$.

Suppose $0 \leq s < s' \leq 1$, and let $j_{s,s'} : H_{\mathcal{D}}^{s'} \hookrightarrow H_{\mathcal{D}}^s$ be the inclusion map. Then $\iota_s = j_{s,s'} \circ \iota_{s'}$, so $\iota_s^\dagger = \iota_{s'}^\dagger \circ j_{s,s'}^\dagger$. Since $j_{s,s'}$ has dense image, $j_{s,s'}^\dagger$ is injective. Consequently $u \in \ker \iota_s^\dagger$ if and only if $\iota_{s'}^\dagger(j_{s,s'}^\dagger(u)) = 0$ and we deduce that $j_{s,s'}^\dagger$ restricts to an injective map $\ker \iota_s^\dagger \rightarrow \ker \iota_{s'}^\dagger$. Identifying $H_{\mathcal{D}^\dagger}^{-s}$ with its image in $H_{\mathcal{D}^\dagger}^{-s'}$ by $j_{s,s'}^\dagger$ this means

$$(3.3) \quad \ker \iota_s^\dagger = H_{\mathcal{D}^\dagger}^{-s} \cap \ker \iota_{s'}^\dagger, \quad 0 \leq s < s'.$$

All that is left is to determine $\ker \iota_1^\dagger$.

Proposition 3.4. *The kernel of ι_1^\dagger consists of all maps $\delta_u : H_{\mathcal{D}}^1 \rightarrow \mathbb{C}$ of the form*

$$(3.5) \quad H_{\mathcal{D}}^1 \ni \psi \mapsto \langle \delta_u, \psi \rangle = [\psi, u]_A \in \mathbb{C}.$$

for some $u \in D^\perp$. Here, as before, D^\perp is the orthogonal complement of D in \mathcal{E} .

Proof. Let $u \in D^\perp$. The functional δ_u is clearly linear. Its continuity as a map $\delta_u : H_{\mathcal{D}}^1 \rightarrow \mathbb{C}$ is an immediate consequence of the Cauchy-Schwarz inequality, the definition of the A -norm and the equivalence of the latter and that of $H_{\mathcal{D}}^1$. If $\psi \in \dot{H}_{\mathcal{D}}^1$, then $[\psi, u]_A = 0$ because $\dot{H}_{\mathcal{D}}^1 = \mathcal{D}_{\min}$ and $D^\perp \subset \mathcal{D}_{\max}$, so $\delta_u \in \ker \iota_1^\dagger$. If $\delta_u = 0$, then $(A\psi, u) - (\psi, Au) = 0$ for all $\psi \in \mathcal{D}$, since $D^\perp + \mathcal{D}_{\min}$ is the domain of the adjoint of $A_{\mathcal{D}}$. So u belongs to the domain of

the adjoint of $A_{\mathcal{D}}$. But since $A_{\mathcal{D}}$ is selfadjoint, we must have $u \in \mathcal{D}$, so $u = 0$. So the map

$$D^\perp \ni u \mapsto \delta_u \in H_{\mathcal{D}^\dagger}^{-1}$$

is an antilinear isomorphism into $\ker \iota_1^\dagger$. The surjectivity follows from the equality of the dimensions of D^\perp and $H_{\mathcal{D}}^1/\dot{H}_{\mathcal{D}}^1 \approx D$. \square

4. ESTIMATES

For $D \in \mathfrak{S}\mathfrak{A}$ we let \mathcal{P}_{D^\perp} be the collection of functionals (3.5):

$$\mathcal{P}_{D^\perp} = \{\delta_u : u \in D^\perp\}.$$

Because of (3.3), elements of \mathcal{P}_{D^\perp} may have better regularity (the number $-s$) than $H_{\mathcal{D}^\dagger}^{-1}$, but of course no element δ_u with $u \neq 0$ belongs to $H_{\mathcal{D}^\dagger}^0$. The following proposition gives an upper bound for the regularity of elements in $\ker \iota_1^\dagger$ in the case where \mathcal{D} is the domain of the Friedrichs extension of A .

Proposition 4.1. *Let $\mathcal{D}_F = \mathcal{D}_F + \mathcal{D}_{\min}$ be the domain of the Friedrichs extension of (1.1). Then $\mathcal{P}_{\mathcal{D}_F^\perp} \cap H_{\mathcal{D}_F^\dagger}^{-1/2} = 0$.*

Proof. We show that $\dot{H}_{\mathcal{D}_F}^{1/2} = H_{\mathcal{D}_F}^{1/2}$ (so also $\dot{H}_{\mathcal{D}_F}^s = H_{\mathcal{D}_F}^s$ if $0 \leq s \leq 1/2$ because of (3.3)), an equality we obtain directly by following the construction of the Friedrichs extension of A . Let

$$\mathfrak{Q}(u, v) = (Au, v) + c(u, v), \quad u, v \in \dot{H}_{\mathcal{D}}^1$$

with a large enough constant c . The norms on $\dot{H}_{\mathcal{D}_F}^1$ induced by \mathfrak{Q} and that of $H_{\mathcal{D}_F}^{1/2}$ are equivalent, so the \mathfrak{Q} -completion of $\dot{H}_{\mathcal{D}}^1$ can be identified with $\dot{H}_{\mathcal{D}_F}^{1/2}$. Let

$$B : H \rightarrow \dot{H}_{\mathcal{D}_F}^{1/2}$$

be the operator such that

$$\mathfrak{Q}(Bu, v) = (u, v) \text{ for all } u \in H, v \in \dot{H}_{\mathcal{D}_F}^{1/2}.$$

Then B is injective and its image is the domain of the Friedrichs extension of $A + cI$, which is the same as that of A . That is, $\mathcal{D}_F \subset \dot{H}_{\mathcal{D}_F}^{1/2}$, which is to say that $H_{\mathcal{D}_F}^1 \subset \dot{H}_{\mathcal{D}_F}^{1/2}$. Since $H_{\mathcal{D}_F}^1$ is dense in $H_{\mathcal{D}_F}^{1/2}$, $\dot{H}_{\mathcal{D}_F}^{1/2}$ is a dense subspace of $H_{\mathcal{D}_F}^{1/2}$. Thus $\dot{H}_{\mathcal{D}_F}^{1/2} = H_{\mathcal{D}_F}^{1/2}$. We note that this equality is standard. \square

Returning to the case of an arbitrary domain \mathcal{D} on which A is selfadjoint, let $\{\lambda_k\}_{k=1}^\infty$ be the sequence of eigenvalues of $A_{\mathcal{D}}$ repeated according to multiplicity and in increasing order, and let $\{\psi_k\} \subset \mathcal{D}$ be an orthonormal basis of H corresponding to these eigenvalues.

The ψ_k are also a complete A -orthogonal system for \mathcal{D} . Therefore, an element $u \in \mathcal{D}_{\max}$ belongs to D^\perp if and only if $(u, \psi_k)_A = 0$ for all k :

$$u \in D^\perp \iff \lambda_k(Au, \psi_k) + (u, \psi_k) = 0 \text{ for all } k.$$

Let $u \in D^\perp$. The relations

$$\begin{cases} \lambda_k(u, \psi_k) - (Au, \psi_k) = \overline{\langle \delta_u, \psi_k \rangle} \\ (u, \psi_k) + \lambda_k(Au, \psi_k) = 0, \end{cases}$$

where the first identity comes from the definition of δ_u and the second is the orthogonality condition just mentioned, give

$$(4.2) \quad (u, \psi_k) = \lambda_k \frac{\overline{\langle \delta_u, \psi_k \rangle}}{1 + \lambda_k^2}, \quad (Au, \psi_k) = -\frac{\overline{\langle \delta_u, \psi_k \rangle}}{1 + \lambda_k^2}.$$

We will now express the elements of \mathcal{P}_{D^\perp} as a Fourier series related to the orthonormal basis $\{\psi_k\}$. Recalling the maps $h_s^\sharp : H_{\mathcal{D}}^s \rightarrow H_{\mathcal{D}^\dagger}^{-s}$ defined in (3.2), let $\psi_k^0 = h_0^\sharp \psi_k$. Since the inclusion map $j_s : H_{\mathcal{D}}^s \hookrightarrow H_{\mathcal{D}}^0$ has dense image, the dual map

$$j_s^\dagger : H_{\mathcal{D}^\dagger}^0 \rightarrow H_{\mathcal{D}^\dagger}^{-s}$$

is injective with dense image. So ψ_k^0 can be regarded as an element of $H_{\mathcal{D}^\dagger}^{-s}$ for any $s \geq 0$. From the definition of the inner product we get $(\psi_k^0, \psi_\ell^0)_0 = \delta_{k\ell}$. For $w \in H_{\mathcal{D}}^s$ we have

$$\langle j_s^\dagger \psi_k^0, w \rangle_s = \langle \psi_k^0, j_s w \rangle_0 = (w, \psi_k) = \frac{(w, \psi_k)_s}{(1 + |\lambda_k|)^{2s}} = \frac{\langle h_s^\sharp \psi_k, w \rangle_s}{(1 + |\lambda_k|)^{2s}}$$

so, using the inverse h_s^\flat of h_s^\sharp ,

$$h_s^\flat(j_s^\dagger \psi_k^0) = (1 + |\lambda_k|)^{-2s} \psi_k.$$

In particular,

$$\|j_s^\dagger \psi_k^0\|_{-s}^2 = (j_s^\dagger \psi_k^0, j_s^\dagger \psi_\ell^0)_{-s} = (1 + |\lambda_k|)^{-2s} \delta_{k\ell}.$$

If $v \in H_{\mathcal{D}^\dagger}^{-s}$, then

$$(v, j_s^\dagger \psi_k^0)_{-s} = (h_s^\flat(j_s^\dagger \psi_k^0), h_s^\flat v)_s = \langle v, h_s^\flat(j_s^\dagger \psi_k^0) \rangle_s = \frac{\langle v, \psi_k \rangle_s}{(1 + |\lambda_k|)^{2s}}.$$

Thus the Fourier series representation of v is

$$v = \sum_k \langle v, \psi_k \rangle_s j_s^\dagger \psi_k^0.$$

The norm of an element $v = \sum_k v_k j_s^\dagger \psi_k^0 \in H_{\mathcal{D}^\dagger}^{-s}$ is given by

$$\|v\|_{-s}^2 = \sum_k (1 + |\lambda_k|)^{-2s} |v_k|^2.$$

Suppose now $u \in D^\perp$ and $\delta_u \in H_{\mathcal{D}^\dagger}^{-s}$. Then

$$\langle \delta_u, \psi_k \rangle_s = (1 + |\lambda_k|)^{2s} (\delta_u, j_m^\dagger \psi_k^0)_{-s},$$

hence

$$(4.3) \quad \|\delta_u\|_{-s}^2 = \sum \frac{|\langle \delta_u, \psi_k \rangle_s|^2}{(1 + |\lambda_k|)^{2s}}.$$

Note that $\langle \delta_u, \psi_k \rangle_s$ is just $\langle \delta_u, \psi_k \rangle$ since $\psi_k \in H_D^s$ for any $0 \leq s \leq 1$.

5. THE BUNDLE OF KERNELS

The background spectrum of A , denoted $\text{bg-spec}(A)$ is the set

$$\{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda \text{ is not injective or } A_{\mathcal{D}_{\max}} - \lambda \text{ is not surjective}\},$$

see [6]. Its complement is denoted $\text{bg-res}(A)$. The background spectrum is of interest in that it is a subset of the spectrum of every extension of A .

In the present case, since A is semibounded and admits an extension with compact resolvent, the set $\text{bg-spec}(A)$ is (if not empty) a discrete subset of the real line with only $+\infty$ as a possible point of accumulation, equal to

$$\text{bg-spec}(A) = \{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda \text{ is not injective}\}.$$

Indeed, if $\lambda \in \mathbb{R}$ then $\ker(A_{\mathcal{D}_{\min}} - \lambda) = \text{rg}(A_{\mathcal{D}_{\max}} - \lambda)^\perp$.

For $\lambda \in \text{bg-res}(A)$ define

$$\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda).$$

Since $A_{\min} - \lambda$ is injective if $\lambda \in \text{bg-res}(A)$, formula (2.1) with $\mathcal{D} = \mathcal{D}_{\max}$ gives $\dim \mathcal{K}_\lambda = d$. For these λ , $\mathcal{K}_\lambda \cap \mathcal{D}_{\min} = 0$. It follows that $K_\lambda = \pi_{\max} \mathcal{K}_\lambda$ also has dimension d for each $\lambda \in \text{bg-res}(A)$. (These spaces are the fibers of a holomorphic vector bundle over $\text{bg-res}(A)$ that extends across $\text{bg-spec}(A)$ as a holomorphic vector bundle. The latter fact, not obvious, will not be proved here as it is not needed.)

The following lemma makes explicit the relevancy of these spaces.

Lemma 5.1. *Let $D \in \text{Gr}_d(\mathcal{E})$. The spectrum of A with domain $\mathcal{D} = D + \mathcal{D}_{\min}$ is*

$$\{\lambda \in \text{bg-res}(A) : K_\lambda \cap D \neq 0\} \cup \text{bg-spec}(A).$$

Indeed, if $\lambda \in \text{spec}(A_{\mathcal{D}})$ and $\lambda \in \text{bg-res}(A)$, then $\ker(A_{\mathcal{D}} - \lambda) = \mathcal{D} \cap \mathcal{K}_\lambda \neq 0$, and $u \in \ker(A_{\mathcal{D}} - \lambda)$ if and only if $\pi_{\max} u \in K_\lambda$ and $\pi_{\max} u \in D$.

Because of the property expressed in the lemma it is of interest to have a formula for the spaces \mathcal{K}_λ when $\lambda \notin \text{bg-spec}(A)$. We get one such formula with the aid of the resolvent of an arbitrary selfadjoint extension $A_{\mathcal{D}}$ of (1.1).

Let then $D \in \mathfrak{S}\mathfrak{A}$, write $\pi_{D^\perp}, \pi_D : \mathcal{D}_{\max} \rightarrow \mathcal{D}_{\max}$ for the A -orthogonal projections on D^\perp and D , respectively, and let $\pi_{\mathcal{D}} : \mathcal{D}_{\max} \rightarrow \mathcal{D}_{\max}$ be the orthogonal projection on \mathcal{D} (so $\pi_{\mathcal{D}} = I - \pi_{D^\perp}$). Let $B_{\mathcal{D}}(\lambda)$ be the resolvent of $A_{\mathcal{D}}$. Suppose $\lambda \in \text{res}(A_{\mathcal{D}})$ and $\phi \in \mathcal{K}_\lambda$. The identity

$$\phi = \pi_{D^\perp} \phi + \pi_D \phi$$

gives

$$0 = (A - \lambda)\pi_{D^\perp} \phi + (A - \lambda)\pi_D \phi.$$

Applying $B_{\mathcal{D}}(\lambda)$ get

$$\pi_D \phi = -B_{\mathcal{D}}(\lambda)(A - \lambda)\pi_{D^\perp} \phi$$

since $\pi_{\mathcal{D}}\phi \in \mathcal{D}$. Thus

$$\phi = \pi_{D^\perp}\phi - B_{\mathcal{D}}(\lambda)(A - \lambda)\pi_{D^\perp}\phi$$

Conversely, it is easily verified that if $u \in D^\perp$, then

$$\phi_u(\lambda) = u - B_{\mathcal{D}}(\lambda)(A - \lambda)u$$

is an element of \mathcal{K}_λ for each $\lambda \in \text{res}(A_{\mathcal{D}})$. Evidently, the map $D^\perp \ni u \mapsto \phi_u(\lambda) \in \mathcal{K}_\lambda$ is bijective and depends holomorphically on $\lambda \notin \text{spec}(A_{\mathcal{D}})$.

Using the orthonormal basis $\{\psi_k\}$ consisting of eigenfunctions of $A_{\mathcal{D}}$, the formula

$$B_{\mathcal{D}}(\lambda)f = \sum_k \frac{(f, \psi_k)}{\lambda_k - \lambda} \psi_k$$

and the formulas (4.2) give

$$\phi_u(\lambda) = u + \sum_k \frac{(1 + \lambda\lambda_k) \overline{\langle \delta_u, \psi_k \rangle}}{(1 + \lambda_k^2)(\lambda_k - \lambda)} \psi_k, \quad \lambda \notin \text{spec}(A_{\mathcal{D}});$$

the series converges absolutely and uniformly in $H_{\mathcal{D}}^1$ on compact subsets of $\text{res}(A_{\mathcal{D}})$. Alternatively, again using (4.2) in the expansion of u in terms of the ψ_k , we have

$$(5.2) \quad \phi_u(\lambda) = \sum_k \frac{\overline{\langle \delta_u, \psi_k \rangle}}{\lambda_k - \lambda} \psi_k, \quad \lambda \notin \text{spec}(A_{\mathcal{D}}).$$

This series converges in $H_{\mathcal{D}}^0$ since

$$\sum_k \frac{|\langle \delta_u, \psi_k \rangle|^2}{(1 + |\lambda_k|)^2}$$

converges (because $\delta_u \in H_{\mathcal{D}^\dagger}^{-1}$).

6. NEGATIVITY AND REGULARITY

We continue our discussion with the selfadjoint operator $A_{\mathcal{D}}$ of the previous section; so $\mathcal{D} = D + \mathcal{D}_{\min}$ with $D \in \mathfrak{SA}$. Let $S : D^\perp \rightarrow D^\perp$ be selfadjoint with respect to the A -inner product, let $T = AS : D^\perp \rightarrow D$, and let

$$\text{graph } T = \{u + Tu : u \in D^\perp\},$$

which by Lemma 2.8 is an element of \mathfrak{SA} . Let

$$\mathcal{D}_T = \text{graph } T + \mathcal{D}_{\min}.$$

By Lemma 5.1, $\lambda \in \text{bg-res}(A)$ belongs to $\text{spec}(A_{\mathcal{D}_T})$ if and only if $\text{graph } T \cap K_\lambda \neq 0$. In particular, $\lambda \in \text{res}(A_{\mathcal{D}})$ belongs to $\text{spec}(A_{\mathcal{D}_T})$ if and only if there is $u \in D^\perp$, $u \neq 0$, such that

$$u - \pi_{\max} B_{\mathcal{D}}(\lambda)(A - \lambda)u = u + Tu,$$

that is, if and only if $-\pi_{\max}B_{\mathcal{D}}(\lambda)(A - \lambda)u = ASu$. Setting

$$F_{\mathcal{D}}(\lambda) = -A\pi_{\max}B_{\mathcal{D}}(\lambda)(A - \lambda)|_{D^{\perp}},$$

an operator $D^{\perp} \rightarrow D^{\perp}$ we thus have

$$(6.1) \quad \lambda \in \text{spec}(A_{\mathcal{D}_T}) \cap \text{res}(A_{\mathcal{D}}) \iff F_{\mathcal{D}}(\lambda) + S \text{ has nontrivial kernel.}$$

Lemma 6.2. *The map $F_{\mathcal{D}}(\lambda)$ satisfies*

$$(6.3) \quad F_{\mathcal{D}}(\lambda)^* = F_{\mathcal{D}}(\bar{\lambda}), \quad \lambda \in \text{res}(A_{\mathcal{D}}).$$

In addition, for any $\lambda \in \text{res}(A_{\mathcal{D}})$,

$$(6.4) \quad (F_{\mathcal{D}}(\lambda)u, u')_A = \sum_{k=0}^{\infty} \frac{\overline{\langle \delta_u, \psi_k \rangle} \langle \delta_{u'}, \psi_k \rangle}{1 + \lambda_k^2} \frac{1 + \lambda \lambda_k}{\lambda_k - \lambda}, \quad u, u' \in D^{\perp}.$$

Proof. Let $u, u' \in D^{\perp}$. Then

$$(6.5) \quad \begin{aligned} (F_{\mathcal{D}}(\lambda)u, u')_A &= (-A\pi_{\max}B_{\mathcal{D}}(\lambda)(A - \lambda)u, u')_A \\ &= (\pi_{\max}B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au')_A = (B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au')_A \end{aligned}$$

where the first equality is the definition of $F_{\mathcal{D}}(\lambda)$, the second because $A|_{\mathcal{E}}$ is an isometry, and the third because $\mathcal{E} \perp \mathcal{D}_{\min}$ in the A -inner product. Using the definition of the A inner product in the last term we thus have

$$\begin{aligned} (F_{\mathcal{D}}(\lambda)u, u')_A &= (AB_{\mathcal{D}}(\lambda)(A - \lambda)u, -u') + (B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au') \\ &= ((A - \lambda)u + \lambda B_{\mathcal{D}}(\lambda)(A - \lambda)u, -u') + (B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au') \\ &= -((A - \lambda)u, u') + (B_{\mathcal{D}}(\lambda)(A - \lambda)u, (A - \bar{\lambda})u'). \end{aligned}$$

Likewise,

$$(u, F_{\mathcal{D}}(\bar{\lambda})u')_A = -(u, (A - \bar{\lambda})u') + ((A - \lambda)u, B_{\mathcal{D}}(\bar{\lambda})(A - \bar{\lambda})u').$$

Then (6.3) follows from noting that $((A - \lambda)u, u') = (u, (A - \bar{\lambda})u')$ because $D^{\perp} + \mathcal{D}_{\min}$ is a selfadjoint domain and $B_{\mathcal{D}}(\lambda)^* = B_{\mathcal{D}}(\bar{\lambda})$. This proves the first assertion of the lemma.

For the second, we have

$$\begin{aligned} (F_{\mathcal{D}}(\lambda)u, u')_A &= (B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au')_A = -(u - B_{\mathcal{D}}(\lambda)(A - \lambda)u, Au')_A \\ &= -(\phi_u(\lambda), Au')_A = \lambda(\phi_u(\lambda), u') - (\phi_u(\lambda), Au') \end{aligned}$$

using (6.5). Using (5.2) and (4.2) we get

$$\lambda(\phi_u(\lambda), u') = \sum_{k=0}^{\infty} \frac{\lambda \lambda_k \overline{\langle \delta_u, \psi_k \rangle} \langle \delta_{u'}, \psi_k \rangle}{(1 + \lambda_k^2)(\lambda_k - \lambda)}$$

and

$$-(\phi_u(\lambda), Au') = \sum_{k=0}^{\infty} \frac{\overline{\langle \delta_u, \psi_k \rangle} \langle \delta_{u'}, \psi_k \rangle}{(1 + \lambda_k^2)(\lambda_k - \lambda)}.$$

The combination of these formulas gives (6.4). \square

The following proposition is the key result:

Proposition 6.6. *Let $\mathcal{D} = D + \mathcal{D}_{\min}$ with $D \in \mathfrak{S}\mathfrak{A}$, let*

$$D_0^\perp = \{u \in D^\perp : \delta_u \in H_{\mathcal{D}^\dagger}^{-1/2}\},$$

let $D_1^\perp \subset D^\perp$ be complementary to D_0^\perp in D^\perp , and let $\pi_{D_1^\perp} : D^\perp \rightarrow D^\perp$ be the orthogonal projection on D_1^\perp . Then for every selfadjoint operator $S : D^\perp \rightarrow D^\perp$ there is $\zeta < 0$ such that $\pi_{D_1^\perp}(F_{\mathcal{D}}(\lambda) + S)|_{D_1^\perp}$ is negative if $\lambda < \zeta$.

Proof. Suppose that the conclusion is false. Then there is a selfadjoint operator $S : D^\perp \rightarrow D^\perp$ and a sequence $\{\zeta_\ell\}_{\ell=1}^\infty$ decreasing to $-\infty$ such that $\pi_{D_1^\perp}(F_{\mathcal{D}}(\zeta_\ell) + S)|_{D_1^\perp}$ has a nonnegative eigenvalue for each ℓ . Let $u_\ell \in D_1^\perp$ be an eigenvector of $F_{\mathcal{D}}(\zeta_\ell) + S$ for such an eigenvalue, with $\|u_\ell\|_A = 1$. Thus

$$(F_{\mathcal{D}}(\zeta_\ell)u_\ell, u_\ell)_A + (Su_\ell, u_\ell)_A \geq 0$$

for all ℓ . Passing to a subsequence, we may assume that $\{u_\ell\}_{\ell=1}^\infty$ converges to some $u \in D_1^\perp$. Using (6.4) we have

$$(Su_\ell, u_\ell)_A \geq -(F_{\mathcal{D}}(\zeta_\ell)u_\ell, u_\ell)_A = -\sum_{k=0}^{\infty} \frac{|\langle \delta_{u_\ell}, \psi_k \rangle|^2}{1 + \lambda_k^2} \frac{1 + \zeta_\ell \lambda_k}{\lambda_k - \zeta_\ell}$$

for every ℓ . If $k_0 = \min\{k : \lambda_k > 0\}$ and $k \geq k_0$, then

$$\frac{1 + \zeta_\ell \lambda_k}{\lambda_k - \zeta_\ell} < 0$$

if $\zeta_\ell < -1/\lambda_{k_0}$, so bearing in mind that the λ_k increase monotonically with k ,

$$\sum_{k=k_0}^{\infty} -\frac{|\langle \delta_{u_\ell}, \psi_k \rangle|^2}{1 + \lambda_k^2} \frac{1 + \zeta_\ell \lambda_k}{\lambda_k - \zeta_\ell}$$

is a series of non-negative terms if $\ell > \ell_0$ so that $\zeta_\ell < -1/\lambda_{k_0}$ for such ℓ . Hence

$$(Su_\ell, u_\ell)_A \geq -\sum_{k=0}^N \frac{|\langle \delta_{u_\ell}, \psi_k \rangle|^2}{1 + \lambda_k^2} \frac{1 + \zeta_\ell \lambda_k}{\lambda_k - \zeta_\ell}$$

for every $N \geq k_0$ and all $\ell > \ell_0$. Taking the limit as $\ell \rightarrow \infty$ gives

$$(Su, u)_A \geq \sum_{k=0}^N \lambda_k \frac{|\langle \delta_u, \psi_k \rangle|^2}{1 + \lambda_k^2}$$

for every N , so

$$\lim_{N \rightarrow \infty} \sum_{k=0}^N \lambda_k \frac{|\langle \delta_u, \psi_k \rangle|^2}{1 + \lambda_k^2} \leq (Su, u)_A.$$

Since only finitely many λ_k can be negative, the estimate implies that

$$\sum_{k=0}^{\infty} |\lambda_k| \frac{|\langle \delta_u, \psi_k \rangle|^2}{1 + \lambda_k^2}$$

converges. This in turn implies that the norm of δ_u as an element of $H_{\mathcal{D}^\dagger}^{-1/2}$ is finite, see (4.3). So $u \in D_0^\perp$, a contradiction since $\|u\|_A = 1$ and $u \in D_1^\perp \cap D_0^\perp$. \square

In particular, if $\mathcal{P}_{D^\perp} \cap H_{\mathcal{D}^\dagger}^{-1/2} = 0$, then for every $c > 0$ there is $\zeta < 0$ such that $F_{\mathcal{D}}(\lambda) + cI$ is negative if $\lambda < \zeta$. Thus:

Corollary 6.7. *If $\mathcal{P}_{D^\perp} \cap H_{\mathcal{D}^\dagger}^{-1/2} = 0$, then $F_{\mathcal{D}}(\lambda)$ is invertible for every sufficiently negative λ , and $\|F_{\mathcal{D}}(\lambda)^{-1}\|_{\mathcal{L}(D^\perp)} \rightarrow 0$ as $\lambda \rightarrow -\infty$.*

The definition of $F_{\mathcal{D}}(\lambda)$ gives

$$K_\lambda = \{u - AF_{\mathcal{D}}(\lambda)u : u \in D^\perp\}.$$

Since $F_{\mathcal{D}}(\lambda)$ is invertible for every sufficiently negative λ , also

$$(6.8) \quad K_\lambda = \{v + F_{\mathcal{D}}(\lambda)^{-1}Av : v \in D\}.$$

Thus, if $\mathcal{P}_{D^\perp} \cap H_{\mathcal{D}^\dagger}^{-1/2} = 0$, Corollary 6.7 and (6.8) give that $K_\lambda \rightarrow D$ as $\lambda \rightarrow -\infty$. Applied to $D = D_F$ and bearing in mind Proposition 4.1 and that the Friedrichs extension of A is bounded below, we get:

Theorem 6.9. *Consider the curve*

$$\mathbb{R}_- \ni \lambda \mapsto K_\lambda \in \text{Gr}_d(\mathcal{E}).$$

Then $K_\lambda \rightarrow D_F$ as $\lambda \rightarrow -\infty$.

The limit $\lim_{\lambda \rightarrow -\infty} K_\lambda$ is of course unique. Since K_λ is independent of its representation, we have that if in (6.8) $K_\lambda \rightarrow D$ then $D = D_F$. Consequently,

Theorem 6.10. *The Friedrichs domain of A is the only selfadjoint domain such that $\mathcal{P}_{D^\perp} \cap H_{\mathcal{D}^\dagger}^{-1/2} = 0$.*

Proposition 6.11. *Suppose $\{D_\ell\}_{\ell=1}^\infty \subset \mathfrak{S}\mathfrak{A}$ is a sequence converging to D and there is $\{\zeta_\ell\} \subset \mathbb{R}$ with $\zeta_\ell \rightarrow -\infty$ as $\ell \rightarrow \infty$ such that $D_\ell \cap K_{\zeta_\ell} \neq 0$. Then $D \cap D_F \neq 0$.*

Proof. For each ℓ pick $v_\ell \in D_\ell \cap K_{\zeta_\ell}$ with $\|v_\ell\|_A = 1$. Passing to a subsequence, assume that $v_\ell \rightarrow v$ as $\ell \rightarrow \infty$. Using $\mathcal{E} = D_F \oplus D_F^\perp$ gives for each ℓ , a unique $w_\ell \in D_F$ such that $v_\ell = w_\ell + F_{\mathcal{D}_F}(\zeta_\ell)^{-1}Aw_\ell$. The continuity of projections gives that w_ℓ converges. Now Corollary 6.7 applied to the Friedrichs domain gives $F_{\mathcal{D}_F}(\zeta)^{-1}Aw_\ell \rightarrow 0$ as $\zeta \rightarrow -\infty$. Thus $w_\ell \rightarrow v$. Since $w_\ell \in D_F$, $v \in D_F$. Now, $D_\ell = \text{graph } T_\ell$ for a unique $T_\ell : D \rightarrow D^\perp$; the statement that $D_\ell \rightarrow D$ means that $T_\ell \rightarrow 0$. Thus $w_\ell = v'_\ell + T_\ell v'_\ell$ for a unique $v'_\ell \in D$ and as before v'_ℓ converges, so w_ℓ converges to an element of D which must be v . Since $\|v\|_A = 1$, $D \cap D_F \neq 0$. \square

7. SPECTRALLY UNSTABLE DOMAINS

The following, a restatement of Theorem 1.4, is our main result.

Theorem 7.1. *Let $\mathcal{D}_F = D_F + \mathcal{D}_{\min}$ be the domain of the Friedrichs extension of A . The element $D \in \mathfrak{SA}$ has the property (1.3) if and only if $D \in \mathfrak{Y}_{\mathcal{D}_F}$.*

We have written $\mathfrak{Y}_{\mathcal{D}_F} = \{D \in \mathfrak{SA} : D \cap D_F \neq 0\}$. This is a real-algebraic subvariety of \mathfrak{SA} of codimension 1.

Proof. If $D \in \mathfrak{SA}$, then either $\pi_{D_F^\perp}|_D : D \rightarrow D_F^\perp$ is injective, or not. In the first case, $D \in U_{D_F^\perp}$, and in the second, $D \in \mathfrak{Y}_{\mathcal{D}_F}$. Thus

$$\mathfrak{SA} = (\mathfrak{SA} \cap U_{D_F^\perp}) \cup \mathfrak{Y}_{\mathcal{D}_F}$$

as a disjoint union.

Proposition 6.11 gives that every element of $\mathfrak{SA} \cap U_{D_F^\perp}$ is spectrally stable, so we only need to show that every element of $\mathfrak{Y}_{\mathcal{D}_F}$ is spectrally unstable.

Suppose $D \in \mathfrak{Y}_{\mathcal{D}_F}$. We will show the existence of curves $\lambda \mapsto D_\lambda$ in \mathfrak{SA} such that $D_\lambda \rightarrow D$ as $\lambda \rightarrow -\infty$ and $D_\lambda \cap K_\lambda \neq 0$. With such a curve we have that if U is a neighborhood of D and $\zeta < 0$, then there is $\zeta' < \zeta$ such that $D_\lambda \in U$ for every $\lambda < \zeta'$. Since $K_\lambda \cap D_\lambda \neq 0$, λ belongs to the spectrum of A with domain $\mathcal{D}_\lambda = D_\lambda + \mathcal{D}_{\min}$, which shows that D is spectrally unstable.

By Corollary 6.7 and Proposition 4.1, the operator $F_{\mathcal{D}_F}(\lambda) : D_F^\perp \rightarrow D_F^\perp$ is invertible for every sufficiently negative λ , so

$$K_\lambda = \{v + F_{\mathcal{D}_F}(\lambda)^{-1}Av : v \in D_F\},$$

see (6.8). Let V be a subspace of $D \cap D_F$, $V \neq 0$. As usual let π_D and π_{D^\perp} be the orthogonal projections on D and D^\perp . If $v \in V$, then

$$\begin{aligned} v + F_{\mathcal{D}_F}(\lambda)^{-1}Av &= \pi_D(v + F_{\mathcal{D}_F}(\lambda)^{-1}Av) + \pi_{D^\perp}(v + F_{\mathcal{D}_F}(\lambda)^{-1}Av) \\ &= (v + \pi_D F_{\mathcal{D}_F}(\lambda)^{-1}Av) + \pi_{D^\perp} F_{\mathcal{D}_F}(\lambda)^{-1}Av. \end{aligned}$$

Let

$$V_\lambda = \{v + \pi_D F_{\mathcal{D}_F}(\lambda)^{-1}Av : v \in V\},$$

a subspace of D . Let W be the orthogonal complement of V in D . The mapping $D \rightarrow D$ given by

$$V \oplus W \ni (v \oplus w) \mapsto v + \pi_D F_{\mathcal{D}_F}(\lambda)^{-1}Av + w \in D$$

is invertible for every sufficiently negative λ because $\|F_{\mathcal{D}_F}(\lambda)^{-1}\| \rightarrow 0$ as $\lambda \rightarrow -\infty$. Its inverse tends to the identity as $\lambda \rightarrow -\infty$ and maps V_λ to V . Let $S_\lambda : V_\lambda \rightarrow V$ be the restriction to V_λ of this inverse and define $T_{\lambda,0} : V_\lambda \rightarrow D^\perp$ by

$$T_{\lambda,0} = \pi_{D^\perp} F_{\mathcal{D}_F}(\lambda)^{-1}AS_\lambda.$$

Then

$$\{v + T_{\lambda,0}v : v \in V_\lambda\} = \{v + F_{\mathcal{D}_F}(\lambda)^{-1}Av : v \in V\} \subset K_\lambda,$$

therefore

$$(7.2) \quad (v + T_{\lambda,0}v, A(v' + T_{\lambda,0}v'))_A = 0 \quad \text{for every } v, v' \in V_\lambda$$

(cf. the proof of Lemma 2.10). Let W_λ be the orthogonal complement of V_λ in D . We now look for $T_{\lambda,1} : W_\lambda \rightarrow D^\perp$ such that with $T_\lambda : D \rightarrow D^\perp$ defined as $T_{\lambda,0}$ on V_λ and as $T_{\lambda,1}$ on W_λ we have that $\text{graph } T_\lambda \in \mathfrak{S}\mathfrak{A}$. Because of (2.7) this will be the case iff for arbitrary $v, v' \in V_\lambda$ and $w, w' \in W_\lambda$ the quantity

$$(v + w + T_{\lambda,0}v + T_{\lambda,1}w, A(v' + w' + T_{\lambda,0}v' + T_{\lambda,1}w'))_A$$

vanishes. Using (7.2) first and then several times that D and D^\perp are both in $\mathfrak{S}\mathfrak{A}$ (so we can take advantage of (2.7)) while keeping in mind that the ranges of $T_{\lambda,0}$ and $T_{\lambda,1}$ lie in D^\perp , the above expression is equivalent to

$$(v, AT_{\lambda,1}w')_A + (T_{\lambda,0}v, Aw')_A + (w, AT_{\lambda,0}v')_A + (T_{\lambda,1}w, Av')_A \\ + (w, AT_{\lambda,1}w')_A + (T_{\lambda,1}w, Aw')_A$$

In order for this to vanish for all v, v', w, w' it is necessary and sufficient that

$(v, AT_{\lambda,1}w')_A + (T_{\lambda,0}v, Aw')_A = 0$ and $(w, AT_{\lambda,1}w')_A + (T_{\lambda,1}w, Aw')_A = 0$ for all $v \in V_\lambda$ and $w, w' \in W_\lambda$. Letting $T_{\lambda,0}^* : D \rightarrow V_\lambda$ be the adjoint of $T_{\lambda,0}$, the first condition is equivalent to the requirement that $AT_{\lambda,1} = -T_{\lambda,0}^*A$, that is,

$$T_{\lambda,1} = AT_{\lambda,0}^*A.$$

With this definition of $T_{\lambda,1}$ both $(w, AT_{\lambda,1}w')_A$ and $(T_{\lambda,1}w, Aw')_A$ vanish because $W_\lambda \perp V_\lambda$ and A is unitary. Thus $AT_\lambda : D \rightarrow D$ is selfadjoint, and since $T_\lambda \rightarrow 0$ as $\lambda \rightarrow -\infty$,

$$D_\lambda = \text{graph } T_\lambda \in \mathfrak{S}\mathfrak{A}, \quad K_\lambda \cap D_\lambda \neq 0, \quad \text{and } D_\lambda \rightarrow D \text{ as } \lambda \rightarrow -\infty.$$

We have shown that \mathfrak{D}_{D_F} consists of spectrally unstable domains. \square

We end with an alternate argument to Proposition 6.11 that all elements of $\mathfrak{S}\mathfrak{A} \cap U_{D_F^\perp}$ are spectrally stable. Let $D_0 \in \mathfrak{S}\mathfrak{A} \cap U_{D_F^\perp}$ be arbitrary, let $T_0 : D_F^\perp \rightarrow D_F$ be such that $D_0 = \text{graph } T_0$, let $S_0 = AT_0$, and let $M > \|S_0\|$. Then

$$U = \{\text{graph } T : T \in \mathcal{L}(D_F^\perp, D_F), S = AT \text{ selfadjoint}, \|S\| < M\}$$

is a neighborhood of D_0 in $\mathfrak{S}\mathfrak{A}$. There is $\zeta < 0$ such that

$$(F_{\mathcal{D}_F}(\lambda)u, u)_A \leq -M\|u\|_A^2 \quad \forall u \in D_F^\perp, \lambda < \zeta.$$

Let $D \in U$, so $D = \text{graph } T$ with $S = AT : D_F^\perp \rightarrow D_F^\perp$ selfadjoint and $\|S\| < M$. Then

$$((F_{\mathcal{D}_F}(\lambda) - S)u, u)_A \leq (-M + \|S\|)\|u\|_A^2 \quad \forall u \in D_F^\perp, \lambda < \zeta$$

hence $\ker(F_{\mathcal{D}_F}(\lambda) - S) = 0$ if $\lambda < \zeta$. Therefore

$$\text{spec}(A_{\mathcal{D}_T}) \subset [\zeta, \infty)$$

by (6.1).

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