On solvability of systems of vector fields
by
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§1. Introduction. Suppose L is a vector field with smooth coefficients defined in a neighborhood \( \Omega \) of \( 0 \) in \( \mathbb{R}^n \), let \( \sigma \) be its principal symbol, and assume \( L \neq 0 \) everywhere. Then

\[ (P) \quad \forall \xi \in \mathcal{C}_0, \exists \xi(\sigma) \text{ does not change sign along null-bicharacteristics of } \mathcal{R}(\sigma) \]

is a sufficient condition for solvability of \( L \), namely, if \( (P) \) holds, then there exists a neighborhood \( \Omega_0 \) of \( 0 \) in \( \Omega \) such that \( \forall \xi \in L^2(\Omega_0) \exists u \in L^2(\Omega_0) \) such that \( Lu = f \) in \( \Omega_0 \), in the distributional sense. Conversely, if \( L \) is solvable in the above sense, then \( (P) \) holds in \( \Omega_0 \). See [NT] and references therein, and [BF], [H].

If \( N = 2 \) and there is a \( C^\infty \) function \( Z \) on \( \Omega \) with \( LZ = 0 \) and \( dZ(0) \neq 0 \), then the solvability of \( L \), or equivalently, condition \( (P) \), can be related to a certain homological condition on the sets \( Z^{-1}(0) \), the "fibers" of \( Z \). In order to clarify this claim, let us assume that \( Z = x + i\phi(x,t) \) with \( \phi \) real valued, a situation to which one is easily reduced by replacing \( Z \) with a suitable function \( \hat{Z} = e^{Z} (F \text{ holomorphic, defined near } Z(0), \text{ with } F'(Z(0)) \neq 0) \), and choosing an appropriate system of coordinates. The solvability of \( L \) near \( 0 \) is equivalent to the solvability of \( A \) when \( a \) is a smooth nonvanishing function, so we may assume \( L = D_1 + i(\phi_x(x,t)) \). The symbol of \( L \) is \( \sigma = x + i(\phi_x(x,t) - \phi_t(x,t)) \), a null-bicharacteristic of \( \mathcal{R}(\sigma) \) is a curve \( s \to (x+t, x_0(t), D_0) \) with \( D_0 \neq 0 \), and along such a curve we have that \( \mathcal{R}(\sigma) \) does not change sign if and only if for each \( x \), \( \phi_t(x,t) \neq 0 \) does not change sign as \( t \) ranges over any interval \( J \) such that \( \{x + iJ \} \subseteq \Omega \). If we take intervals \( I \) and \( J \) containing \( 0 \), with \( 0 < I < \Omega \), a neighborhood of \( 0 \), we conclude that \( (P) \) implies that \( \phi \) is monotone in \( t \in J \), for each \( x \in I \), or equivalently, that the sets

\[ \{x, t \in I: x = x_0, \phi(x,t) = y_0\} \]

are connected, for each \( y_0 = x_0 + i\phi_0 \). Conversely, if this last statement is true, then \( \phi \) is monotone and thus condition \( (P) \) is satisfied by \( \sigma \) in \( U \).

Connectedness is a homological property and we shall describe in Section 3 sufficient conditions of a similar nature for the solvability of certain systems of vector fields. The latter will be introduced in Section 2, where we also state the problem more precisely. Complete proofs of the theorems stated in Section 3 can be found in [MT].

Related results, concerning either necessary or sufficient conditions, can be found in [T1], [T3], [CH], [T5] and [CT]. In particular, [CT] contains results on necessary conditions for the existence

of solutions, for systems of vector fields which are more general than the ones we treat in [MT].

§2. Statement of the problem. The systems of vector fields we are interested in are local expressions of certain complexes

\[ L_q(C^\infty(M,M_1)) \to C^\infty(M,M_{q+1}), q = 0, 1, ... \]

of first order differential operators between bundles over a \( (C^\infty \text{ paracomplex } N \text{-dimensional}) \) manifold \( M \) which are generalizations of the Cauchy-Riemann complex. The basic datum needed for the construction of the complexes, see [T2] or [T4], is a rank \( n \) subbundle \( V \) of \( CTM \), the complexified tangent bundle of \( M \), having the property that given two vector fields \( X, Y \) of the distribution \( V \), the Lie bracket \([X,Y]\) is also a vector field of \( V \). Given such \( V \), let \( E = V^* \) be the dual bundle, let \( E_0 = \mathcal{A} \mathcal{E} \), and define \( L_q(C^\infty(M,E_0)) \to C^\infty(M,E_{q+1}) \) as follows. If \( u \) is a section of \( E_0 \), i.e., a function, then \( L_q(u)(v) = du(v) \) for \( v \in V \), and if \( u \in C^\infty(M,E_0), q > 0 \), and \( V_1,...,V_{q+1} \) \( \in V_p \), the fiber of \( V \) over \( p \), then with smooth extensions \( V_1,...,V_{q+1} \) in \( V \) we set

\[ (2.1) \quad L_q(u)(V_1,...,V_{q+1}) = (q+1) \sum_{i=0}^{q+1} (-1)^{q+1+i} V_1 u(V_1,...,\hat{V}_i,...,V_{q+1}) + \sum_{i=0}^{q+1} (-1)^{q+1+i} u(V_1,...,\hat{V}_i,...,V_{q+1}) \]

with the right hand side computed at \( p \). Using the formal integrability of \( V \) one verifies that this definition is independent of the extensions \( V_i \) of the \( v_i \) and that \( L_q1L_q = 0 \). We shall drop the subindex from the notation and write simply \( L \).

Many questions may be asked about the complex (2.1). Here we are interested in the local exactness at a point \( p_0 \in M \). Namely, under what conditions is the following true:

\[ (2.2) \quad \text{Given a neighborhood } \Omega \text{ of } p_0 \text{, there exists a neighborhood } \Omega' \text{ of } p_0 \text{ in } \Omega \text{ such that } \forall f \in C^\infty(\Omega,E_0) \text{ with } \mathcal{L}f = 0 \exists u \in C^\infty(\Omega,E_{-1}) \text{ such that } Lu = f \text{ in } \Omega'. \]

Our most fundamental hypothesis on \( L \) or \( V \), will be that the structure bundle \( V \) be "locally integrable" near \( p_0 \). This means that the annihilator \( V^1 \) of \( V \) in \( CTM \) is generated, near \( p_0 \), by the differentials of \( m = n \) smooth functions. That is, there are smooth function \( Z_1,...,Z_m \) defined in a neighborhood of \( \Omega \) with \( dZ_1,...,dZ_m \) independent at every point, such that \( L[Z_j - f] \quad \text{for } j = 1,...,m \). The existence of such sets of "solutions" is by no means automatic: there are examples (see [N], [JT], [T3]) of structures \( V \) where no such sets exist in any neighborhood of a point.

We shall also assume \( m = 1 \), although there are still a number of general statements one can make concerning the situation \( m > 1 \). We write \( Z \) instead of \( Z_1 \). Since the sufficient condition for solvability will involve the sets \( Z^{-1}(0), z_0 \in C \), it is important to point out that these sets are determined by \( V \) itself, in the following sense: any point \( p \) in the domain of \( Z \) has a neighborhood
U such that if h is a C^1 solution of Lh = 0 with the same domain as Z then \( \forall z_0 \in \mathbb{C}, h \) is constant on the sets \( Z^{-1}(z_0) \cap U \). This was shown in [BT].

If \( d\Omega \) generates \( V^1 \) over \( \Omega \), then the same is true of \( \zeta + \eta, \xi + \sigma, \eta + \tau \in \mathbb{C} \). Thus by choosing \( \zeta \) and \( \eta \) we may assume \( Z(p_0) = 0 \) and \( d\mathcal{R}(Z) \neq 0 \) at \( p_0 \). Let \( X = \mathcal{R}(Z), t_1, \ldots, t_n \) be coordinates centered at \( p_0 \), let \( Z = Z(x, t)Z_x(0, 0) \). Then \( x = \mathcal{R}(Z), t_1, \ldots, t_n \) is a local chart centered at \( p_0 \), \( d\mathcal{Z} \) spans \( V^1 \) over \( \Omega \), and \( Z = x + i\varphi(x, t), \varphi \) real valued, smooth and \( \varphi(0,0) = \varphi_x(0,0) = 0 \). We shall assume that we have coordinates \( x, t_1, \ldots, t_n \in \Omega \) centered at \( p_0 \) with respect to which \( Z = x + i\varphi \) as described above. A basis for \( V \) over \( \Omega \) is then given by the commuting vector fields

\[
(2.3) \quad L_1 = \frac{-i\varphi(x, t)}{1 + \varphi^2(x, t)} \partial_x.
\]

The forms \( d\Omega, dt_1, \ldots, dt_n \) span \( C^\ast(\Omega) \), so \( V^1 \subset \Omega \) is isomorphic to the span of the \( dt_i \), and any element \( \varphi \in C^\infty(\Omega, \mathbb{R}) \) can be identified as an element \( \bar{\varphi} = \sum \int_{t_1}^{t_n} \bar{\varphi}(x, t_i) dt_i \) with \( \bar{\varphi} \in C^\infty(\Omega) \).

With this identification we have

\[
(2.4) \quad L_1 \sum \int_{t_1}^{t_n} \bar{u}(x, t_i) dt_i = \sum \int_{t_1}^{t_n} \int_{t_1}^{t_n} \bar{L}_1 \bar{u}(x, t_i) dt_i \partial_{t_i}.
\]

Notice that if \( \varphi(0,0) \neq 0 \) then the complex \( L_1 \) is elliptic near \( p_0 \) and (2.2) holds for all \( q \). Thus we shall always assume \( \varphi(0,0) = 0 \).

§3. Results. Throughout this section \( \Omega \) will be a neighborhood of 0 in \( \mathbb{R}^n \). It is assumed that \( \varphi = \Phi(\Omega) \), real valued and satisfying \( \varphi(0,0) = \Phi(0,0) = 0 \). The vector bundle \( V \) is spanned by the vector fields (2.3). \( E \) is the subbundle of \( CT^\ast \Omega \) spanned by the differentials \( dt_i \) and \( L_1 C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}) \), \( q = 0, 1, \ldots, n \) is given by (2.4).

It was shown in [T3] that if the coefficients of the vector fields \( L_i \) are real-analytic and every point \( p \in \Omega \) has a basis \( N_p \) of neighborhoods such that \( Z^{-1}(z_0) \cap U \) is connected whenever \( U \in N_p \), then (2.2) holds. As regards to top degree, [CH] showed that if there exists a neighborhood \( U \) of 0 such that the sets \( Z^{-1}(z_0) \cap U \) have trivial \( (n-1) \)-homology, and again \( Z \) is real-analytic, then (2.2) holds. The condition that \( Z \) be real analytic in the case of top degree forms was removed in [T5]. In [MT] we consider the condition

\[
(3.1) \quad \text{There exist two bases of neighborhoods } \{U_k\}_{k \in \mathbb{N}}, \{U_k\}_{k \in \mathbb{N}} \text{ of } 0 \text{ with } U_k \subset U_k, \quad U_k = \text{isomorphic with } \Phi(\Omega) \text{ homologically trivial, such that } \forall z \in \mathbb{C}, \text{ if } Z^{-1}(z_0) \cap U_k \neq \emptyset, \text{ then the } d\text{-homology of } Z^{-1}(z_0) \cap U_k \text{ is trivial.}
\]

We show

**THEOREM 3.1.** If (3.1) holds, then (2.2) holds.

**THEOREM 3.2.** Theorem 3.1 holds if in (3.1) holds, then (2.2) holds.

**Theorem (3.1) improves the result on sufficiency in [T3] in that the vector fields need not have real-analytic coefficients. On the other hand, Theorem 3.2, in the case \( q = n \), is weaker than the results in [CH] and [T5]; the additional condition on the \( (n-2) \)-homology of the fibers is necessary there, and in fact, there are examples of solvable systems in degree \( n \) where (3.1) holds but not (3.2).

We shall end with an outline of our proof of theorem 3.1. Assume (3.1) holds. There is no loss in assuming that the neighborhoods \( U_k \) have the form \( \mathbb{R}^n \), where the \( B_k \) are balls, so we shall assume this choice from now on. Fix \( x_0, y_0 \in \mathbb{R}, \) let \( z_0 = x + iy_0 \). We claim that if \( \varphi \) is a \( C^\infty \) locally constant function defined in the set \( F = \{ t \in \mathbb{R}; \varphi(x_0, t) > y_0 \} \) then \( \varphi \) is constant on \( C^\infty(F) = \{ t \in B_k; \varphi(x_0, t) < y_0 \} \) (if this set is nonempty). Indeed, either \( \varphi(x_0, t) < y_0 \) when \( t \in B_k \) and \( \varphi \) is constant on the connected set \( B_k \) or there exists \( t_0 \in B_k \) where \( \varphi(x_0, t_0) = y_0 \) in which case one can show, using the augmented Mayer-Vietoris sequence, Sard's Lemma and the conditions on \( U_k \), \( \Phi \) and the fibers of \( Z \) expressed by (3.1), that \( F \) is itself a connected. A similar statement holds for the sets \( F^2 \). Let \( t: \Omega \to \mathbb{C} \) be the map \( i(x, t) = (2t(x, t)) \), note that

\[
i_0(L_1) = \partial_t - \frac{2\varphi \varphi_x}{1 + \varphi^2}.
\]

Denote by \( \pi: \mathbb{C} \to \mathbb{C} \) the projection and let \( \rho: \mathbb{C} \to \mathbb{R} \) be the map \( \rho(z, t) = (x, t) \); we write \( z = x + iy \). Let

\[
O^+ = \{ (z, t); \rho(z, t) \in \Omega, \varphi > \varphi(z, t) \}
\]

and \( O^- = \{ (z, t); \rho(z, t) \in \Omega, \varphi < \varphi(z, t) \} \).

and let \( \Sigma = \vdash (\Omega) \). There exists a neighborhood \( W \) of 0 in \( \mathbb{C} \) such that for any \( f = \sum \int_{t_1}^{t_n} \int_{t_1}^{t_n} \bar{f}(x, t_i) dt_i \) \( f \in C^\infty(\Omega), \) there exists \( f^\delta = \sum \int_{t_1}^{t_n} \int_{t_1}^{t_n} \bar{f}^\delta(x, t_i) dt_i \in C^\infty(\Omega) \) smooth up to \( \partial \Omega \), holomorphic in \( z \) and such that \( \int f^\delta f^\varphi_1 \varphi_2 = \int \varphi_1 f^\varphi_2 \) (\( \varphi \) smooth). Furthermore, on can choose \( W \) so that if \( L_f = 0 \) in \( \Omega \) then \( d\varphi^\delta = 0 \). Here \( d \) indicates differentiation in the \( t \) variables only, \( z \) is a parameter. The construction of \( f^\varphi \) and \( f^\varphi \) in [MT] is such that not only is \( d\varphi^\delta = 0 \), but in fact, for each \( \varphi \), \( f^\varphi = i \) is exact. With this information and the validity of (3.1) one can find another neighborhood \( W_1 \) of 0 in \( W \), independent of \( f \), and \( u^2 \in C^\infty(\Omega) \) such that

\[
d^2 u^2 = f^2 \in W_1 \cap \mathbb{C}^2.
\]
with $C, \kappa, N$ positive, depending on $\alpha, \beta$ but not on $f$, and $K$ some fixed compact set in $\Omega$. Of course $u^2$ need not be holomorphic. However, $v^2 = \partial u^2/\partial \xi$ is necessarily locally constant on each set $\pi^{-1}(\eta_0) \cap W_{\eta} \cap \mathbf{O}^2$, since its differential in $\xi$ vanishes. Choose $k$ and $s$ such that $W_2 = \{ (x,t) : p(t)x + \xi \in U_k \subseteq J_\kappa \mathbf{X}_k, |\xi| < s \} \subseteq \Omega$ and let $W_3 = \{ (x,t) : p(t)x + \xi \in J_\kappa \mathbf{X}_k, |\xi| < s \}$. Then $v^2|_{W_2 \cap \mathbf{O}^2}$ is constant on the sets $\pi^{-1}(\eta_0) \cap W_{\eta} \cap \mathbf{O}^2$. Indeed, $v^2(\eta_0)$ is locally constant on $\{ t \in K : \phi_{\eta_0}(x) < \gamma_0 \} = \pi^{-1}(\eta_0) \cap W_{\eta} \cap \mathbf{O}^2$, thus constant on $\{ t \in K : \phi_{\eta_0}(x) < \gamma_0 \} = \pi^{-1}(\eta_0) \cap W_{\eta} \cap \mathbf{O}^2$, and similarly for $\eta$. Let $D^2 = \eta(W_{\eta} \cap \mathbf{O}^2)$, $\eta^2$ the function on $D^2$ defined by $v^2$. If $\Delta$ is a sufficiently small neighborhood of 0 in $C$, depending only on $W_3$, one can find $v^2 \in C^{(\Delta^2 \mathbf{O}^2)}$ such that $\partial v^2/\partial \xi = \eta^2$ in $\Delta^2 \mathbf{O}^2$ and such that

$$D^2 v^2 \leq C \chi - \phi(x) \Gamma(x) \leq C N $$

where $W_4 = \pi^{-1}(\Delta^2 \mathbf{O}^2)$, for certain numbers $C, M$ and $N$ and compact set $N$ in $\Omega$ independent of $f$. Let $w^2 = \pi^2 v^2$.

If $\bar{u}^2 = u^2 - w^2$, in $W_{\eta} \cap \mathbf{O}^2$, then $\bar{u}^2$ is smooth, holomorphic in $z$, and satisfies an estimate of the form $(3.2)$, again with $C, \kappa, N$ and $K$ independent of $f$. This implies that $\bar{u}^2$ has distributional boundary values $\mu^2$ on $\partial W_4 = \pi^{-1}(\eta_0) \cap \mathbf{O}^2$, and $L(\mu^2) = f^2$ in $\eta_0$, thus $L(\mu^2 + \mu_1^2) = f^2$ in that set. The distribution $\mu^2 + \mu_1^2$ has finite order independent of $f$, and one can use this together with the integrability of the structure $V$ to show that in a smaller neighborhood $\Omega'$ of 0 in $\Omega_0$ there is in fact a $C^2$ solution $u$ of $Lu = f$.

References


