A GYSIN SEQUENCE FOR MANIFOLDS WITH $\mathbb{R}$-ACTION

GERARDO A. MENDOZA

ABSTRACT. We associate an exact sequence involving the cohomology groups of a pair of differential complexes to any pair $(\mathcal{N}, \mathcal{T})$ where $\mathcal{N}$ is a closed connected smooth manifold and $\mathcal{T}$ a real nowhere vanishing smooth vector field on $\mathcal{N}$ that admits an invariant metric. The orbits of $\mathcal{T}$ need not be closed. The sequence is a natural generalization of the classical Gysin sequence (for circle bundles) in real cohomology.

1. INTRODUCTION

In this note we shall associate an exact sequence in certain cohomology to any pair $(\mathcal{N}, \mathcal{T})$ where $\mathcal{N}$ is a smooth compact manifold without boundary and $\mathcal{T}$ a real nowhere vanishing smooth vector field on $\mathcal{N}$ that admits an invariant Riemannian metric. The orbits of $\mathcal{T}$ need not be closed. The procedure leads to the classical Gysin sequence in real cohomology when $\mathcal{N}$ is the circle bundle of a complex line bundle, somewhat more generally, to an exact sequence in real cohomology whenever the orbits of the action are circles.

A brief discussion of, and comments about, the classical case will place the results in context. Let $\mathcal{B}$ be a connected compact manifold and $E \to \mathcal{B}$ a complex vector bundle of rank $r \leq \dim \mathcal{B}/2$ with sphere bundle $\rho : SE \to \mathcal{B}$. The Gysin sequence of SE (see Gysin [2], Milnor and Stasheff [7], Bott and Tu [1], Hatcher [3]) in integral cohomology,

$$\cdots \to H^{q-2r}(\mathcal{B}, \mathbb{Z}) \xrightarrow{\mathbf{e}\smile\cdot} H^q(\mathcal{B}, \mathbb{Z}) \xrightarrow{\rho^*} H^q(SE, \mathbb{Z}) \xrightarrow{\rho_*} H^{q-2r+1}(\mathcal{B}, \mathbb{Z}) \to \cdots \tag{1.1}$$

serves to define the Euler class, $\mathbf{e} \in H^{2r}(\mathcal{B}, \mathbb{Z})$, of E. This class is also the r-th Chern class of E. By a well known induction process this definition can be used to define the total Chern class, $\mathrm{c}(E) = 1 + c_1(E) + \cdots + c_r(E)$, of E. The total Chern class represents an obstruction to the vector bundle being trivial (although the vanishing of $\mathrm{c}_j(E)$ for all $j \geq 1$ is not sufficient to ascertain the triviality of E, see [9]). In the special case where the vector bundle is a line bundle, its Euler class, $\mathbf{e} = \mathrm{c}_1(E)$, does determine it; this property is attributable to the fact that U(1), the unitary group in dimension 1, is abelian. Thus the first Chern class of E plays, in the case of a line bundle, the dual role of being the element (except for sign) which makes the sequence (1.1) exact, and of classifier of the line bundle. Of course to discuss the Euler class $\mathcal{B}$ need not be a manifold, nor compact; it is only required that $E \to \mathcal{B}$ be of finite type. The more restricted context of this paragraph fits the assumptions on the rest of the paper with $\mathcal{N}$ being SE and $\mathcal{T}$ being the infinitesimal generator of the $\mathbb{R}$-action $t \mapsto e^{it}p$, $p \in SE$.

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Denote by $\mathscr{F}$ the class of pairs $(\mathcal{N},\mathcal{T})$ as in the first paragraph:

> $\mathcal{N}$ is a connected smooth closed manifold and $\mathcal{T}$ a real nowhere vanishing smooth vector field on $\mathcal{N}$ that admits an invariant metric.

For a class of examples of this kind of pairs see the end of this introduction.

Let $(\mathcal{N},\mathcal{T})\in\mathscr{F}$. Write $\mathcal{O}_p$ for the orbit of $\mathcal{T}$ through p. Defining $p_1\sim p_2$ if $p_1\in\overline{\mathcal{O}}_{p_2}$ gives a relation of equivalence on $\mathcal{N}$ because of the existence of a $\mathcal{T}$-invariant metric. We will write $\mathcal{B}$ for the quotient space and refer to it as the base space of $\mathcal{N}$ (more precisely, of $(\mathcal{N},\mathcal{T})$); it is a compact Hausdorff space but may not be a manifold.

Define the following two relations of equivalence on $\mathscr{F}$ (see [6]). Say that $(\mathcal{N}',\mathcal{T}')$ is locally equivalent to $(\mathcal{N},\mathcal{T})$ if there are open covers $\{U'_a\}_{a\in A}$ of $\mathcal{N}'$, $\{U_a\}_{a\in A}$ of $\mathcal{N}$ by $\mathcal{T}'$, resp. $\mathcal{T}$ invariant sets, and diffeomorphisms $h_a: U'_a\to U_a$ such that $dh_a(\mathcal{T}')=\mathcal{T}$ and

$$\forall a,b\in A\ \forall p\in U_a\cap U_b: h_a\circ h_b^{-1}(p)\in\overline{\mathcal{O}}_p.$$

Also say that $(\mathcal{N}',\mathcal{T}')$ and $(\mathcal{N},\mathcal{T})$ are globally equivalent if there is a diffeomorphism $h:\mathcal{N}'\to\mathcal{N}$ such that $dh(\mathcal{T}')=\mathcal{T}$. Fix $(\mathcal{N},\mathcal{T})\in\mathscr{F}$, let $\mathscr{F}_{\mathcal{N},\mathcal{T}}$ be the set of elements of $\mathscr{F}$ which are locally equivalent to $(\mathcal{N}',\mathcal{T}')$. Global equivalence defines a relation of equivalence within this set. Let $\mathcal{B}$ be the base space of $\mathcal{N}$. The base space of each element of $\mathscr{F}_{\mathcal{N},\mathcal{T}}$ is homeomorphic to $\mathcal{B}$.

Theorem 1.1 (Theorem 3.9 of [6])**.** *The set of global equivalence classes of elements of $\mathscr{F}_{\mathcal{N},\mathcal{T}}$ is in one to one correspondence with the elements of $H^2(\mathcal{B};\mathbb{Z}^d)$ for some d.*

The number d is defined in (7.1). Thus the elements of $H^2(\mathcal{B};\mathbb{Z}^d)$ are like first Chern classes in their role as classifiers of elements of $\mathscr{F}_{\mathcal{N},\mathcal{T}}$. They are relative Chern classes because elements within a local equivalence class are being compared to an arbitrarily fixed element in the class, in contrast with line bundles and their regular Chern classes where the comparison is with the trivial line bundle.

We shall define an Euler class for $(\mathcal{N},\mathcal{T})\in\mathscr{F}$ as an element of the second cohomology group of certain subcomplex, see (2.3), of the de Rham complex of $\mathcal{N}$. This subcomplex reduces to the de Rham complex of the base when $\mathcal{N}$ is a circle bundle over a smooth manifold. The defining property of $\mathbf{e}$ is the following theorem, the central result of this paper.

Theorem 1.2. *Let $(\mathcal{N},\mathcal{T})\in\mathscr{F}$, let $\mathcal{B}$ be the base space of $\mathcal{N}$ and let $\rho:\mathcal{N}\to\mathcal{B}$ be the projection map. There is an exact sequence*

$$\cdots\to H^{q-2}_{\mathrm{dR}}(\mathcal{B})\xrightarrow{\mathbf{e}\wedge\cdot}H^q_{\mathrm{dR}}(\mathcal{B})\xrightarrow{\rho^*}H^q_{\mathrm{dR}}(\mathcal{N})\xrightarrow{\rho_*}H^{q-1}_{\mathrm{dR}}(\mathcal{B})\to\cdots. \tag{1.2}$$

The spaces $H^q_{\mathrm{dR}}(\mathcal{N})$ are the de Rham cohomology groups of $\mathcal{N}$, the groups $H^q_{\mathrm{dR}}(\mathcal{B})$ are the cohomology groups of the complex (2.3) defined in the next section, the maps ρ^* and ρ_* are defined in Section 5, and the theorem is proved in Section 6.

The theorem is somewhat unsatisfactory in that it appears to be, to some extent, a theorem in the scalar realm (however, see Theorem 7.1) that does not reflect the fact (notwithstanding that the present context is in the realm of real cohomology) that the classifier group established in Theorem 1.1 is cohomology with values in $\mathbb{Z}^d$ rather than $\mathbb{Z}$. Note also that since the coefficients are real, any nonzero multiple of $\mathbf{e}$ also works.

We shall in fact prove what appears to be a more general result than Theorem 1.2, motivated by the following considerations in the context of the classical case of a manifold $\mathcal{B}$ and a complex line bundle $E \to \mathcal{B}$ with a given Hermitian metric and circle bundle SE. The space SE is an m-fold covering space of the circle bundle of the tensor product $E^{\otimes m}$ with covering map $\wp_m : SE \to SE^{\otimes m}$,

$$\wp_m(p) = \begin{cases} p \otimes \ldots \otimes p & \text{if } m \geq 0 \\ p^* \otimes \ldots \otimes p^* & \text{if } m < 0, \end{cases}$$

($|m|$ factors in either case) with $p^* \in E^*_{\rho(p)}$ such that $\langle p^*, p\rangle = 1$. The space $C^\infty(SE^{\otimes m})$ is isomorphic to $\wp_m^* C^\infty(SE^{\otimes m})$. This and the property $\wp_m(e^{it}p) = e^{imt}\wp_m(p)$ allows us to view the space of smooth functions on $SE^{\otimes m}$ as the space

$$\{\phi \in C^\infty(SE) : t \mapsto \phi(e^{it}p) \text{ is } 2\pi/m\text{-periodic for each } p \in SE\}.$$

Letting $\mathfrak{a}_t(p) = e^{it}p$ we have, more generally, that $C^\infty(SE^{\otimes m}; \bigwedge^q SE^{\otimes m})$ is isomorphic to

$$\{\phi \in C^\infty(SE, \textstyle\bigwedge^q E) : t \mapsto \mathfrak{a}_t^*\phi \text{ is } 2\pi/m\text{-periodic}\}. \tag{1.3}$$

We also note that if ψ is a section of $E^{\otimes m}$, then it determines (and is determined by) a liner map on each fiber of $E^{-\otimes m}$, the dual bundle, so $\wp_{-m}^*\psi$ is defined as a function on SE and satisfies

$$(\wp_{-m}^*\psi)(e^{it}p) = e^{-imt}(\wp_{-m}^*\psi)(p)$$

equivalently, solves

$$\mathcal{T}(\wp_{-m}^*\psi) - im\wp_{-m}^*\psi = 0.$$

More generally, smooth q-forms on $\mathcal{B}$ with values in $E^{\otimes m}$ can be viewed as elements of the space

$$\{\phi \in C^\infty(SE; \rho^* \textstyle\bigwedge^q \mathcal{B}) : \mathcal{L}_\mathcal{T}\phi - im\phi = 0\}. \tag{1.4}$$

On the plus side, these observations, certainly familiar to experts in algebraic geometry, permit the analysis of objects such as de Rham cohomology to be carried out in SE rather than $SE^{\otimes m}$. We take advantage of these observations, especially (1.3), to define the analogue of the de Rham complex for tensor products in our more general context, see (2.2). On the minus side, in what pertains this paper there is no real gain because, as a consequence of exactness of the Gysin sequence (or merely by the fact that the Euler class of E^m is m times that of E), the de Rham cohomology of $SE^{\otimes m}$ is identical to that of SE. Also in our more general context the cohomology groups will be the same (Theorem 2.2).

We end this introduction with an example of an element of $\mathscr{F}$.

Example 1.3. Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$, let $\tau^1, \ldots, \tau^{n+1}$ be nonzero real numbers. The vector field

$$\mathcal{T}' = i\sum_{j=1}^{n+1} \tau^j \left(z^j \frac{\partial}{\partial z^j} - \overline{z}^j \frac{\partial}{\partial \overline{z}^j} \right)$$

is tangent to the sphere: $\mathcal{T}'|z|^2 = 0$. To verify that the pair $(S^{2n+1}, \mathcal{T}')$ belongs to $\mathscr{F}$ we only need to observe that the standard Riemannian metric of the sphere is $\mathcal{T}'$-invariant.

Remark 1.4. On can show that if $(\mathcal{N},\mathcal{T})\in\mathscr{F}$, then there is $N\in\mathbb{N}$ and a smooth embedding $F:\mathcal{N}\to S^{2N+1}\subset\mathbb{C}^{N+1}$ such that

$$F_*\mathcal{T}=\mathcal{T}'$$

with $\mathcal{T}'$ as in the example. The assertion, whose proof will appear elsewhere, can be viewed as an analogue of the statement that every complex line bundle over a compact manifold $\mathcal{B}$ can be realized as the pullback of the canonical line bundle of complex projective space of some dimension.

2. Set up

Throughout the rest of the paper we work with a fixed element $(\mathcal{N},\mathcal{T})$ of $\mathscr{F}$. We write $\mathfrak{a}_t$ for the one-parameter group of diffeomorphisms defined by $\mathcal{T}$ and $\mathcal{O}_p$ for the orbit of $\mathcal{T}$ through p. Given any pair $(p,p')\in\mathcal{N}\times\mathcal{N}$ define $p'\sim p$ if $p'\in\overline{\mathcal{O}}_p$. As was already mentioned, $\sim$ is a relation of equivalence because of the of the assumption on the existence of a $\mathcal{T}$-invariant metric. The space $\mathcal{B}=\mathcal{N}/{\sim}$ is a compact Hausdorff space, not necessarily a manifold. We let

$$\rho:\mathcal{N}\to\mathcal{B}$$

be the quotient map.

Let

$$\mathbf{i}_\mathcal{T}:{\textstyle\bigwedge}^q\mathcal{N}\to{\textstyle\bigwedge}^{q-1}\mathcal{N} \tag{2.1}$$

be interior multiplication by $\mathcal{T}$. If $\mathcal{H}^*$ is the kernel of $\mathbf{i}_\mathcal{T}$ in $\mathbb{C}T^*\mathcal{N}$, then the kernel of (2.1) is ${\textstyle\bigwedge}^q\mathcal{H}^*$. Let $\mathcal{L}_\mathcal{T}$ denote Lie derivative with respect to $\mathcal{T}$. For any number τ and nonnegative integers q and k or $k=\infty$ define, consistently with (1.4),

$$C^k(\mathcal{B};E^\tau\otimes{\textstyle\bigwedge}^q\mathcal{B})=\{\phi\in C^k(\mathcal{N};{\textstyle\bigwedge}^q\mathcal{H}^*):\mathcal{L}_\mathcal{T}\phi-i\tau\phi=0\}. \tag{2.2}$$

The symbol E is a place holder which becomes a Hermitian line bundle with circle bundle $\mathcal{N}$ if the action determined by $\mathcal{T}$ make $\mathcal{N}$ into a principal S^1-bundle. The symbol E^τ is a reference to a virtual (not in the sense of K-theory) line bundle which becomes the m-th tensor power of a line bundle when $\tau=m$ and E is an actual line bundle. The number τ can be any complex number but must be a real number if $C^k(\mathcal{B};E^\tau\otimes{\textstyle\bigwedge}^q\mathcal{B})\neq 0$; it need not be an integer, but the spaces are nonzero only for τ in a countable subset of $\mathbb{R}$.

If τ, q, and k are all zero, the space (2.2) is canonically isomorphic to $C(\mathcal{B})$. In general when $\tau=0$ we will use the notation $C^k(\mathcal{B})$ if $q=0$ and $C^k(\mathcal{B};{\textstyle\bigwedge}^q\mathcal{B})$ when $q>0$ for the space (2.2).

The space $C(\mathcal{B};E^\tau\otimes{\textstyle\bigwedge}^q\mathcal{B})$ is a $C(\mathcal{B})$-module. It is not necessarily a finitely generated projective module, so by a theorem of Swan [12] it need not be the space of continuous sections of a vector bundle over $\mathcal{B}$.

Since $\mathcal{L}_\mathcal{T}$ commutes with differentiation and $\mathbf{i}_\mathcal{T}d=d\mathbf{i}_\mathcal{T}-\mathcal{L}_\mathcal{T}$, there is a complex

$$\cdots\to C^\infty(\mathcal{B};{\textstyle\bigwedge}^q\mathcal{B})\xrightarrow{d_\mathcal{B}}C^\infty(\mathcal{B};{\textstyle\bigwedge}^{q+1}\mathcal{B})\to\cdots \tag{2.3}$$

where $d_\mathcal{B}$ is the restriction of d to $C^\infty(\mathcal{B};{\textstyle\bigwedge}^q\mathcal{B})$. We shall denote the cohomology groups of (2.3) by $H^q_{\mathrm{dR}}(\mathcal{B})$ even though (2.3) need not be the de Rham complex, nor its cohomology groups isomorphic to the real cohomology groups of $\mathcal{B}$. The groups $H^q_{\mathrm{dR}}(\mathcal{B})$ are the ones appearing in (1.2)

The following proposition links, in a special case, the cohomology groups of (2.3) with the real cohomology groups of $\mathcal{B}$.

Theorem 2.1. *If the orbits of $\mathcal{T}$ are circles then the cohomology groups of the complex* (2.3) *are canonically isomorphic to the Čech cohomology groups of $\mathcal{B}$ with real coefficients.*

This reduces to the de Rham Theorem [10] (see also [11]) when $\mathcal{N}$ is the circle bundle of a complex line bundle (in which case $\mathcal{B}$ is a smooth compact manifold and de Rham's theorem makes sense). The proofs of this proposition, the next, and subsequent theorem in this sections will be given later.

Let $\mathfrak{a}_t$ be the one-parameter group generated by $\mathcal{T}$. Following (1.3), define, for any τ, q and k as above (but now $\tau \neq 0$),

$$C^k(\mathcal{N}^\tau; {\textstyle\bigwedge}^q\mathcal{N}^\tau) = \{\phi \in C^k(\mathcal{N}; {\textstyle\bigwedge}^q\mathcal{N}) : t \mapsto \mathfrak{a}_t^*\phi \text{ is } 2\pi/\tau\text{-periodic}\}$$

The notation $\mathcal{N}^\tau$ is a reference to the circle bundle of a tensor power of a complex line bundle.

Using that $\mathfrak{a}_t^*$ commutes with d we see that there is a complex

$$\cdots \to C^\infty(\mathcal{N}^\tau; {\textstyle\bigwedge}^q\mathcal{N}^\tau) \xrightarrow{d_\tau} C^\infty(\mathcal{N}^\tau; {\textstyle\bigwedge}^{q+1}\mathcal{N}^\tau) \to \cdots \tag{2.4}$$

where d_τ is the restriction of d to $C^\infty(\mathcal{N}^\tau; \bigwedge^q\mathcal{N}^\tau)$. We will write $H^q_{\mathrm{dR}}(\mathcal{N}^\tau)$ for the cohomology groups of (2.4). These are not a priori the real cohomology groups of $\mathcal{N}$ (nor of $\mathcal{N}^\tau$, which is a virtual object anyway). That they are, is a consequence of the following proposition and de Rham's theorem.

Proposition 2.2. *The cohomology groups of the complex* (2.4) *are canonically isomorphic to those of the de Rham complex of $\mathcal{N}$.*

Let θ be any smooth one-form on $\mathcal{N}$ with the properties

$$\langle\theta, \mathcal{T}\rangle = 1 \text{ and } \mathcal{L}_\mathcal{T}\theta = 0. \tag{2.5}$$

Then $d\theta \in C^\infty(\mathcal{B}, \bigwedge^q\mathcal{B})$: $\mathcal{L}_\mathcal{T} d\theta = d(\mathcal{L}_\mathcal{T}\theta) = 0$. Since $d\theta$ is obviously $d_\mathcal{B}$-closed, it defines an element $\mathbf{e}_\tau = [\tau/2\pi\, d\theta] \in H^2_{\mathrm{dR}}(\mathcal{B})$. The factor $\tau/2\pi$ guarantees consistency with the classical case but is otherwise irrelevant.

In view of Proposition 2.2, the following theorem is equivalent to Theorem 1.2.

Theorem 2.3. *Suppose that $\tau \neq 0$. There is an exact sequence*

$$\cdots \to H^q_{\mathrm{dR}}(\mathcal{B}) \xrightarrow{\mathbf{e}_\tau\wedge\cdot} H^{q+2}_{\mathrm{dR}}(\mathcal{B}) \xrightarrow{\rho^*} H^{q+2}_{\mathrm{dR}}(\mathcal{N}^\tau) \xrightarrow{\rho_*} H^{q+1}_{\mathrm{dR}}(\mathcal{B}) \to \cdots \tag{2.6}$$

The homomorphisms ρ^*, ρ_* are defined in Section 5. When $\mathcal{N}$ is the circle bundle of a complex line bundle $E \to \mathcal{B}$ this is the Gysin sequence in real cohomology for E^τ where τ is an integer. Note that the specific value of τ (which is nonzero) does not play an essential role.

3. Decompositions

Let g be an arbitrarily chosen $\mathcal{T}$-invariant Riemannian metric. Then $g(\mathcal{T}, \mathcal{T})$ is smooth, positive and $\mathcal{T}g(\mathcal{T}, \mathcal{T}) = 0$, so we may assume $g(\mathcal{T}, \mathcal{T}) = 1$. Assuming this, the 1-form $\theta = \mathcal{T}\rfloor g$ satisfies (2.5).

Define

$$\Pi_\theta : {\textstyle\bigwedge}^q\mathcal{N} \to {\textstyle\bigwedge}^q\mathcal{H}^*$$

by

$$\Pi_\theta\phi = \phi - \theta \wedge \mathbf{i}_\mathcal{T}\phi.$$

Since $\mathbf{i}_{\mathcal{T}}(\theta)=1$, Π_θ is the projection on $\bigwedge^q\mathcal{H}^*$ with kernel $\theta\wedge\bigwedge^{q-1}\mathcal{N}$. If $\phi\in\bigwedge^q\mathcal{N}$, then

$$\phi=\Pi_\theta\phi+\theta\wedge\mathbf{i}_{\mathcal{T}}\phi$$

gives the decomposition

$$\textstyle\bigwedge^q\mathcal{N}=\bigwedge^q\mathcal{H}^*\oplus(\theta\wedge\bigwedge^{q-1}\mathcal{H}^*).$$

If $\phi\in C^\infty(\mathcal{N};\bigwedge^q\mathcal{H}^*)$, then

$$d\phi=\Pi_\theta d\phi+\theta\wedge\mathcal{L}_{\mathcal{T}}\phi.$$

Define

$$d_\theta:C^\infty(\mathcal{N};\textstyle\bigwedge^q\mathcal{H}^*)\to C^\infty(\mathcal{N};\bigwedge^{q+1}\mathcal{H}^*),\quad d_\theta=\Pi_\theta\circ d.$$

Thus, if $\phi^0\in C^\infty(\mathcal{N};\bigwedge^q\mathcal{H}^*)$ and $\phi^1\in C^\infty(\mathcal{N};\bigwedge^{q-1}\mathcal{H}^*)$, then

$$d(\phi^0+\theta\wedge\phi^1)=d_\theta\phi^0+d\theta\wedge\phi^1+\theta\wedge(\mathcal{L}_{\mathcal{T}}\phi^0-d_\theta\phi^1).$$

Define $\mathbf{\Theta}\phi=d\theta\wedge\phi$. Since $\mathbf{i}_{\mathcal{T}}d\theta=0$,

$$\mathbf{\Theta}:C^\infty(\mathcal{N};\textstyle\bigwedge^{q-1}\mathcal{H}^*)\to C^\infty(\mathcal{N};\bigwedge^{q+1}\mathcal{H}^*),\quad d\circ\mathbf{\Theta}=\mathbf{\Theta}\circ d$$

and we may view d as the operator

$$d=\begin{bmatrix}d_\theta & \mathbf{\Theta}\\ \mathcal{L}_{\mathcal{T}} & -d_\theta\end{bmatrix}:\begin{matrix}C^\infty(\mathcal{N};\bigwedge^q\mathcal{H}^*)\\ \oplus\\ C^\infty(\mathcal{N};\bigwedge^{q-1}\mathcal{H}^*)\end{matrix}\to\begin{matrix}C^\infty(\mathcal{N};\bigwedge^{q+1}\mathcal{H}^*)\\ \oplus\\ C^\infty(\mathcal{N};\bigwedge^{q}\mathcal{H}^*).\end{matrix}\tag{3.1}$$

The operators d_θ do not form a complex unless $d\theta=0$, since $d_\theta^2+\mathbf{\Theta}\mathcal{L}_{\mathcal{T}}=0$. However the restrictions of d_θ to the spaces $C^\infty(\mathcal{B};\bigwedge^q\mathcal{B})$ do form a complex and are in fact identical to the operators $d_{\mathcal{B}}$ already introduced. Indeed, if $\phi\in C^\infty(\mathcal{B};\bigwedge^q\mathcal{B})$ then by definition $\mathbf{i}_{\mathcal{T}}\phi=0$ and $\mathcal{L}_{\mathcal{T}}\phi=0$. Using $\phi^0=\Pi_\theta\phi=\phi$ and $\phi^1=\mathbf{i}_{\mathcal{T}}\phi$ in (3.1) we get $d\phi=d_\theta\phi$.

Let $\mathfrak{m}$ be the Riemannian density on $\mathcal{N}$. The spaces $L^2(\mathcal{N};\bigwedge^q\mathcal{N})$ is defined using the Riemannian metric and the density $\mathfrak{m}$. The Laplacian of the de Rham complex is

$$\Delta=\begin{bmatrix}\Delta_\theta-\mathcal{L}_{\mathcal{T}}^2+\mathbf{\Theta}\mathbf{\Theta}^* & d_\theta^\star\mathbf{\Theta}-\mathbf{\Theta}d_\theta^\star\\ \mathbf{\Theta}^\star d_\theta-d_\theta\mathbf{\Theta}^\star & \Delta_\theta-\mathcal{L}_{\mathcal{T}}^2+\mathbf{\Theta}^*\mathbf{\Theta}.\end{bmatrix}\tag{3.2}$$

where $\Delta_\theta=d_\theta d_\theta^\star+d_\theta^\star d_\theta$. Since Δ is elliptic, so is $\Delta_\theta-\mathcal{L}_{\mathcal{T}}^2$ in each degree.

Let

$$\ker_q\Delta_\theta=\{\phi\in L^2(\mathcal{N};\textstyle\bigwedge^q\mathcal{N}):\Delta_\theta\phi=0\}$$

This is a closed subspace of $L^2(\mathcal{N};\bigwedge^q\mathcal{N})$, therefore a Hilbert space on its own right. Let

$$\operatorname{Dom}(\mathcal{L}_{\mathcal{T}})=\{\phi\in\ker_q\Delta_\theta:\mathcal{L}_{\mathcal{T}}\phi\in L^2(\mathcal{N};E)\}.$$

Since Δ_θ commutes with $\mathcal{L}_{\mathcal{T}}$, $\mathcal{L}_{\mathcal{T}}\operatorname{Dom}(\mathcal{L}_{\mathcal{T}})\subset\ker_q(\delta_\theta)$ and we have that

$$-i\mathcal{L}_{\mathcal{T}}^0:\operatorname{Dom}(\mathcal{L}_{\mathcal{T}})\subset\ker_q\Delta_\theta\to\ker_q\Delta_\theta,\tag{3.3}$$

the restriction of $\mathcal{L}_{\mathcal{T}}$ to $\operatorname{Dom}(\mathcal{L}_{\mathcal{T}})$, is an unbounded closed operator. The ellipticity of $\Delta_\theta-\mathcal{L}_{\mathcal{T}}^2$ gives:

Theorem 3.1. *The operator operator* (3.3) *is selfadjoint and Fredholm with discrete spectrum.*

Thus $\ker_q \Delta_\theta$ decomposes as a direct sum

$$\ker_q \Delta_\theta = \bigoplus_{\tau \in \operatorname{spec}(-i\mathcal{L}^0_{\mathcal{T}})} \mathscr{E}^q_\tau$$

4. Fourier series

Suppose $\tau \neq 0$. For arbitrary $\phi \in C^\infty(\mathcal{N}, \bigwedge^q \mathcal{N})$ define

$$(\pi_{\tau,m}\phi)(p) = \frac{\tau}{2\pi} \int_0^{2\pi/\tau} \mathfrak{a}_s^*\big(\phi(\mathfrak{a}_s p)\big)\, e^{-im\tau s}\, ds. \tag{4.1}$$

This gives a smooth section of $\bigwedge^q \mathcal{N}$. To see this, let $\mathfrak{a} : \mathbb{R} \times \mathcal{N} \to \mathcal{N}$ be the action defined by $\mathcal{T}$, $\mathfrak{a}(t,p) = \mathfrak{a}_t p$, let $\pi : \mathbb{R} \times \mathcal{N} \to \mathcal{N}$ be the canonical projection, and let $\chi : \mathbb{R} \to \mathbb{R}$ be the characteristic function of $[0, 2\pi/\tau]$, regarded as a function on $\mathbb{R} \times \mathcal{N}$. Then

$$\pi_{\tau,m}\phi = \pi_*\big(\chi \mathfrak{a}^* \phi e^{-im\tau t}\, dt\big).$$

An analysis of the behavior of the wavefront set under pull-back and push-forward, see Hörmander [5], gives that $\pi_{\tau,m}\phi$ is smooth if ϕ is.

Suppose now that $\phi \in C^\infty(\mathcal{N}^\tau, \bigwedge^q \mathcal{N}^\tau)$ and let $\phi_m = \pi_{\tau,m}\phi$. Then

$$\mathfrak{a}_t^*(\phi_m(\mathfrak{a}_t p)) = \phi_m(p) e^{i\tau m t}. \tag{4.2}$$

Indeed, using the the definition of $\pi_{\tau,m}$, the group property of $\mathfrak{a}$ and a change of variables we have that $(2\pi/2\pi)\mathfrak{a}_t^*(\phi_m(\mathfrak{a}_t p))$ is equal to

$$\begin{aligned} \int_0^{2\pi/\tau} \mathfrak{a}_t^*\big(\mathfrak{a}_s^*\big(\phi(\mathfrak{a}_s p)\big)\big)(\mathfrak{a}_t p)\, e^{-im\tau s}\, ds &= \int_0^{2\pi/\tau} \mathfrak{a}_{s+t}^*\big(\phi(\mathfrak{a}_{s+t} p)\big)\, e^{-im\tau s}\, ds \\ &= \int_0^{2\pi/\tau} \mathfrak{a}_s^*\big(\phi(\mathfrak{a}_s p)\big)\, e^{-im\tau s}\, ds\, e^{i\tau m t}. \end{aligned}$$

Consequently

$$\mathfrak{a}_t^*(\phi(\mathfrak{a}_t p)) = \sum_{m \in \mathbb{Z}} \phi_m(p) e^{im\tau t}$$

for each $p \in \mathcal{N}$ with convergence in $C^\infty(\mathbb{R})$. In particular

$$\phi = \sum_{m \in \mathbb{Z}} \phi_m$$

with pointwise convergence of the series, in fact with convergence in C^∞. It also follows that

$$\mathcal{L}_{\mathcal{T}} \phi_m = im\tau \phi_m. \tag{4.3}$$

Finally note that

$$\mathbf{i}_{\mathcal{T}}(\phi_m) = (\mathbf{i}_{\mathcal{T}}\phi)_m, \quad \mathbf{\Theta}\phi_m = (\mathbf{\Theta}\phi)_m, \quad \Pi_\theta \phi_m = (\Pi_\theta \phi)_m, \quad d\phi_m = (d\phi)_m. \tag{4.4}$$

The first two properties are immediate, the third is a consequence of the previous two. The last follows from (4.2), $d\mathfrak{a}_t^* = \mathfrak{a}_t^* d$ and uniqueness of the Fourier coefficients. In particular, if ϕ is a section of $\bigwedge^q \mathcal{H}^*$, then so is ϕ_m. Thus the Fourier series representation is compatible with the decomposition 3.1 of d.

5. Homomorphisms

We will first remove τ from the picture. Let

$$C^\infty_{(0)}(\mathcal{N};\textstyle\bigwedge^q\mathcal{N}_0)=\{\phi\in C^\infty(\mathcal{N};\textstyle\bigwedge^q\mathcal{N}):\mathcal{L}_\mathcal{T}\phi=0\}$$

Since d commutes with $\mathcal{L}_\mathcal{T}$, we have (yet another) complex,

$$\cdots\to C^\infty_{(0)}(\mathcal{N};\textstyle\bigwedge^q\mathcal{N})\xrightarrow{d_0}C^\infty_{(0)}(\mathcal{N};\textstyle\bigwedge^{q+1}\mathcal{N})\to\cdots,\tag{5.1}$$

where d_0 is d restricted to $C^\infty_{(0)}(\mathcal{N};\bigwedge^q\mathcal{N})$. This is a subcomplex of (2.4) for every $\tau>0$ as well as of the de Rham complex of $\mathcal{N}$. We will write $H^q_{\mathrm{dR},0}(\mathcal{N})$ for its cohomology groups.

The restriction to $C^\infty(\mathcal{N}^\tau;\bigwedge^q\mathcal{N}^\tau)$ of the map $\pi_{\tau,m}$ in (4.1) satisfies

$$\mathcal{L}_\mathcal{T}\circ\pi_{\tau,m}=i\tau m\,\pi_{\tau,m}$$

(see (4.3)), so with $m=0$ we have

$$\pi_{\tau,0}:C^\infty(\mathcal{N}^\tau;\textstyle\bigwedge^q\mathcal{N}^\tau)\to C^\infty_{(0)}(\mathcal{N};\textstyle\bigwedge^q\mathcal{N}).$$

This defines a chain map from the complex (2.4) to the complex (5.1).

Lemma 5.1. *The cohomology groups of the complex* (5.1) *are isomorphic to those of the complex* (2.4)*.*

Proof. We show that there is a homotopy from the complex (2.4) to the subcomplex (5.1). Note that Π_θ and $\mathbf{i}_\mathcal{T}$ preserve periodicity because they commute with $\mathfrak{a}_t^*$. Using (4.4) we see that if $\phi\in C^\infty(\mathcal{N}^\tau;\bigwedge^q\mathcal{N}^\tau)$ then in the representation $\phi=\phi^0+\theta\wedge\phi^1$ with $\phi^0=\Pi_\theta\phi$ and $\phi^1=\mathbf{i}_\mathcal{T}\phi$, both $t\mapsto\mathfrak{a}_t^*\phi^0$ and $t\mapsto\mathfrak{a}_t^*\phi^1$ are $2\pi/\tau$-periodic. Let

$$\phi^0=\sum_m\phi^0_m,\quad \phi^1=\sum_m\phi^1_m$$

be their Fourier series expansions. Thus $\pi_{\tau,0}\phi=\phi^1_0+\theta\wedge\phi^1_0$. Defining

$$h^q_\tau\phi=\sum_{m\neq0}\frac{1}{i\tau m}\phi^1_m$$

we get a homomorphism

$$h^q_\tau:C^\infty(\mathcal{N}^\tau;\textstyle\bigwedge^q\mathcal{N}^\tau)\to C^\infty(\mathcal{N}^\tau;\textstyle\bigwedge^{q-1}\mathcal{N}^\tau).$$

whose image is actually contained in $C^\infty(\mathcal{N};\bigwedge^{q-1}\mathcal{H}^*)$. It is easily verified that

$$\mathcal{L}_\mathcal{T}h^q_\tau\phi=h^q_\tau\mathcal{L}_\mathcal{T}\phi=\phi^1-\phi^1_0,\quad h^{q+1}_\tau d_\theta=d_\theta h^q_\tau.$$

Using (3.1) we have

$$dh^q_\tau\phi=\begin{bmatrix}d_\theta&\mathbf{\Theta}\\\mathcal{L}_\mathcal{T}&-d_\theta\end{bmatrix}\begin{bmatrix}h^q_\tau\phi\\0\end{bmatrix}=\begin{bmatrix}d_\theta h^q_\tau\phi\\\mathcal{L}_\tau h^q_\tau\phi\end{bmatrix}=\begin{bmatrix}d_\theta h^q_\tau\phi\\\phi^1-\phi^1_0\end{bmatrix},$$

whereas

$$h^{q+1}_\tau d\phi=\begin{bmatrix}h^{q+1}_\tau(\mathcal{L}_\mathcal{T}\phi^0-d_\theta\phi^1)\\0\end{bmatrix}=\begin{bmatrix}\phi^0-\phi^0_0-d_\theta h^q_\tau\phi^1)\\0\end{bmatrix}.$$

Adding these two formulas we obtain

$$(h^{q+1}_\tau d+dh^q_\tau)=I-\pi_{\tau,0}.$$

The induced homomorphism $\pi_{\tau,0} : H^q_{\mathrm{dR}}(\mathcal{N}^\tau) \to H^q_{\mathrm{dR},0}(\mathcal{N})$ has as inverse the map induced by the inclusion

$$\iota : C^\infty(\mathcal{N}; \textstyle\bigwedge^q \mathcal{N}) \to C^\infty_{(0)}(\mathcal{N}^\tau; \textstyle\bigwedge^q \mathcal{N}^\tau).$$

This completes the proof of the lemma. □

Consequently we have isomorphisms

$$\pi_{\tau,0} : H^q_{\mathrm{dR}}(\mathcal{N}^\tau) \to H^q_{\mathrm{dR},0}(\mathcal{N}).$$

Thus we may, and shall, use the groups $H^q_{\mathrm{dR},0}(\mathcal{N})$ in place of the groups $H^q_{\mathrm{dR}}(\mathcal{N}^\tau)$.

The space $C^\infty(\mathcal{B}; \bigwedge^q \mathcal{B})$ is a subspace of $C^\infty_{(0)}(\mathcal{N}; \bigwedge^q \mathcal{N})$ and the inclusion map, which we shall denote by

$$\rho^* : C^\infty(\mathcal{B}; \textstyle\bigwedge^q \mathcal{B}) \to C^\infty_{(0)}(\mathcal{N}; \textstyle\bigwedge^q \mathcal{N})$$

satisfies $d_0 \rho^* = \rho^* d_{\mathcal{B}}$. So there is an induced map in cohomology:

$$\rho^* : H^q_{\mathrm{dR}}(\mathcal{B}) \to H^q_{\mathrm{dR},0}(\mathcal{N}).$$

If $\phi \in C^\infty_{(0)}(\mathcal{N}; \bigwedge^q \mathcal{N})$, then $\phi^1 = \mathbf{i}_{\mathcal{T}}\phi$ obviously satisfies $\mathbf{i}_{\mathcal{T}}\phi^1 = 0$ and $\mathcal{L}_{\mathcal{T}}\phi^1 = 0$. So ϕ^1 is an element of $C^\infty(\mathcal{B}; \bigwedge^{q-1}\mathcal{B})$. Since

$$\begin{array}{ccc} C^\infty_{(0)}(\mathcal{N}, \bigwedge^q \mathcal{N}) & \xrightarrow{d_0} & C^\infty_{(0)}(\mathcal{N}; \bigwedge^{q+1} \mathcal{N}) \\ \mathbf{i}_{\mathcal{T}} \big\downarrow & & \big\downarrow \mathbf{i}_{\mathcal{T}} \\ C^\infty(\mathcal{B}; \bigwedge^{q-1} \mathcal{B}) & \xrightarrow{d_{\mathcal{B}}} & C^\infty(\mathcal{B}; \bigwedge^q \mathcal{B}) \end{array}$$

commutes, there is a map

$$\rho_* : H^q_{\mathrm{dR},0}(\mathcal{N}; \textstyle\bigwedge^q \mathcal{N}) \to H^q_{\mathrm{dR}}(\mathcal{B}, \textstyle\bigwedge^{q-1} \mathcal{B}) \tag{5.2}$$

induced by $\mathbf{i}_{\mathcal{T}}$.

Finally, since $\mathcal{L}_{\mathcal{T}} d\theta = 0$ and $\mathbf{i}_{\mathcal{T}} d\theta = 0$, $d\theta \wedge \phi \in C^\infty_{(0)}(\mathcal{B}; \bigwedge^{q+2}\mathcal{B})$ whenever $\phi \in C^\infty_{(0)}(\mathcal{B}; \bigwedge^q \mathcal{B})$. Since $\Theta \circ d_{\mathcal{B}} = d_{\mathcal{B}} \circ \Theta$, there is an induced map in cohomology:

$$H^q_{\mathrm{dR}}(\mathcal{B}) \ni [\phi] \overset{\mathbf{e}_\tau \wedge \cdot}{\longmapsto} [\frac{\tau}{2\pi} d\theta \wedge \phi] \in H^{q+2}_{\mathrm{dR}}(\mathcal{B}).$$

6. Exactness

Exactness of

$$H^{q-2}_{\mathrm{dR}}(\mathcal{B}) \xrightarrow{\mathbf{e}_\tau \wedge \cdot} H^q_{\mathrm{dR}}(\mathcal{B}) \xrightarrow{\rho^*} H^q_{\mathrm{dR},0}(\mathcal{N}).$$

Let $\psi \in C^\infty(\mathcal{B}; \bigwedge^{q-2}\mathcal{B})$ represent an element of $H^{q-1}_{\mathrm{dR}}(\mathcal{B})$. Then $\mathbf{e}_\tau[\psi]$ is represented by $\frac{\tau}{\pi} d\theta \wedge \psi \in C^\infty(\mathcal{B}, \bigwedge^q \mathcal{B})$ and $\rho^*(\mathbf{e}_\tau[\psi])$ is represented by the same form but in $C^\infty_{(0)}(\mathcal{N}; \bigwedge^q \mathcal{N})$. But this form is d_0-exact:

$$\frac{\tau}{2\pi} d\theta \wedge \psi = \frac{\tau}{2\pi} d_0(\theta \wedge \psi), \quad \theta \wedge \psi \in C^\infty_{(0)}(\mathcal{N}, \textstyle\bigwedge^{q-1} \mathcal{N}).$$

So $\rho^*(\mathbf{e}_\tau[\psi]) = 0$.

Now suppose $\phi \in C^\infty(\mathcal{B};\bigwedge^q\mathcal{B})$ represents an element in $H^q_{\mathrm{dR}}(\mathcal{B})$ and $\rho^*[\phi]=0$. Thus $\phi^1=\mathbf{i}_\phi=0$, $\mathcal{L}_\mathcal{T}\psi=0$, $d_\theta\phi^1=0$, and there exists $\chi\in C^\infty_{(0)}(\mathcal{N},\bigwedge^{q-1}\mathcal{N})$ such that with $\psi^0=\Pi_\theta\psi$, $\psi^1=\mathbf{i}_\mathcal{T}\psi$ likewise we have

$$\phi=\begin{bmatrix}\phi^0\\0\end{bmatrix}=d\psi=\begin{bmatrix}d_\theta & \mathbf{\Theta}\\ \mathcal{L}_\mathcal{T} & -d_\theta\end{bmatrix}\begin{bmatrix}\psi^0\\ \psi^1\end{bmatrix}$$

So $\phi^0=d_\theta\psi^0+d\theta\wedge\psi^1$, $0=d_\theta\psi^1$. Since $d_\theta\psi^0=d\psi^0=d_0\psi^0$, the class of ϕ in $H^0_{\mathrm{dR},0}(\mathcal{N})$ is equal to that of $\phi-d\psi^0$. We may thus assume that $\mathbf{i}_\mathcal{T}\phi=0$ to begin with. So $\phi=d\theta\wedge\psi^1$. From $d\phi=0$ get $d\psi^1=0$, and we conclude that ϕ is in the image of $\mathbf{e}_\tau\wedge\cdot$.

Exactness of

$$H^q_{\mathrm{dR}}(\mathcal{B})\xrightarrow{\rho^*}H^q_{\mathrm{dR},0}(\mathcal{N})\xrightarrow{\rho_*}H^{q-1}_{\mathrm{dR}}(\mathcal{B}).$$

Suppose $\psi\in C^\infty(\mathcal{B};\bigwedge^q\mathcal{B})$ is $d_\mathcal{B}$-closed. Then it is d_0 (or just d)-closed as an element of $C^\infty_{(0)}(\mathcal{N};\bigwedge^q\mathcal{N})$. Since $\mathbf{i}_\mathcal{T}\psi=0$, $\rho_*\rho^*[\psi]=[\mathbf{i}_\mathcal{T}\psi]=0$.

Now suppose $[\phi]\in H^q_{\mathrm{dR},0}(\mathcal{N})$ and $\rho_*[\phi]=0$, that is,

$$\mathbf{i}_\mathcal{T}\phi=d_\mathcal{B}\chi,\ \ \chi\in C^\infty(\mathcal{B};\textstyle\bigwedge^{q-2}\mathcal{B}).$$

The form ϕ is cohomologous in the complex (5.1) to $\psi=\phi+d_0(\theta\wedge\chi)$. But

$$\phi+d_0(\theta\wedge\chi)=\begin{bmatrix}\phi^0\\ \phi^1\end{bmatrix}+\begin{bmatrix}d_\theta & \mathbf{\Theta}\\ \mathcal{L}_\mathcal{T} & -d_\theta\end{bmatrix}\begin{bmatrix}0\\ \chi\end{bmatrix}=\begin{bmatrix}\phi^0\\ \phi^1\end{bmatrix}+\begin{bmatrix}d\theta\wedge\chi\\ d_\theta\chi\end{bmatrix}=\begin{bmatrix}\phi^0+d\theta\wedge\chi\\ 0\end{bmatrix}.$$

Since $\phi^0+d\theta\wedge\chi\in C^\infty(\mathcal{B};\bigwedge^q\mathcal{B})$, $[\phi]=\rho^*[\phi^0+d\theta\wedge\phi]$.

Exactness of

$$H^q_{\mathrm{dR},0}(\mathcal{N})\xrightarrow{\rho_*}H^{q-1}_{\mathrm{dR}}(\mathcal{B})\xrightarrow{\mathbf{e}_\tau\wedge\cdot}H^{q+1}_{\mathrm{dR}}(\mathcal{B}).$$

Suppose $[\psi]\in H^q_{\mathrm{dR},0}(\mathcal{N})$. Thus $\psi\in C^\infty(\mathcal{N};\bigwedge^q\mathcal{N})$, $\mathcal{L}_\mathcal{T}\psi=0$, and $d\psi=0$. We show that $d\theta\wedge\psi^1$ is exact in the complex (2.3). With ψ^0, ψ^1 as usual we have

$$\begin{bmatrix}d_\theta\psi^1+d\theta\wedge\psi^1\\ -d\psi^1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

The top line gives $d\theta\wedge\psi^1=-d_\theta\psi^0$.

Now suppose $[\phi]\in H^{q-1}_{\mathrm{dR}}(\mathcal{B})$ and $d\theta\wedge\phi=0$. Then $\psi=\theta\wedge\phi$ is a d_0-closed element of $C^\infty_{(0)}(\mathcal{N};\bigwedge^q\mathcal{N})$ such that $\mathbf{i}_\mathcal{T}\psi=\phi$.

This completes the proof of exactness of the sequence (2.6).

7. The groups $H^q_{\mathrm{dR}}(\mathcal{B})$

We shall now analyze the cohomology groups $H^q_{\mathrm{dR}}(\mathcal{B})$ of the complex (2.3) from two viewpoints. In the first we will exhibit them (for $q\geq d$) as the Čech cohomology with coefficients in a suitable sheaf. The arguments are essentially those used to relate, for example, Čech and Rham cohomology. The sheaf in question reduces to the sheaf of locally constant real-valued functions when the orbits of $\mathcal{T}$ are circles, thus proving Theorem 2.1, namely that $H^q_{\mathrm{dR}}(\mathcal{B})$ is isomorphic to $H^q(\mathcal{B},\mathbb{R})$. In the second characterization we present the groups, for any q, as spaces of harmonic forms for a Laplace-like operator

Let g be some $\mathcal{T}$-invariant metric. Since $\mathcal{L}_\mathcal{T}g=0$, $\{\mathfrak{a}_t\}$, the one-parameter group of diffeomorphisms generated by $\mathcal{T}$ consists of isometries of g. The full group of isometries is a compact Lie group (see Myers and Steenrod [8], Helgason [4]) because $\mathcal{N}$ is compact, and $\{\mathfrak{a}_t : t \in \mathbb{R}\}$ is a subgroup. Let G be its closure, a compact abelian Lie group. The closure of $\mathcal{O}_p$ is the orbit of G through p, thus an embedded submanifold of $\mathcal{N}$ diffeomorphic to a torus (whose dimension may change form point to point, however). If $x \in \mathcal{B}$, denote by $\mathcal{F}_x$ the set $\rho^{-1}(x)$. Thus $\mathcal{F}_x = \overline{\mathcal{O}}_p$ for some $p \in \mathcal{N}$. Define

$$d = \max\{\dim \mathcal{F}_x : x \in \mathcal{B}\}. \tag{7.1}$$

Let $\mathcal{H}^*_U$ denote the part of $\mathcal{H}^*$ over $U \subset \mathcal{N}$. The family whose elements are

$$C^\infty(V; \textstyle\bigwedge^q V) = \{\phi \in C^\infty(U; \bigwedge^q \mathcal{H}^*_U) : \mathcal{L}_\mathcal{T} \phi = 0\}, \quad V \subset \mathcal{B} \text{ open},\ U = \rho^{-1}(V)$$

together with the canonical restriction maps forms a presheaf whose associated sheaf we denote by $\mathscr{E}^q$. The operator $d_\mathcal{B}$ induces a sheaf map $d_\mathcal{B} : \mathscr{E}^q \to \mathscr{E}^{q+1}$. If $d = 1$ and $q = 0$ then $\mathscr{Z}^q$ is the sheaf of locally constant real-valued functions on $\mathcal{B}$

The family whose elements are

$$Z^q(V) = \{\phi \in C^\infty(V, \textstyle\bigwedge^q V) : d_\mathcal{B}\phi = 0\}, \quad V \subset \mathcal{B} \text{ open}$$

with its restriction maps is also a presheaf. The associated sheaf, $\mathscr{Z}^q$, is a subsheaf of $\mathscr{E}^q$ containing the image of $d_\mathcal{B}$.

Theorem 7.1. *If $q - p \geq d - 1$ then $H^q_{\mathrm{dR}}(\mathcal{B})$ is isomorphic to $\check{H}^p(\mathcal{B}; \mathscr{Z}^{q-p})$. Thus if $d = 1$, then $H^q_{\mathrm{dR}}(\mathcal{B}) \approx \check{H}^q(\mathcal{B}; \mathbb{R})$.*

The proof requires an analysis of the extent to which surjectivity of

$$d_\mathcal{B} : \mathscr{E}^{q-1} \to \mathscr{Z}^q \tag{7.2}$$

holds.

Fix $x_0 \in \mathcal{B}$ arbitrarily. As just discussed, $\mathcal{F}_{x_0}$ is a smooth embedded submanifold of $\mathcal{N}$. The surjectivity of (7.2) is linked to the vanishing of the cohomology groups in large degree of the complex

$$\cdots \to C^\infty(\mathcal{F}_{x_0}; \textstyle\bigwedge^q \mathcal{H}^*_{\mathcal{F}_{x_0}}) \xrightarrow{d} C^\infty(\mathcal{F}_{x_0}; \bigwedge^{q+1} \mathcal{H}^*_{\mathcal{F}_{x_0}}) \to \dots, \tag{7.3}$$

We will write $H^*(\mathcal{F}_{x_0}; \mathcal{H}^*_{\mathcal{F}_{x_0}})$ for the cohomology groups of this complex.

Pick a point $p_0 \in \mathcal{F}_{x_0}$ arbitrarily. We may then view $\mathcal{F}_{x_0}$ as an actual torus with identity element p_0 and invariant metric induced by g. Identifying the de Rham cohomology groups of $\mathcal{F}_{x_0}$ with the spaces of harmonic forms in the various degrees and using that the vector field $\mathcal{T}$ along $\mathcal{F}_{x_0}$ is translation invariant one gets that $H^q(\mathcal{F}_{x_0}; \mathcal{H}^*_{\mathcal{F}_{x_0}})$ can be identified with the kernel of $\mathbf{i}_\mathcal{T}$ on $H^q_{\mathrm{dR}}(\mathcal{F}_{x_0})$. We thus have

$$\dim H^q(\mathcal{F}_{x_0}; \mathcal{H}^*_{\mathcal{F}_{x_0}}) = \dim H^q_{\mathrm{dR}}(\mathcal{F}_{x_0}) - 1.$$

Consequently $H^q(\mathcal{F}_{x_0}; \mathcal{H}^*_{\mathcal{F}_{x_0}}) = 0$ if $q \geq \dim \mathcal{F}_{x_0}$.

Let $N\mathcal{F}_{x_0} \subset T_{\mathcal{F}_{x_0}}\mathcal{N}$ be the orthogonal bundle to $T\mathcal{F}_{x_0}$ according to the metric. Define

$$B_\varepsilon = \{v \in N\mathcal{F}_{x_0} : g(v, v) < \varepsilon^2\}$$

and pick $\varepsilon > 0$ so small that $\exp : B_\varepsilon \to \mathcal{N}$ is a diffeomorphism onto $U = \exp(B_\varepsilon)$. The latter, an open subset of $\mathcal{N}$, is closed under orbits of $\mathcal{T}$ because as a consequence of g being $\mathcal{T}$-invariant, $\mathfrak{a}_t \circ \exp = \exp \circ d\mathfrak{a}_t$. By the definition of the quotient topology and map, $V = \rho(U)$ is an open subset of $\mathcal{B}$ and $\rho^{-1}(V) = U$.

Lemma 7.2. *Let $x_0 \in \mathcal{B}$, $\mathcal{F}_{x_0} = \rho^{-1}(x_0)$, and pick $V \subset \mathcal{B}$ be as just described. The cohomology of the complex*

$$\cdots \to C^\infty(V;{\textstyle\bigwedge}^q V) \xrightarrow{d_{\mathcal{B}}} C^\infty(V;{\textstyle\bigwedge}^{q+1} V) \to \cdots \tag{7.4}$$

is isomorphic to that of the complex (7.3)*. In particular, these cohomology groups are zero in all degrees $q \geq \dim \mathcal{F}_{x_0}$.*

As a corollary we obtain the following version of the Poincaré Lemma:

Proposition 7.3. *Let $d = \max\{\dim \mathcal{F}_x : x \in \mathcal{B}\}$. The map $d_{\mathcal{B}} : \mathscr{E}^{q-1} \to \mathscr{Z}^q$ is surjective for $q \geq d$.*

Proof of Lemma 7.2. Write $H^q_{\mathrm{dR}}(V)$ for the q-th cohomology group of the complex (7.4). Let $\iota : \mathcal{F}_{x_0} \to U$ be the inclusion map and $\wp : U \to \mathcal{F}_{x_0}$ the composition

$$U \xrightarrow{\exp^{-1}} B_\varepsilon \xrightarrow{\;\pi\;} \mathcal{F}_{x_0}.$$

Then $\wp$ commutes with $\mathbf{i}_{\mathcal{T}}$, so there is an induced map

$$\wp^* : H^q(\mathcal{F}_{x_0}; \mathcal{H}^*_{\mathcal{F}x_0}) \to H^q_{\mathrm{dR}}(V) \tag{7.5}$$

in each degree. Since $\wp \circ \iota = I$, $\wp^*$ is an isomorphism onto its image.

Let $\kappa_{-\infty} : U \to U$ be the map $\iota \circ \wp$. We will exhibit a homotopy from the identity map of the complex (7.4) to the cochain map given by

$$\kappa^*_{-\infty} : C^\infty(V;{\textstyle\bigwedge}^q V) \to C^\infty(V;{\textstyle\bigwedge}^q V), \tag{7.6}$$

The existence of this homotopy implies that (7.5) is also surjective. In constructing the homotopy we closely follow the proof of the Poincaré Lemma in [13].

Let R be the radial vector field of $N\mathcal{F}_{x_0}$; its integral curve through v is $s \mapsto e^s v$. Since $\mathfrak{a}_t$ is an isometry, it sends geodesics to geodesics and so $\exp \circ d\mathfrak{a}_t = \mathfrak{a}_t \circ \exp$. Thus

$$\mathfrak{a}_t \exp(e^s v) = \exp(e^s d a_t v). \tag{7.7}$$

This says that if $X = d\exp R$, a well defined smooth vector field on U, then $d\mathfrak{a}_t X = X$. Write κ_s for the flow of X in U. This is defined for every $p \in U$ and every $s < \delta_p$ where $\delta_p > 0$. In particular κ_s maps U into U if $s \leq 0$.

It follows from (7.7) that $\mathfrak{a}_t \circ \kappa_s = \kappa_s \circ \mathfrak{a}_t$. As a consequence, if $\phi \in C^\infty(V;\bigwedge^q V)$ then also $\kappa_s^*\phi \in C^\infty(V;\bigwedge^q V)$ for every $s \leq 0$.

The family

$$\kappa_s^* : C^\infty(V;{\textstyle\bigwedge}^q V) \to C^\infty(V;{\textstyle\bigwedge}^q V)$$

converges pointwise as $s \to -\infty$ to the map (7.6). To see this, pick local coordinates $x^1 \dots, x^{n-d_0}, y^1, \dots, y^{d_0}$ ($n = \dim \mathcal{N}$, $d_0 = \dim \mathcal{F}_{x_0}$) near an arbitrary $p \in \mathcal{F}_{x_0}$ such that, near p_0, the x^j are defining functions for $\mathcal{F}_{x_0}$, the vector fields ∂_{x_j} form an orthonormal basis of $N\mathcal{F}_{x_0}$ and $d\exp(\partial_{x^j}|_v) = \partial_{x^j}|_{\exp(v)}$ if $v \in B_\varepsilon$. Then $X = \sum_j x^j \partial_{x^j}$, $\kappa_s(x,y) = (e^s x, y)$, and so if

$$\phi = {\sum_{|I|+|J|=q}}' \phi_{IJ}(x,y) dx^I \wedge dy^J$$

then

$$(\kappa_s^*\phi)(x,y) = {\sum_{|I|+|J|=q}}' e^{s|I|}\phi_{IJ}(e^s x, y)\, dx^I \wedge dy^J.$$

Suppose $\iota^*\phi = 0$. Then $\kappa^*_{-\infty}\phi = 0$ in which case one has $\kappa^*_s\phi = O(e^s)$ as $s \to -\infty$. This estimate implies that the integral

$$\alpha_q(\phi)(p) = \int_{-\infty}^{0} \kappa_s^*(\phi(\kappa_s p))\, ds, \quad p \in U$$

is finite and that the resulting form $\alpha_q(\phi)$is smooth on U. Furthermore, $\alpha_{q+1}d\phi$ is also defined because $\iota^* d\phi = d\iota^*\phi = 0$. In addition,

$$\alpha_{q+1}d\phi = d\alpha_q\phi \quad \text{if } \iota^*\phi = 0 \tag{7.8}$$

using $\kappa_s^* d\phi = d\kappa_s^*\phi$.

Suppose now that ϕ is a general element of $C^\infty(V; \bigwedge^q V)$. Obviously $\iota^*\mathcal{L}_X\phi$ vanishes and

$$\kappa_s^*(\mathcal{L}_X\phi)(p) = \frac{d}{ds}\kappa_s^*(\phi(\kappa_s p)).$$

So $\alpha_q(\mathcal{L}_X\phi)$ is defined and

$$\alpha_q(\mathcal{L}_X\phi)(p) = \int_{-\infty}^{0} \frac{d}{ds}\kappa_s^*(\phi(\kappa_s p))\, ds = \phi(p) - (\kappa^*_{-\infty}\phi)(p).$$

This gives

$$d\alpha_{q-1}\mathbf{i}_X\phi + \alpha_q\mathbf{i}_X d\phi = I - \kappa^*_{-\infty}$$

using $\mathcal{L}_X = d\mathbf{i}_X + \mathbf{i}_X d$, $\iota^*\mathbf{i}_X\phi = 0$, and (7.8). Thus defining $h_q = \alpha_{q-1} \circ \mathbf{i}_X$ we have

$$d \circ h_q + h_{q+1} \circ d = I - \kappa^*_{-\infty}.$$

The map $\kappa^*_{-\infty}$ vanishes on $C^\infty(V; \bigwedge^q V) \to C^\infty(\mathcal{F}_{x_0}; \bigwedge^q \mathcal{F}_{x_0})$ when $q \geq \dim \mathcal{F}_{x_0}$ for the simple reason that the target space in

$$\iota^* : C^\infty(V; \textstyle\bigwedge^q V) \to C^\infty(\mathcal{F}_{x_0}; \textstyle\bigwedge^q \mathcal{F}_{x_0})$$

is zero when $q \geq \dim \mathcal{F}_{x_0}$. □

Thus there is an exact sequence

$$0 \to \mathscr{Z}^{q-1} \xrightarrow{\iota} \mathscr{E}^{q-1} \xrightarrow{d_{\mathcal{B}}} \mathscr{Z}^q \to 0$$

in each degree $q \geq d$. Therefore for such q we have the long exact sequence

(7.9)

$$0 \to \Gamma(\mathcal{B}, \mathscr{Z}^{q-1}) \to \Gamma(\mathcal{B}, \mathscr{E}^{q-1}) \xrightarrow{d_{\mathcal{B}}} \Gamma(\mathcal{B}, \mathscr{Z}^q) \to \check{H}^1(\mathcal{B}, \mathscr{Z}^{q-1}) \to \check{H}^1(\mathcal{B}, \mathscr{E}^{q-1}) \to$$
$$\cdots \to \check{H}^p(\mathcal{B}, \mathscr{E}^{q-1}) \to \check{H}^p(\mathcal{B}, \mathscr{Z}^q) \to \check{H}^{p+1}(\mathcal{B}, \mathscr{Z}^{q-1}) \to \check{H}^{p+1}(\mathcal{B}, \mathscr{E}^{q-1}) \to \dots$$

in Čech cohomology.

Lemma 7.4. *The sheaf $\mathscr{E}^q$ is fine.*

This is a consequence of the existence of partitions of unity in $C^\infty(\mathcal{B})$. The following argument proving the existence of such partitions is taken from [6]. Define $\mathfrak{A} : G \times \mathcal{N} \to \mathcal{N}$ by $\mathfrak{A}(h, p) = h(p)$. This is a smooth map. Let also $\pi : G \times \mathcal{N} \to \mathcal{N}$ be the canonical projection. Finally, let $\mathfrak{m}$ be the Haar measure on G. Define $\mathrm{av} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ by

$$\mathrm{av} f = \pi_*(\mathfrak{A}^* f\, \mathfrak{m})$$

Again an analysis of wavefront sets gives immediately that av dos indeed send smooth functions to smooth functions, and since $\mathrm{av} f$ is constant on orbits of G, av in fact maps into $C^\infty(\mathcal{B})$.

Now, if $\mathfrak{V} = \{V_a\}_{a\in A}$ is an open cover of $\mathcal{B}$ and $\{\chi_b\}_{b\in B}$ is a smooth partition of unity subordinate to the open cover $\{\rho^{-1}(V_a)\}$ of $\mathcal{N}$ (so $\chi_b \in C^\infty(\mathcal{N})$ for all b), then the family $\{\mathrm{av}\chi_b\}_{b\in B}$ is another such partition, but now with functions in $C^\infty(\mathcal{B})$. Interpreting these functions as functions on $\mathcal{B}$, they form a partition of unity subordinate to the open cover $\mathfrak{V}$ of $\mathcal{B}$: given any open cover $\mathfrak{V} = \{V_a\}_{a\in A}$ of $\mathcal{B}$ there is a partition of unity $\{\chi_b\}_{b\in B}$ with $\chi_b \in C^\infty(\mathcal{B})$ for all $b \in B$ subordinate to the given cover.

Since $\mathscr{E}^q$ is a fine sheaf, $\check{H}^p(\mathcal{B}, \mathscr{E}^q) = 0$ for $p > 0$. As in the standard case we deduce from the long exact sequence that

$$\check{H}^1(B, \mathscr{Z}^{q-1}) \approx \Gamma(\mathcal{B}, \mathscr{Z}^q)/d_\mathcal{B}\Gamma(\mathcal{B}, \mathscr{E}^{q-1}), \quad q \geq d,$$

that is

$$\check{H}^1(B, \mathscr{Z}^{q-1}) \approx H^q_{\mathrm{dR}}(\mathcal{B}) \quad \text{if } q \geq d$$

and

$$\check{H}^p(\mathcal{B}, \mathscr{Z}^q) \approx \check{H}^{p+1}(\mathcal{B}, \mathscr{Z}^{q-1}) \text{ for } p > 0 \text{ and } q \geq d.$$

from which we reach the thesis of the theorem.

We now turn to a purely analytic description of the groups cohomology groups of the complex (2.3)

Proposition 7.5. *The cohomology group of the complex* (2.3) *in degree q is isomorphic to the kernel, $\mathscr{H}^q(\mathcal{B})$, of Δ_θ in $C^\infty(\mathcal{B}; \bigwedge^q\mathcal{B})$. The spaces $\mathscr{H}^q(\mathcal{B})$ are finite-dimensional.*

Proof. The proof is similar to that of the analogous statement in Hodge theory. Since $\Delta_\theta - \mathcal{L}^2_\mathcal{T}$ is elliptic, its kernel in $C^\infty(\mathcal{N}; \bigwedge^q\mathcal{H}^*)$, namely $\mathscr{H}^q(\mathcal{B})$, is finite dimensional.

Now, $\mathscr{H}^q(\mathcal{B})$ is in fact the kernel of $\Delta^2_\theta - \mathcal{L}^2_\mathcal{T}$. Clearly $\mathscr{H}^q(\mathcal{B}) \subset \ker(\Delta^2_\theta - \mathcal{L}^2_\mathcal{T})$. On the other hand, if $\phi \in \ker(\Delta^2_\theta - \mathcal{L}^2_\mathcal{T})$, then

$$0 = \big((\Delta_\theta - \mathcal{L}^2_\mathcal{T})\phi, \phi\big) = \|d_\theta\phi\|^2 + \|d^\star_\theta\phi\|^2 + \|\mathcal{L}_\mathcal{T}\phi\|^2.$$

hence in particular $\mathcal{L}_\mathcal{T}\phi = 0$, so $\phi \in \mathscr{H}^q(\mathcal{B})$.

Let

$$\Pi^q_\mathcal{B} : L^2(\mathcal{N}; \textstyle\bigwedge^q\mathcal{H}^*) \to \mathscr{H}^q(\mathcal{B})$$

be the orthogonal projection. Then $\mathcal{L}_\mathcal{T} \circ \Pi^q_\mathcal{B} = 0$, while also, trivially, $\Pi^q_\mathcal{B} \circ \mathcal{L}_\mathcal{T} = 0$ on $C^\infty(\mathcal{B}; \bigwedge^q\mathcal{B})$. For each q there is $\mathcal{G} : L^2(\mathcal{N}; \bigwedge^q\mathcal{H}^*) \to L^2(\mathcal{N}; \bigwedge^q\mathcal{H}^*)$, a selfadjoint pseudodifferential operator of order -2, such that $\mathcal{G}(\Delta_\theta - \mathcal{L}^2_\mathcal{T}) = (\Delta_\theta - \mathcal{L}^2_\mathcal{T})\mathcal{G} = I - \Pi^q_\mathcal{B}$. Since $\mathcal{L}_\mathcal{T}$ commutes with $\Pi^q_\mathcal{B}$ and with $(\Delta_\theta - \mathcal{L}^2_\mathcal{T})$, $\mathcal{G}$ commutes with $\mathcal{L}_\mathcal{T}$. Thus $\mathcal{G}$ maps $C^\infty(\mathcal{B}; \bigwedge^q\mathcal{B})$ to itself. The formula

$$d_\theta(\Delta_\theta - \mathcal{L}^2_\mathcal{T}) = (\Delta_\theta - \mathcal{L}^2_\mathcal{T})d_\theta.$$

gives

$$d_\theta\mathcal{G}\phi = \mathcal{G}d_\theta\phi, \quad \phi \in C^\infty(\mathcal{B}; \textstyle\bigwedge^q\mathcal{B}).$$

The rest of the proof consists of using this to show that $H^q_{\mathrm{dR}}(\mathcal{B})$ is isomorphic to $\mathscr{H}^q(\mathcal{B})$. The details are identical to the corresponding proof in Hodge theory. $\square$

8. The groups $H^q_{\mathrm{dR},0}(\mathcal{N})$

We now show that $H^q_{\mathrm{dR},0}(\mathcal{N})$, the q-th cohomology group of the complex (5.1), is isomorphic to $H^q(\mathcal{N},\mathbb{R})$ by exhibiting a homotopy from the identity morphism of the de Rham complex of $\mathcal{N}$ to a projection with range the subcomplex (5.1) inducing an invertible map in cohomology. Together with Lemma 5.1 this completes the proof of Proposition 2.2.

Let $\mathfrak{g}$ be the Lie algebra of G. There is a basis $\hat{Y}_1,\dots,\hat{Y}_d$ of $\mathfrak{g}$ such that each of the curves $s\mapsto \exp(s\hat{Y}_j)$ is periodic of period 2π. The map $(s^1,\dots,s^d)\mapsto \varrho(s^1,\dots,s^d)=\exp(s^1\hat{Y}_1+\dots+s^d\hat{Y}_d)$ is surjective, a covering map of period 2π in each component. Setting

$$\omega(\varrho(s^1,\dots,s^d))=(e^{is^1},\dots,e^{is^d})$$

we get well defined functions $\omega^j:G\to\mathbb{C}^d$. If $\phi\in C^\infty(\mathcal{N};\bigwedge^q\mathcal{N})$ define

$$\pi_\alpha\phi=\pi_*(\overline{\omega}^\alpha\mathfrak{A}^*\phi\,\mathfrak{m}),\quad \alpha\in\mathbb{Z}^d.$$

That is, if $v_1,\dots,v_d\in T_p\mathcal{N}$ and $V_1,\dots,V_d$ denotes the canonical liftings of the respective v_j as tangent vector fields of $G\times\mathcal{N}$ along $\pi^{-1}(p)$, then $\pi_\alpha\phi(p)=\phi_\alpha(p)$ is the q-covector such that

$$\pi_\alpha\phi(p)(v_1,\dots,v_d)=\int_G\overline{\omega}^\alpha(g)\,\mathfrak{A}^*_g(\phi(\mathfrak{A}_g p))(V_1,\dots,V_d)\,d\mathfrak{m}$$

Then $\pi_\alpha\phi$ is smooth if ϕ is smooth and $d\pi_\alpha=\pi_\alpha d$. Since

$$\mathfrak{A}^*_{\exp\sum s_j\hat{Y}_j}\pi_\alpha\phi=e^{i\sum s_j\alpha_j}\pi_\alpha\phi,$$

$\phi=\sum_{\alpha\in\mathbb{Z}^d}\phi_\alpha$ with convergence in $C^\infty(\mathcal{N};\bigwedge^q\mathcal{N})$.

For $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{Z}^d$ let $J_\alpha=\{j:\alpha_j\neq 0\}$ and let m_α be the cardinality of J_α. Let Y_j be the vector field on $\mathcal{N}$ determined by $\hat{Y}^j$:

$$Y_j(p)=\frac{d}{ds}\Big|_{s=0}\mathfrak{A}_{\exp(s\hat{y}_j)}p.$$

Then $\mathcal{L}_{Y_j}\pi_\alpha=i\alpha_j\,\pi_\alpha$. Defining

$$h_q\phi=\sum_{\alpha\neq 0}\frac{1}{m_\alpha}\sum_{j\in J_\alpha}\frac{1}{i\alpha_j}\mathbf{i}_{Y_j}\pi_\alpha\phi$$

we have

$$dh_q\phi=\sum_{\alpha\neq 0}\frac{1}{m_\alpha}\sum_{j\in J_\alpha}\frac{1}{i\alpha_j}(\mathcal{L}_{Y_j}\pi_\alpha\phi-\mathbf{i}_{Y_j}d\pi_\alpha\phi)=\sum_{\alpha\neq 0}\phi_\alpha-h_{q+1}d\phi,$$

that is,

$$dh_q+h_{q+1}d=I-\pi_0.$$

It follows immediately that any class in $H^q_{\mathrm{dR}}(\mathcal{N})$ has a representative in the space $C^\infty_{(0)}(\mathcal{N};\bigwedge^q\mathcal{N})$. If $\phi\in C^\infty_{(0)}(\mathcal{N};\bigwedge^q\mathcal{N})$ and $\phi=d\psi$ for some $\psi\in C^\infty(\mathcal{N};\bigwedge^{q-1}\mathcal{N})$, then

$$\phi-d\pi_0\psi=dh_qd\psi.$$

Applying π_0 to both sides of this identity gives $\pi_0dh_qd\psi=0$. Since ϕ, $d\pi_0\psi\in C^\infty_{(0)}(\mathcal{N};\bigwedge^q\mathcal{N})$, $dh_qd\psi$ must be zero. Thus there is a well defined injective map

$$\boldsymbol{\pi}_0:H^q_{\mathrm{dR}}(\mathcal{N})\to H^q_{\mathrm{dR},0}(\mathcal{N}).$$

The same argument shows that the map $\boldsymbol{\iota} : H^q_{\mathrm{dR},0}(\mathcal{N}) \to H^q_{\mathrm{dR}}(\mathcal{N})$ obtained from the inclusions $C^\infty_{(0)}(\mathcal{N};\bigwedge^*\mathcal{N}) \to C^\infty(\mathcal{N};\bigwedge^*\mathcal{N})$, which is a left inverse of $\boldsymbol{\pi}_0$, is injective. So $\boldsymbol{\pi}_0$ is an isomorphism.

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E-mail address: gmendoza@math.temple.edu

Department of Mathematics, Temple University, Philadelphia, PA 19122