

COMPLEX b -MANIFOLDS

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ABSTRACT. A complex b -structure on a manifold \mathcal{M} with boundary is an involutive subbundle ${}^bT^{0,1}\mathcal{M}$ of the complexification of ${}^bT\mathcal{M}$ with the property that $\mathbb{C}{}^bT\mathcal{M} = {}^bT^{0,1}\mathcal{M} + \overline{{}^bT^{0,1}\mathcal{M}}$ as a direct sum; the interior of \mathcal{M} is a complex manifold. The complex b -structure determines an elliptic complex of b -operators and induces a rich structure on the boundary of \mathcal{M} . We study the cohomology of the indicial complex of the b -Dolbeault complex.

1. INTRODUCTION

A complex b -manifold is a smooth manifold with boundary together with a complex b -structure. The latter is a smooth involutive subbundle ${}^bT^{0,1}\mathcal{M}$ of the complexification $\mathbb{C}{}^bT\mathcal{M}$ of Melrose's b -tangent bundle [5, 6] with the property that

$$\mathbb{C}{}^bT\mathcal{M} = {}^bT^{0,1}\mathcal{M} + \overline{{}^bT^{0,1}\mathcal{M}}$$

as a direct sum. Manifolds with complex b -structures generalize the situation that arises as a result of spherical and certain anisotropic (not complex) blowups of complex manifolds at a discrete set of points or along a complex submanifold, cf. [7, Section 2], [9], as well as (real) blow-ups of complex analytic varieties with only point singularities.

The interior of \mathcal{M} is a complex manifold. Its $\bar{\partial}$ -complex determines a b -elliptic complex, the ${}^b\bar{\partial}$ -complex, on sections of the exterior powers of the dual of ${}^bT^{0,1}\mathcal{M}$, see Section 2. The indicial families $\bar{\mathcal{D}}(\sigma)$ of the ${}^b\bar{\partial}$ -operators at a connected component \mathcal{N} of $\partial\mathcal{M}$ give, for each σ , an elliptic complex, see Section 6. Their cohomology at the various values of σ determine the asymptotics at \mathcal{N} of tempered representatives of cohomology classes of the ${}^b\bar{\partial}$ -complex, in particular of tempered holomorphic functions.

Each boundary component \mathcal{N} of \mathcal{M} inherits from ${}^bT^{0,1}\mathcal{M}$ the following objects in the C^∞ category:

- (1) an involutive vector subbundle $\bar{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ such that $\mathcal{V} + \bar{\mathcal{V}} = \mathbb{C}T\mathcal{N}$;
- (2) a real nowhere vanishing vector field \mathcal{T} such that $\mathcal{V} \cap \bar{\mathcal{V}} = \text{span}_{\mathbb{C}} \mathcal{T}$;
- (3) a class $\boldsymbol{\beta}$ of sections of $\bar{\mathcal{V}}^*$,

where the elements of $\boldsymbol{\beta}$ have additional properties, described in (4) below. The vector bundle $\bar{\mathcal{V}}$, being involutive, determines a complex of first order differential operators $\bar{\mathbb{D}}$ on sections of the exterior powers of $\bar{\mathcal{V}}^*$, elliptic because of the second property in (1) above. To that list add

- (4) If $\beta \in \boldsymbol{\beta}$ then $\bar{\mathbb{D}}\beta = 0$ and $\Im\langle\beta, \mathcal{T}\rangle = -1$, and if $\beta, \beta' \in \boldsymbol{\beta}$, then $\beta' - \beta = \bar{\mathbb{D}}u$ with u real-valued.

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These properties, together with the existence of a Hermitian metric on $\overline{\mathcal{V}}$ invariant under \mathcal{T} make \mathcal{N} behave in many ways as the circle bundle of a holomorphic line bundle over a compact complex manifold. These analogies are investigated in [10, 11, 12, 13]. The last of these papers contains a detailed account of circle bundles from the perspective of these boundary structures. The paper [8], a predecessor of the present one, contains some facts studied here in more detail.

The paper is organized as follows. Section 2 deals with the definition of complex b -structure and Section 3 with holomorphic vector bundles over complex b -manifolds (the latter term just means that the b -tangent bundle takes on a primary role over that of the usual tangent bundle). The associated Dolbeault complexes are defined in these sections accordingly.

Section 4 is a careful account of the structure inherited by the boundary.

In Section 5 we show that complex b -structures have no formal local invariants at boundary points. The issue here is that we do not have a Newlander-Nirenberg theorem that is valid in a neighborhoods of a point of the boundary, so no explicit local model for b -manifolds.

Section 6 is devoted to general aspects of b -elliptic first order complexes A . We introduce here the set $\text{spec}_{b,\mathcal{N}}^q(A)$, the boundary spectrum of the complex in degree q at the component \mathcal{N} of \mathcal{M} , and prove basic properties of the boundary spectrum (assuming that the boundary component \mathcal{N} is compact), including some aspects concerning Mellin transforms of A -closed forms. Some of these ideas are illustrated using the b -de Rham complex.

Section 7 is a systematic study of the $\overline{\partial}_b$ -complex of CR structures on \mathcal{N} associated with elements of the class β . Each $\beta \in \beta$ defines a CR structure, $\overline{\mathcal{K}}_\beta = \ker \beta$. Assuming that $\overline{\mathcal{V}}$ admits a \mathcal{T} -invariant Hermitian metric, we show that there is $\beta \in \beta$ such that the CR structure $\overline{\mathcal{K}}_\beta$ is \mathcal{T} -invariant.

In Section 8 we assume that $\overline{\mathcal{V}}$ is \mathcal{T} -invariant and show that for \mathcal{T} -invariant CR structures, a theorem proved in [13] gives that the cohomology spaces of the associated $\overline{\partial}_b$ -complex, viewed as the kernel of the Kohn Laplacian at the various degrees, split into eigenspaces of $-i\mathcal{L}_\mathcal{T}$. The eigenvalues of the latter operator are related to the indicial spectrum of the ${}^b\overline{\partial}$ -complex.

In Section 9 we prove a precise theorem on the indicial cohomology and spectrum for the ${}^b\overline{\partial}$ -complex under the assumption that $\overline{\mathcal{V}}$ admits a \mathcal{T} -invariant Hermitian metric.

Finally, we have included a very short appendix listing a number of basic definitions in connection with b -operators.

2. COMPLEX b -STRUCTURES

Let \mathcal{M} be a smooth manifold with smooth boundary. An almost CR b -structure on \mathcal{M} is a subbundle $\overline{\mathcal{W}}$ of the complexification, $\mathbb{C}{}^bT\mathcal{M} \rightarrow \mathcal{M}$ of the b -tangent bundle of \mathcal{M} (Melrose [5, 6]) such that

$$(2.1) \quad \mathcal{W} \cap \overline{\mathcal{W}} = 0$$

with $\mathcal{W} = \overline{\overline{\mathcal{W}}}$. If in addition

$$(2.2) \quad \mathcal{W} + \overline{\mathcal{W}} = \mathbb{C}{}^bT\mathcal{M}$$

then we say that $\overline{\mathcal{W}}$ is an almost complex b -structure and write ${}^bT^{0,1}\mathcal{M}$ instead of $\overline{\mathcal{W}}$ and ${}^bT^{1,0}\mathcal{M}$ for its conjugate. As is customary, the adverb ‘‘almost’’ is dropped

if \mathcal{W} is involutive. Note that since $C^\infty(\mathcal{M}; {}^bT\mathcal{M})$ is a Lie algebra, it makes sense to speak of involutive subbundles of ${}^bT\mathcal{M}$ (or its complexification).

Definition 2.3. A complex b -manifold is a manifold together with a complex b -structure.

By the Newlander-Nirenberg Theorem [14], the interior of complex b -manifold is a complex manifold. However, its boundary is not a CR manifold; rather, as we shall see, it naturally carries a family of CR structures parametrized by the defining functions of $\partial\mathcal{M}$ in \mathcal{M} which are positive in $\overset{\circ}{\mathcal{M}}$.

That $C^\infty(\mathcal{M}; {}^bT\mathcal{M})$ is a Lie algebra is an immediate consequence of the definition of the b -tangent bundle, which indeed can be characterized as being a vector bundle ${}^bT\mathcal{M} \rightarrow \mathcal{M}$ together with a vector bundle homomorphism

$$\text{ev} : {}^bT\mathcal{M} \rightarrow T\mathcal{M}$$

covering the identity such that the induced map

$$\text{ev}_* : C^\infty(\mathcal{M}; {}^bT\mathcal{M}) \rightarrow C^\infty(\mathcal{M}; T\mathcal{M})$$

is a $C^\infty(\mathcal{M}; \mathbb{R})$ -module isomorphism onto the submodule $C_{\text{tan}}^\infty(\mathcal{M}; T\mathcal{M})$ of smooth vector fields on \mathcal{M} which are tangential to the boundary of \mathcal{M} . Since $C_{\text{tan}}^\infty(\mathcal{M}, T\mathcal{M})$ is closed under Lie brackets, there is an induced Lie bracket on $C^\infty(\mathcal{M}; {}^bT\mathcal{M})$. The homomorphism ev is an isomorphism over the interior of \mathcal{M} , and its restriction to the boundary,

$$(2.4) \quad \text{ev}_{\partial\mathcal{M}} : {}^bT_{\partial\mathcal{M}}\mathcal{M} \rightarrow T\partial\mathcal{M}$$

is surjective. Its kernel, a fortiori a rank-one bundle, is spanned by a canonical section denoted $\mathfrak{r}\partial_{\mathfrak{r}}$. Here and elsewhere, \mathfrak{r} refers to any smooth defining function for $\partial\mathcal{M}$ in \mathcal{M} , by convention positive in the interior of \mathcal{M} .

Associated with a complex b -structure on \mathcal{M} there is a Dolbeault complex. Let ${}^b\Lambda^{0,q}\mathcal{M}$ denote the q -th exterior power of the dual of ${}^bT^{0,1}\mathcal{M}$. Then the operator

$$\dots \rightarrow C^\infty(\mathcal{M}; {}^b\Lambda^{0,q}\mathcal{M}) \xrightarrow{\bar{\partial}} C^\infty(\mathcal{M}; {}^b\Lambda^{0,q+1}\mathcal{M}) \rightarrow \dots$$

is define by

$$(2.5) \quad (q+1) \bar{\partial}\phi(V_0, \dots, V_q) = \sum_{j=0}^q V_j \phi(V_0, \dots, \hat{V}_j, \dots, V_q) + \sum_{j < k} (-1)^{j+k} \phi([V_j, V_k], V_0, \dots, \hat{V}_j, \dots, \hat{V}_k, \dots, V_q)$$

as with the standard de Rham differential (see Helgason [3, p. 21]) whenever ϕ is a smooth section of ${}^b\Lambda^q\mathcal{M}$ and $V_0, \dots, V_q \in C^\infty(\mathcal{M}; {}^bT^{0,1}\mathcal{M})$. In this formula V_j acts on functions via the vector field ev_*V_j . The involutivity of ${}^bT^{0,1}\mathcal{M}$ is used in the the terms involving brackets, of course. The same proof that $d \circ d = 0$ works here to give that $\bar{\partial}^2 = 0$. The formula

$$(2.6) \quad \bar{\partial}(f\phi) = f \bar{\partial}\phi + \bar{\partial}f \wedge \phi \text{ for } \phi \in C^\infty(\mathcal{M}; {}^b\Lambda^q\mathcal{M}) \text{ and } f \in C^\infty(\mathcal{M}),$$

implies that $\bar{\partial}$ is a first order operator.

Since we do not have at our disposal holomorphic frames (near the boundary) for the bundles of forms of type (p, q) for $p > 0$, we define $\bar{\partial}$ on forms of type (p, q) with $p > 0$ with the aid of the b -de Rham complex, exactly as in Folland and Kohn [2] for

standard complex structures and de Rham complex. The b -de Rham complex, we recall from Melrose [6], is the complex associated with the dual, $\mathbb{C}^{bT^*}\mathcal{M}$, of $\mathbb{C}^{bT}\mathcal{M}$,

$$\dots \rightarrow C^\infty(\mathcal{M}; {}^b\Lambda^r \mathcal{M}) \xrightarrow{{}^b d} C^\infty(\mathcal{M}; {}^b\Lambda^{r+1} \mathcal{M}) \rightarrow \dots$$

where ${}^b\Lambda^q \mathcal{M}$ denotes the r -th exterior power of $\mathbb{C}^{bT^*}\mathcal{M}$. The operators ${}^b d$ are defined by the same formula as (2.5), now however with the $V_j \in C^\infty(\mathcal{M}; \mathbb{C}^{bT}\mathcal{M})$. On functions f we have

$${}^b df = \text{ev}^* df.$$

More generally,

$$\text{ev}^* \circ d = {}^b d \circ \text{ev}^*$$

in any degree. Also,

$$(2.7) \quad {}^b d(f\phi) = f {}^b d\phi + {}^b df \wedge \phi \text{ for } \phi \in C^\infty(\mathcal{M}; {}^b\Lambda^r \mathcal{M}) \text{ and } f \in C^\infty(\mathcal{M}).$$

It is convenient to note here that for $f \in C^\infty(\mathcal{M})$,

$$(2.8) \quad {}^b df \text{ vanishes on } \partial\mathcal{M} \text{ if } f \text{ does.}$$

Now, with the obvious definition,

$$(2.9) \quad {}^b\Lambda^r \mathcal{M} = \bigoplus_{p+q=r} {}^b\Lambda^{p,q} \mathcal{M}.$$

Using the special cases

$$\begin{aligned} {}^b d : C^\infty(\mathcal{M}; {}^b\Lambda^{0,1}) &\rightarrow C^\infty(\mathcal{M}; {}^b\Lambda^{1,1}) + C^\infty(\mathcal{M}; {}^b\Lambda^{0,2}), \\ {}^b d : C^\infty(\mathcal{M}; {}^b\Lambda^{1,0}) &\rightarrow C^\infty(\mathcal{M}; {}^b\Lambda^{2,0}) + C^\infty(\mathcal{M}; {}^b\Lambda^{1,1}), \end{aligned}$$

consequences of the involutivity of ${}^bT^{0,1}\mathcal{M}$ and its conjugate, one gets

$${}^b d\phi \in C^\infty(\mathcal{M}; {}^b\Lambda^{p+1,q} \mathcal{M}) \oplus C^\infty(\mathcal{M}; {}^b\Lambda^{p,q+1} \mathcal{M}) \quad \text{if } \phi \in C^\infty(\mathcal{M}; {}^b\Lambda^{p,q} \mathcal{M})$$

for general (p, q) . Let $\pi_{p,q} : {}^b\Lambda^k \mathcal{M} \rightarrow {}^b\Lambda^{p,q} \mathcal{M}$ be the projection according to the decomposition (2.9), and define

$${}^b \partial = \pi_{p+1,q} {}^b d, \quad \bar{{}^b \partial} = \pi_{q,p+1} {}^b d,$$

so ${}^b d = {}^b \partial + \bar{{}^b \partial}$. The operators $\bar{{}^b \partial}$ are identical to the $\bar{\partial}$ -operators over the interior of \mathcal{M} and with the previously defined ${}^b \bar{\partial}$ operators on $(0, q)$ -forms, and give a complex

$$(2.10) \quad \dots \rightarrow C^\infty(\mathcal{M}; {}^b\Lambda^{p,q} \mathcal{M}) \xrightarrow{\bar{{}^b \partial}} C^\infty(\mathcal{M}; {}^b\Lambda^{p,q+1} \mathcal{M}) \rightarrow \dots$$

for each p . On functions $f : \mathcal{M} \rightarrow \mathbb{C}$,

$$(2.11) \quad \bar{{}^b \partial} f = \pi_{0,1} {}^b df.$$

The formula

$$(2.7') \quad \bar{{}^b \partial} f \phi = \bar{{}^b \partial} f \wedge \phi + f \bar{{}^b \partial} \phi, \quad f \in C^\infty(\mathcal{M}), \quad \phi \in C^\infty(\mathcal{M}; {}^b\Lambda^{p,q} \mathcal{M}),$$

a consequence of (2.7), implies that $\bar{{}^b \partial}$ is a first order operator. As a consequence of (2.8),

$$(2.8') \quad \bar{{}^b \partial} f \text{ vanishes on } \partial\mathcal{M} \text{ if } f \text{ does.}$$

The operators of the b -de Rham complex are first order operators because of (2.7), and (2.8) implies that these are b -operators, see (A.1). Likewise, (2.7') and

(2.8') imply that in any bidegree, the operator $\phi \mapsto \tau^{-1} \bar{\partial} \tau \phi$ has coefficients smooth up to the boundary, so

$$(2.12) \quad \bar{\partial} \in \text{Diff}_b^1(\mathcal{M}; {}^b\Lambda^{p,q}\mathcal{M}, {}^b\Lambda^{p,q+1}\mathcal{M}),$$

see (A.1). We also get from these formulas that the b -symbol of $\bar{\partial}$ is

$$(2.13) \quad {}^b\sigma(\bar{\partial})(\xi)(\phi) = i\pi_{0,1}(\xi) \wedge \phi, \quad x \in \mathcal{M}, \xi \in {}^bT_x^*\mathcal{M}, \phi \in {}^b\Lambda_x^{p,q}\mathcal{M},$$

see (A.2). Since $\pi_{0,1}$ is injective on the real b -cotangent bundle (this follows from (2.2)), the complex (2.10) is b -elliptic.

3. HOLOMORPHIC VECTOR BUNDLES

The notion of holomorphic vector bundle in the b -category is a translation of the standard one using connections. Let $\rho : F \rightarrow \mathcal{M}$ be a complex vector bundle. Recall from [6] that a b -connection on F is a linear operator

$${}^b\nabla : C^\infty(\mathcal{M}; F) \rightarrow C^\infty(\mathcal{M}; {}^b\Lambda^1\mathcal{M} \otimes F)$$

such that

$$(3.1) \quad {}^b\nabla f\phi = f {}^b\nabla\phi + {}^bdf \otimes \phi$$

for each $\phi \in C^\infty(\mathcal{M}; F)$ and $f \in C^\infty(\mathcal{M})$. This property automatically makes ${}^b\nabla$ a b -operator.

A standard connection $\nabla : C^\infty(\mathcal{M}; F) \rightarrow C^\infty(\mathcal{M}; \Lambda^1\mathcal{M} \otimes F)$ determines a b -connection by composition with

$$\text{ev}^* \otimes I : \Lambda^1\mathcal{M} \otimes F \rightarrow {}^b\Lambda^1\mathcal{M} \otimes F,$$

but b -connections are more general than standard connections. Indeed, the difference between the latter and the former can be any smooth section of the bundle $\text{Hom}(F, {}^b\Lambda^1\mathcal{M} \otimes F)$. A b -connection ${}^b\nabla$ on F arises from a standard connection if and only if ${}^b\nabla_{\tau\partial_\tau} = 0$ along $\partial\mathcal{M}$.

As in the standard situation, the b -connection ${}^b\nabla$ determines operators

$$(3.2) \quad {}^b\nabla : C^\infty(\mathcal{M}; {}^b\Lambda^k\mathcal{M} \otimes F) \rightarrow C^\infty(\mathcal{M}; {}^b\Lambda^{k+1}\mathcal{M} \otimes F)$$

by way of the usual formula translated to the b setting:

$$(3.3) \quad {}^b\nabla(\alpha \otimes \phi) = (-1)^k \alpha \wedge {}^b\nabla\phi + {}^bd\alpha \wedge \phi, \quad \phi \in C^\infty(\mathcal{M}; F), \alpha \in {}^b\Lambda^k\mathcal{M}.$$

Since

$${}^b\nabla\tau\alpha \otimes \phi = \tau {}^b\nabla(\alpha \otimes \phi) + {}^bd\tau \wedge \alpha \otimes \phi$$

is smooth and vanishes on $\partial\mathcal{M}$, also

$${}^b\nabla \in \text{Diff}_b^1(\mathcal{M}; {}^b\Lambda^k\mathcal{M} \otimes F, {}^b\Lambda^{k+1}\mathcal{M} \otimes F).$$

The principal b -symbol of (3.2), easily computed using (3.3) and

$${}^b\sigma({}^b\nabla)({}^bdf)(\phi) = \lim_{\tau \rightarrow \infty} \frac{e^{-i\tau f}}{\tau} {}^b\nabla e^{i\tau f} \phi$$

for $f \in C^\infty(\mathcal{M}; \mathbb{R})$ and $\phi \in C^\infty(\mathcal{M}; {}^b\Lambda^k\mathcal{M} \otimes F)$, is

$${}^b\sigma({}^b\nabla)(\xi)(\phi) = i\xi \wedge \phi, \quad \xi \in {}^bT_x^*\mathcal{M}, \phi \in {}^b\Lambda_x^k\mathcal{M} \otimes F_x, x \in \mathcal{M}.$$

As expected, the connection is called holomorphic if the component in ${}^b\Lambda^{0,2}\mathcal{M} \otimes F$ of the curvature operator

$$\Omega = {}^b\nabla^2 : C^\infty(\mathcal{M}; F) \rightarrow C^\infty(\mathcal{M}; {}^b\Lambda^2\mathcal{M} \otimes F),$$

vanishes. Such a connection gives F the structure of a complex b -manifold. Its complex b -structure can be described locally as in the standard situation, as follows. Fix a frame η_μ for F and let the ω_μ^ν be the local sections of ${}^b\Lambda^{0,1}\mathcal{M}$ such that

$${}^b\bar{\partial}\eta_\mu = \sum_\nu \omega_\mu^\nu \otimes \eta_\nu.$$

Denote by ζ^μ the fiber coordinates determined by the frame η_μ . Let V_1, \dots, V_{n+1} be a frame of ${}^bT^{0,1}\mathcal{M}$ over U , denote by \tilde{V}_j the sections of $\mathbb{C}{}^bTF$ over $\rho^{-1}(U)$ which project on the V_j and satisfy $\tilde{V}_j\zeta^\mu = \tilde{V}_j\bar{\zeta}^\mu = 0$ for all μ , and by ∂_{ζ^μ} the vertical vector fields such that $\partial_{\zeta^\mu}\zeta^\nu = \delta_\mu^\nu$ and $\partial_{\zeta^\mu}\bar{\zeta}^\nu = 0$. Then the sections

$$(3.4) \quad \tilde{V}_j - \sum_{\mu,\nu} \zeta^\mu \langle \omega_\mu^\nu, V_j \rangle \partial_{\zeta^\nu}, \quad j = 1, \dots, n+1, \quad \partial_{\bar{\zeta}^\nu}, \quad \nu = 1, \dots, k$$

of $\mathbb{C}{}^bTF$ over $\rho^{-1}(U)$ form a frame of ${}^bT^{0,1}F$. As in the standard situation, the involutivity of this subbundle of $\mathbb{C}{}^bTF$ is equivalent to the condition on the vanishing of the $(0,2)$ component of the curvature of ${}^b\nabla$. A vector bundle $F \rightarrow \mathcal{M}$ together with the complex b -structure determined by a choice of holomorphic b -connection (if one exists at all) is a holomorphic vector bundle.

The $\bar{\partial}$ operator of a holomorphic vector bundle is

$${}^b\bar{\partial} = (\pi_{0,q+1} \otimes I) \circ {}^b\nabla : C^\infty(\mathcal{M}; {}^b\Lambda^{0,q}\mathcal{M} \otimes F) \rightarrow C^\infty(\mathcal{M}; {}^b\Lambda^{0,q+1}\mathcal{M} \otimes F).$$

As is the case for standard complex structures, the condition on the curvature of ${}^b\nabla$ implies that these operators form a complex, b -elliptic since

$${}^b\partial({}^b\bar{\partial})(\xi)(\phi) = i\pi_{0,1}(\xi) \wedge \phi, \quad \xi \in {}^bT_x^*\mathcal{M}, \quad \phi \in {}^b\Lambda_x^k\mathcal{M} \otimes F_x, \quad x \in \mathcal{M}$$

and $\pi_{0,1}(\xi) = 0$ for $\xi \in {}^bT^*\mathcal{M}$ if and only if $\xi = 0$.

Also as usual, a b -connection ${}^b\nabla$ on a Hermitian vector bundle $F \rightarrow \mathcal{M}$ with Hermitian form h is Hermitian if

$${}^bdh(\phi, \psi) = h({}^b\nabla\phi, \psi) + h(\phi, {}^b\nabla\psi)$$

for every pair of smooth sections ϕ, ψ of F . In view of the definition of bd this means that for every $v \in \mathbb{C}{}^bT\mathcal{M}$ and sections as above,

$$\text{ev}(v)h(\phi, \psi) = h({}^b\nabla_v\phi, \psi) + h(\phi, {}^b\nabla_{\bar{v}}\psi)$$

On a complex b -manifold \mathcal{M} , if an arbitrary connection ${}^b\nabla'$ and the Hermitian form h are given for a vector bundle F , holomorphic or not, then there is a unique Hermitian b -connection ${}^b\nabla$ such that $\pi_{0,1}{}^b\nabla = \pi_{0,1}{}^b\nabla'$. Namely, let η_μ be a local orthonormal frame of F , let

$$(\pi_{0,1} \otimes I) \circ {}^b\nabla'\eta_\mu = \sum_\nu \omega_\mu^\nu \otimes \eta_\nu,$$

and let ${}^b\nabla$ be the connection defined in the domain of the frame by

$$(3.5) \quad {}^b\nabla\eta_\mu = (\omega_\mu^\nu - \bar{\omega}_\nu^\mu) \otimes \eta_\nu.$$

If the matrix of functions $Q = [q_\lambda^\mu]$ is unitary and $\tilde{\eta}_\lambda = \sum_\mu q_\lambda^\mu \eta_\mu$, then

$$(\pi_{0,1} \otimes I) \circ {}^b\nabla' \tilde{\eta}_\lambda = \sum_\nu \tilde{\omega}_\lambda^\sigma \otimes \tilde{\eta}_\sigma$$

with

$$\tilde{\omega}_\lambda^\sigma = \sum_\mu \bar{q}_\sigma^\mu {}^b\bar{\partial} q_\lambda^\mu + \sum_{\mu,\nu} \bar{q}_\sigma^\mu q_\lambda^\nu \omega_\nu^\mu,$$

using (3.1), that $Q^{-1} = [\bar{q}_\lambda^\mu]$, and that $\pi_{0,1} {}^bdf = {}^b\bar{\partial}f$. Thus

$$\begin{aligned} \tilde{\omega}_\lambda^\sigma - \bar{\omega}_\sigma^\lambda &= \sum_\mu (\bar{q}_\sigma^\mu {}^b\bar{\partial} q_\lambda^\mu - q_\lambda^\mu {}^b\partial \bar{q}_\sigma^\mu) + \sum_{\mu,\nu} (\bar{q}_\sigma^\mu q_\lambda^\nu \omega_\nu^\mu - q_\lambda^\mu \bar{q}_\sigma^\nu \bar{\omega}_\nu^\mu) \\ &= \sum_\mu ({}^b\bar{\partial} q_\lambda^\mu + {}^b\partial q_\lambda^\mu) \bar{q}_\sigma^\mu + \sum_{\mu,\nu} q_\lambda^\nu (\omega_\nu^\mu - \bar{\omega}_\mu^\nu) \bar{q}_\sigma^\mu \\ &= \sum_\mu {}^b d q_\lambda^\mu + \bar{q}_\sigma^\mu + \sum_{\mu,\nu} q_\lambda^\nu (\omega_\nu^\mu - \bar{\omega}_\mu^\nu) \bar{q}_\sigma^\mu \end{aligned}$$

using that $\overline{{}^b\bar{\partial}f} = {}^b\partial\bar{f}$ and that $\sum_\mu q_\lambda^\mu {}^b\partial \bar{q}_\sigma^\mu = -\sum_\mu {}^b\partial q_\lambda^\mu \bar{q}_\sigma^\mu$ because $\sum_\mu q_\lambda^\mu \bar{q}_\sigma^\mu$ is constant, and that ${}^b\bar{\partial} q_\lambda^\mu + {}^b\partial q_\lambda^\mu = {}^b d q_\lambda^\mu$. Thus there is a globally defined Hermitian connection locally given by (3.5). We leave to the reader to verify that this connection is Hermitian. Clearly ${}^b\nabla$ is the unique Hermitian connection such that $\pi_{0,1} {}^b\nabla = \pi_{0,1} {}^b\nabla'$. When ${}^b\nabla'$ is a holomorphic connection, ${}^b\nabla$ is the unique Hermitian holomorphic connection.

Lemma 3.6. *The vector bundles ${}^b\Lambda^{p,0}\mathcal{M}$ are holomorphic.*

We prove this by exhibiting a holomorphic b -connection. Fix an auxiliary Hermitian metric on ${}^b\Lambda^{p,0}\mathcal{M}$ and pick an orthonormal frame (η_μ) of ${}^b\Lambda^{p,0}\mathcal{M}$ over some open set U . Let ω_μ^ν be the unique sections of ${}^b\Lambda^{0,1}\mathcal{M}$ such that

$${}^b\bar{\partial}\eta_\mu = \sum_\nu \omega_\mu^\nu \wedge \eta_\nu,$$

and let ${}^b\nabla$ be the b -connection defined on U by the formula (3.5). As in the previous paragraph, this gives a globally defined b -connection. That it is holomorphic follows from

$${}^b\bar{\partial}\omega_\mu^\nu + \sum_\lambda \omega_\lambda^\nu \wedge \omega_\mu^\lambda = 0,$$

a consequence of $\overline{{}^b\bar{\partial}^2} = 0$. Evidently, with the identifications ${}^b\Lambda^{0,q}\mathcal{M} \otimes {}^b\Lambda^{p,0}\mathcal{M} = {}^b\Lambda^{p,q}\mathcal{M}$, $\pi_{p,q+1} {}^b\nabla$ is the ${}^b\bar{\partial}$ operator in (2.12).

4. THE BOUNDARY A COMPLEX b -MANIFOLD

Suppose that \mathcal{M} is a complex b -manifold and \mathcal{N} is a component of its boundary. We shall assume \mathcal{N} compact, although for the most part this is not necessary.

The homomorphism

$$\text{ev} : \mathbb{C}{}^bT\mathcal{M} \rightarrow \mathbb{C}T\mathcal{M}$$

is an isomorphism over the interior of \mathcal{M} , and its restriction to \mathcal{N} maps onto $\mathbb{C}T\mathcal{N}$ with kernel spanned by $\tau\partial_\tau$. Write

$$\text{ev}_\mathcal{N} : \mathbb{C}{}^bT_\mathcal{N}\mathcal{M} \rightarrow \mathbb{C}T\mathcal{N}$$

for this restriction and

$$(4.1) \quad \Phi : {}^bT_{\mathcal{N}}^{0,1}\mathcal{M} \rightarrow \bar{\mathcal{V}}$$

for of the restriction of $\text{ev}_{\mathcal{N}}$ to ${}^bT_{\mathcal{N}}^{0,1}\mathcal{M}$. From (2.1) and the fact that the kernel of $\text{ev}_{\mathcal{N}}$ is spanned by the real section $\mathfrak{r}\partial_{\mathfrak{r}}$ one obtains that Φ is injective, so its image,

$$\bar{\mathcal{V}} = \Phi({}^bT_{\mathcal{N}}^{0,1}\mathcal{M})$$

is a subbundle of $\mathbb{C}T\mathcal{N}$.

Since ${}^bT_{\mathcal{N}}^{0,1}\mathcal{M}$ is involutive, so is $\bar{\mathcal{V}}$, see [7, Proposition 3.12]. From (2.2) and the fact that $\text{ev}_{\mathcal{N}}$ maps onto $\mathbb{C}T\mathcal{N}$, one obtains that

$$(4.2) \quad \mathcal{V} + \bar{\mathcal{V}} = \mathbb{C}T\mathcal{N},$$

see [7, Lemma 3.13]. Thus

Lemma 4.3. *$\bar{\mathcal{V}}$ is an elliptic structure.*

This just means what we just said: $\bar{\mathcal{V}}$ is involutive and (4.2) holds, see Treves [16, 17]; the sum need not be direct. All elliptic structures are locally of the same kind, depending only on the dimension of $\mathcal{V} \cap \bar{\mathcal{V}}$. This is a result of Nirenberg [15] (see also Hörmander [4]) extending the Newlander-Nirenberg theorem. In the case at hand, $\bar{\mathcal{V}} \cap \mathcal{V}$ has rank 1 because of the relation

$$\text{rank}_{\mathbb{C}}(\mathcal{V} \cap \bar{\mathcal{V}}) = 2 \text{rank}_{\mathbb{C}} \bar{\mathcal{V}} - \dim \mathcal{N}$$

which holds whenever (4.2) holds.

Every $p_0 \in \mathcal{N}$ has a neighborhood in which there coordinates x^1, \dots, x^{2n}, t such that with $z^j = x^j + \mathfrak{m}x^{j+n}$, the vector fields

$$(4.4) \quad \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial t}$$

span $\bar{\mathcal{V}}$ near p_0 . The function (z^1, \dots, z^n, t) is called a hypoanalytic chart (Baouendi, Chang, and Treves [1], Treves [17]).

The intersection $\bar{\mathcal{V}} \cap \mathcal{V}$ is, in the case we are discussing, spanned by a canonical globally defined real vector field. Namely, let $\mathfrak{r}\partial_{\mathfrak{r}}$ be the canonical section of ${}^bT\mathcal{M}$ along \mathcal{N} . There is a unique section $J\mathfrak{r}\partial_{\mathfrak{r}}$ of ${}^bT\mathcal{M}$ along \mathcal{N} such that $\mathfrak{r}\partial_{\mathfrak{r}} + iJ\mathfrak{r}\partial_{\mathfrak{r}}$ is a section of ${}^bT^{0,1}\mathcal{M}$ along \mathcal{N} . Then

$$\mathcal{T} = \text{ev}_{\mathcal{N}}(J\mathfrak{r}\partial_{\mathfrak{r}})$$

is a nonvanishing real vector field in $\mathcal{V} \cap \bar{\mathcal{V}}$, (see [8, Lemma 2.1]). Using the isomorphism (4.1) we have

$$\mathcal{T} = \Phi(J(\mathfrak{r}\partial_{\mathfrak{r}}) - i\mathfrak{r}\partial_{\mathfrak{r}}).$$

Because $\bar{\mathcal{V}}$ is involutive, there is yet another complex, this time associated with the exterior powers of the dual of $\bar{\mathcal{V}}$:

$$(4.5) \quad \dots \rightarrow C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{V}}^*) \xrightarrow{\mathbb{D}} C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{V}}^*) \rightarrow \dots,$$

where \mathbb{D} is defined by the formula (2.5) where now the V_j are sections of $\bar{\mathcal{V}}$. The complex (4.5) is elliptic because of (4.2). For a function f we have $\mathbb{D}f = \iota^* df$, where $\iota^* : \mathbb{C}T^*\mathcal{N} \rightarrow \bar{\mathcal{V}}^*$ is the dual of the inclusion homomorphism $\iota : \bar{\mathcal{V}} \rightarrow \mathbb{C}T\mathcal{N}$.

For later use we show:

Lemma 4.6. *Suppose that \mathcal{N} is compact and connected. If $\zeta : \mathcal{N} \rightarrow \mathbb{C}$ solves $\mathbb{D}\zeta = 0$, then ζ is constant.*

Proof. Let p_0 be an extremal point of $|\zeta|$. Fix a hypoanalytic chart (z, t) for $\bar{\mathcal{V}}$ centered at p_0 . Since $\bar{\mathbb{D}}\zeta = 0$, $\zeta(z, t)$ is independent of t and $\partial_{\bar{z}^\nu}\zeta = 0$. So there is a holomorphic function Z defined in a neighborhood of 0 in \mathbb{C}^n such that $\zeta = Z \circ z$. Then $|Z|$ has a maximum at 0, so Z is constant near 0. Therefore ζ is constant, say $\zeta(p) = c$, near p_0 . Let $C = \{p : \zeta(p) = c\}$, a closed set. Let $p_1 \in C$. Since p_1 is also an extremal point of ζ , the above argument gives that ζ is constant near p_1 , therefore equal to c . Thus C is open, and consequently ζ is constant on \mathcal{N} . \square

Since the operators $\bar{\partial} : C^\infty(\mathcal{M}, b\Lambda^{0,q}\mathcal{M}) \rightarrow C^\infty(\mathcal{M}, b\Lambda^{0,q+1}\mathcal{M})$ are totally characteristic, they induce operators

$$\bar{\partial}_b : C^\infty(\mathcal{N}, b\Lambda_{\mathcal{N}}^{0,q}\mathcal{M}) \rightarrow C^\infty(\mathcal{M}, b\Lambda_{\mathcal{N}}^{0,q+1}\mathcal{M}),$$

see (A.3); these boundary operators define a complex because of (A.4). By way of the dual

$$(4.7) \quad \Phi^* : \bar{\mathcal{V}}^* \rightarrow b\Lambda_{\mathcal{N}}^{0,1}\mathcal{M}$$

of the isomorphism (4.1) the operators $\bar{\partial}_b$ become identical to the operators of the $\bar{\mathbb{D}}$ -complex (4.5): The diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{V}}^*) & \xrightarrow{\bar{\mathbb{D}}} & C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{V}}^*) & \longrightarrow & \dots \\ & & \Phi^* \downarrow & & \downarrow \Phi^* & & \\ \dots & \longrightarrow & C^\infty(\mathcal{N}, b\Lambda_{\mathcal{N}}^{0,q}\mathcal{M}) & \xrightarrow{\bar{\partial}_b} & C^\infty(\mathcal{M}, b\Lambda_{\mathcal{N}}^{0,q+1}\mathcal{M}) & \longrightarrow & \dots \end{array}$$

is commutative and the vertical arrows are isomorphisms. This can be proved by writing the $\bar{\partial}$ operators using Cartan's formula (2.5) for $\bar{\partial}$ and $\bar{\mathbb{D}}$ and comparing the resulting expressions.

Let $\mathfrak{r} : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth defining function for $\partial\mathcal{M}$, $\mathfrak{r} > 0$ in the interior of \mathcal{M} . Then $\bar{\partial}\mathfrak{r}$ is smooth and vanishes on $\partial\mathcal{M}$, so $\frac{\bar{\partial}\mathfrak{r}}{\mathfrak{r}}$ is also a smooth $\bar{\partial}$ -closed section of $b\Lambda^{0,1}\mathcal{M}$. Thus we get a $\bar{\mathbb{D}}$ -closed element

$$(4.8) \quad \beta_{\mathfrak{r}} = [\Phi^*]^{-1} \frac{\bar{\partial}\mathfrak{r}}{\mathfrak{r}} \in C^\infty(\partial\mathcal{M}; \bar{\mathcal{V}}^*).$$

By definition,

$$\langle \beta_{\mathfrak{r}}, \mathcal{T} \rangle = \left\langle \frac{\bar{\partial}\mathfrak{r}}{\mathfrak{r}}, J(\mathfrak{r}\partial_{\mathfrak{r}}) - i\mathfrak{r}\partial_{\mathfrak{r}} \right\rangle.$$

Extend the section $\mathfrak{r}\partial_{\mathfrak{r}}$ to a section of ${}^bT\mathcal{M}$ over a neighborhood U of \mathcal{N} in \mathcal{M} with the property that $\mathfrak{r}\partial_{\mathfrak{r}}\mathfrak{r} = \mathfrak{r}$. In U we have

$$\langle \bar{\partial}\mathfrak{r}, J(\mathfrak{r}\partial_{\mathfrak{r}}) - i\mathfrak{r}\partial_{\mathfrak{r}} \rangle = (J(\mathfrak{r}\partial_{\mathfrak{r}}) - i\mathfrak{r}\partial_{\mathfrak{r}})\mathfrak{r} = J(\mathfrak{r}\partial_{\mathfrak{r}})\mathfrak{r} - i\mathfrak{r}.$$

The function $J(\mathfrak{r}\partial_{\mathfrak{r}})\mathfrak{r}$ is smooth, real-valued, and vanishes along the boundary. So $\mathfrak{r}^{-1}J(\mathfrak{r}\partial_{\mathfrak{r}})\mathfrak{r}$ is smooth, real-valued. Thus

$$\langle \beta_{\mathfrak{r}}, \mathcal{T} \rangle = a_{\mathfrak{r}} - i$$

on \mathcal{N} for some smooth function $a_{\mathfrak{r}} : \mathcal{N} \rightarrow \mathbb{R}$, see [8, Lemma 2.5].

If \mathfrak{r}' is another defining function for $\partial\mathcal{M}$, then $\mathfrak{r}' = \mathfrak{r}e^u$ for some smooth function $u : \mathcal{M} \rightarrow \mathbb{R}$. Then

$$\bar{\partial}\mathfrak{r}' = e^u \bar{\partial}\mathfrak{r} + e^u \mathfrak{r} \bar{\partial}u$$

and it follows that

$$\beta_{\mathfrak{r}'} = \beta_{\mathfrak{r}} + \bar{\mathbb{D}}u.$$

In particular,

$$a_{\mathfrak{r}'} = a_{\mathfrak{r}} + \mathcal{T}u.$$

Let \mathbf{a}_t denote the one-parameter group of diffeomorphisms generated by \mathcal{T} .

Proposition 4.9. *The functions $a_{\text{av}}^{\text{sup}}, a_{\text{av}}^{\text{inf}} : \mathcal{N} \rightarrow \mathbb{R}$ defined by*

$$a_{\text{av}}^{\text{sup}}(p) = \limsup_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t a_{\mathfrak{r}}(\mathbf{a}_s(p)) ds, \quad a_{\text{av}}^{\text{inf}}(p) = \liminf_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t a_{\mathfrak{r}}(\mathbf{a}_s(p)) ds$$

are invariants of the complex b -structure, that is, they are independent of the defining function \mathfrak{r} . The equality $a_{\text{av}}^{\text{sup}} = a_{\text{av}}^{\text{inf}}$ holds for some \mathfrak{r} if and only if it holds for all \mathfrak{r} .

Indeed,

$$\lim_{t \rightarrow \infty} \left(\frac{1}{2t} \int_{-t}^t a_{\mathfrak{r}'}(\mathbf{a}_s(p)) ds - \frac{1}{2t} \int_{-t}^t a_{\mathfrak{r}}(\mathbf{a}_s(p)) ds \right) = \lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t \frac{d}{ds} u(\mathbf{a}_s(p)) ds = 0$$

because u is bounded (since \mathcal{N} is compact).

The functions $a_{\text{av}}^{\text{sup}}, a_{\text{av}}^{\text{inf}}$ are constant on orbits of \mathcal{T} , but they may not be smooth.

Example 4.10. Let \mathcal{N} be the unit sphere in \mathbb{C}^{n+1} centered at the origin. Write (z^1, \dots, z^{n+1}) for the standard coordinates in \mathbb{C}^{n+1} . Fix $\tau_1, \dots, \tau_{n+1} \in \mathbb{R} \setminus \{0\}$, all of the same sign, and let

$$\mathcal{T} = i \sum_{j=1}^{n+1} \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}).$$

This vector field is real and tangent to \mathcal{N} . Let $\bar{\mathcal{K}}$ be the standard CR structure of \mathcal{N} as a submanifold of \mathbb{C}^{n+1} (the part of $T^{0,1}\mathbb{C}^{n+1}$ tangential to \mathcal{N}). The condition that the τ_j are different from 0 and have the same sign ensures that \mathcal{T} is never in $\mathcal{K} \oplus \bar{\mathcal{K}}$. Indeed, the latter subbundle of $\mathbb{C}T\mathcal{N}$ is the annihilator of the pullback to \mathcal{N} of $i\bar{\partial} \sum_{\ell=1}^{n+1} |z^\ell|^2$. The pairing of this form with \mathcal{T} is

$$\left\langle i \sum_{\ell=1}^{n+1} z^\ell d\bar{z}^\ell, i \sum_{j=1}^{n+1} \tau_j (z^j \partial_{z^j} - \bar{z}^j \partial_{\bar{z}^j}) \right\rangle = \sum_{j=1}^{n+1} \tau_j |z^j|^2,$$

a function that vanishes nowhere if and only if all τ_j are different from zero and have the same sign. Thus $\bar{\mathcal{V}} = \bar{\mathcal{K}} \oplus \text{span}_{\mathbb{C}} \mathcal{T}$ is a subbundle of $\mathbb{C}T\mathcal{N}$ of rank $n+1$ with the property that $\mathcal{V} + \bar{\mathcal{V}} = \mathbb{C}T\mathcal{N}$. To show that $\bar{\mathcal{V}}$ is involutive we first note that $\bar{\mathcal{K}}$ is the annihilator of the pullback to \mathcal{N} of the span of the differentials dz^1, \dots, dz^{n+1} . Let $\mathcal{L}_{\mathcal{T}}$ denote the Lie derivative with respect to \mathcal{T} . Then $\mathcal{L}_{\mathcal{T}} dz^j = i\tau_j dz^j$, so if L is a CR vector field, then so is $[L, \mathcal{T}]$. Since in addition $\bar{\mathcal{K}}$ and $\text{span}_{\mathbb{C}} \mathcal{T}$ are themselves involutive, $\bar{\mathcal{V}}$ is involutive. Thus $\bar{\mathcal{V}}$ is an elliptic structure with $\mathcal{V} \cap \bar{\mathcal{V}} = \text{span}_{\mathbb{C}} \mathcal{T}$. Let β be the section of $\bar{\mathcal{V}}^*$ which vanishes on $\bar{\mathcal{K}}$ and satisfies $\langle \beta, \mathcal{T} \rangle = -i$. Let \mathbb{D} denote the operators of the associated differential complex. Then $\mathbb{D}\beta = 0$, since β vanishes on commutators of sections of $\bar{\mathcal{K}}$ (since $\bar{\mathcal{K}}$ is involutive) and on commutators of \mathcal{T} with sections of $\bar{\mathcal{K}}$ (since such commutators are in $\bar{\mathcal{K}}$).

If the τ_j are positive (negative), this example may be viewed as the boundary of a blowup (compactification) of \mathbb{C}^{n+1} , see [9].

Let now $\rho : F \rightarrow \mathcal{M}$ be a holomorphic vector bundle. Its $\bar{\partial}$ -complex also determines a complex along \mathcal{N} ,

$$(4.11) \quad \cdots \rightarrow C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{V}}^* \otimes F_{\mathcal{N}}) \xrightarrow{\bar{\mathbb{D}}} C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{V}}^* \otimes F_{\mathcal{N}}) \rightarrow \cdots,$$

where $\bar{\mathbb{D}}$ is defined using the boundary operators $\bar{\partial}_b$ and the isomorphism (4.7):

$$(4.12) \quad \bar{\mathbb{D}}(\phi \otimes \eta) = (\Phi^*)^{-1} \bar{\partial}_b[\Phi^*(\phi \otimes \eta)]$$

where Φ^* means $\Phi^* \otimes I$. These operators can be expressed locally in terms of the operators of the complex (4.5). Fix a smooth frame η_μ , $\mu = 1, \dots, k$, of F in a neighborhood $U \subset \mathcal{M}$ of $p_0 \in \mathcal{N}$, and suppose

$$\bar{\partial} \eta_\mu = \sum_\nu \omega_\mu^\nu \otimes \eta_\nu.$$

The ω_μ^ν are local sections of ${}^b\Lambda^{0,1} \mathcal{M}$, and if $\sum_\mu \phi^\mu \otimes \eta_\mu$ is a section of ${}^b\Lambda^{0,q} \mathcal{M} \otimes F$ over U , then

$$\bar{\partial} \sum \phi^\mu \otimes \eta_\mu = \sum_\nu (\bar{\partial} \phi^\nu + \sum_\mu \omega_\mu^\nu \wedge \phi^\mu) \otimes \eta_\nu.$$

Therefore, using the identification (4.7), the boundary operator $\bar{\partial}_b$ is the operator given locally by

$$(4.13) \quad \bar{\mathbb{D}} \sum \phi^\mu \otimes \eta_\mu = \sum_\nu (\bar{\mathbb{D}} \phi^\nu + \sum_\mu \omega_\mu^\nu \wedge \phi^\mu) \otimes \eta_\nu$$

where now the ϕ^μ are sections of ${}^b\Lambda^q \bar{\mathcal{V}}^*$, the ω_μ^ν are the sections of $\bar{\mathcal{V}}^*$ corresponding to the original ω_μ^ν via Φ^* , and $\bar{\mathbb{D}}$ on the right hand side of the formula is the operator associated with $\bar{\mathcal{V}}$.

The structure bundle ${}^bT^{0,1}F$ is locally given as the span of the sections (3.4). Applying the evaluation homomorphism $\mathbb{C}^bT_{\partial F}F \rightarrow \mathbb{C}T\partial F$ (over \mathcal{N}) to these sections gives vector fields on $F_{\mathcal{N}}$ forming a frame for the elliptic structure $\bar{\mathcal{V}}_F$ inherited by $F_{\mathcal{N}}$. Writing $V_j^0 = \text{ev}V_j$, this frame is just

$$(4.14) \quad \tilde{V}_j^0 - \sum_{\mu, \nu} \zeta^\mu \langle \omega_\mu^\nu, V_j^0 \rangle \partial_{\zeta^\nu}, \quad j = 1, \dots, n+1, \quad \partial_{\bar{\zeta}^\nu}, \quad \nu = 1, \dots, k,$$

where now the ω_μ^ν are the forms associated to the $\bar{\mathbb{D}}$ operator of $F_{\mathcal{N}}$. Alternatively, one may take the $\bar{\mathbb{D}}$ operators of $F_{\mathcal{N}}$ and use the formula (4.13) to define a subbundle of $\mathbb{C}TF$ locally as the span of the vector fields (4.14), a fortiori an elliptic structure on $F_{\mathcal{N}}$, involutive because

$$\bar{\mathbb{D}}\omega^\nu + \sum_\lambda \omega_\lambda^\nu \wedge \omega_\mu^\lambda = 0.$$

To obtain a formula for the canonical real vector field \mathcal{T}_F in $\bar{\mathcal{V}}_F$, let J_F be the almost complex b -structure of bTF and consider again the sections (3.4); they are defined in an open set $\rho^{-1}(U)$, U a neighborhood \mathcal{M} of a point of \mathcal{N} . Since the elements $\partial_{\bar{\zeta}^\nu}$ are sections of ${}^bT^{0,1}F$,

$$(4.15) \quad J_F \Re \partial_{\bar{\zeta}^\nu} = \Im \partial_{\bar{\zeta}^\nu}.$$

Pick a defining function \mathfrak{r} for \mathcal{N} . Then $\tilde{\mathfrak{r}} = \rho^* \mathfrak{r}$ is a defining function for $F_{\mathcal{N}}$. We may take $V_{n+1} = \mathfrak{r} \partial_{\mathfrak{r}} + iJ\mathfrak{r} \partial_{\mathfrak{r}}$ along $U \cap \mathcal{N}$. Then $\tilde{V}_{n+1} = \tilde{\mathfrak{r}} \partial_{\tilde{\mathfrak{r}}} + i\tilde{J}\tilde{\mathfrak{r}} \partial_{\tilde{\mathfrak{r}}}$ along

$\rho^{-1}(U) \cap F_{\mathcal{N}}$ and so

$$\begin{aligned} J_F \Re(\tilde{\mathfrak{r}}\partial_{\tilde{\mathfrak{t}}} + iJ\tilde{\mathfrak{r}}\partial_{\tilde{\mathfrak{t}}} - \sum_{\mu,\nu} \zeta^\mu \langle \omega_\mu^\nu, \mathfrak{r}\partial_{\mathfrak{t}} + iJ\mathfrak{r}\partial_{\mathfrak{t}} \rangle \partial_{\zeta^\nu}) = \\ \Im(\tilde{\mathfrak{r}}\partial_{\tilde{\mathfrak{t}}} + iJ\tilde{\mathfrak{r}}\partial_{\tilde{\mathfrak{t}}} - \sum_{\mu,\nu} \zeta^\mu \langle \omega_\mu^\nu, \mathfrak{r}\partial_{\mathfrak{t}} + iJ\mathfrak{r}\partial_{\mathfrak{t}} \rangle \partial_{\zeta^\nu}) \end{aligned}$$

along $\rho^{-1}(U) \cap F_{\mathcal{N}}$. Using (4.15) this gives

$$J_F \tilde{\mathfrak{r}}\partial_{\tilde{\mathfrak{t}}} = \widetilde{J\mathfrak{r}\partial_{\mathfrak{t}}} - 2\Im \sum_{\mu,\nu} \zeta^\mu \langle \omega_\mu^\nu, \mathfrak{r}\partial_{\mathfrak{t}} + iJ\mathfrak{r}\partial_{\mathfrak{t}} \rangle \partial_{\zeta^\nu}.$$

Applying the evaluation homomorphism gives

$$(4.16) \quad \mathcal{T}_F = \tilde{\mathcal{T}} - 2\Im \sum_{\mu,\nu} \zeta^\mu \langle \omega_\mu^\nu, \mathfrak{r}\partial_{\mathfrak{t}} + iJ\mathfrak{r}\partial_{\mathfrak{t}} \rangle \partial_{\zeta^\nu}$$

where $\tilde{\mathcal{T}}$ is the real vector field on $\rho^{-1}(U \cap \mathcal{N}) = \rho^{-1}(U) \cap F_{\mathcal{N}}$ which projects on \mathcal{T} and satisfies $\tilde{\mathcal{T}}\zeta^\mu = 0$ for all μ .

Let h be a Hermitian metric on F , and suppose that the frame η_μ is orthonormal. Applying \mathcal{T}_E as given in (4.16) to the function $|\zeta|^2 = \sum |\zeta^\mu|^2$ we get that \mathcal{T}_F is tangent to the unit sphere bundle of F if and only if

$$\langle \omega_\mu^\nu, \mathfrak{r}\partial_{\mathfrak{t}} + iJ\mathfrak{r}\partial_{\mathfrak{t}} \rangle - \overline{\langle \omega_\nu^\mu, \mathfrak{r}\partial_{\mathfrak{t}} + iJ\mathfrak{r}\partial_{\mathfrak{t}} \rangle} = 0$$

for all μ, ν . Equivalently, in terms of the isomorphism (4.7),

$$(4.17) \quad \langle (\Phi^*)^{-1}\omega_\mu^\nu, \mathcal{T} \rangle + \overline{\langle (\Phi^*)^{-1}\omega_\nu^\mu, \mathcal{T} \rangle} = 0 \quad \text{for all } \mu, \nu.$$

Definition 4.18. The Hermitian metric h will be called exact if (4.17) holds.

The terminology in Definition 4.18 is taken from the notion of exact Riemannian b -metric of Melrose [6, pg. 31]. For such metrics, the Levi-Civita b -connection has the property that ${}^b\nabla_{\mathfrak{r}\partial_{\mathfrak{t}}} = 0$ [op. cit., pg. 58]. We proceed to show that the Hermitian holomorphic connection of an exact Hermitian metric on F also has this property. Namely, suppose that h is an exact Hermitian metric, and let η_μ be an orthonormal frame of F . Then for the Hermitian holomorphic connection we have

$$\langle \omega_\mu^\nu - \bar{\omega}_\nu^\mu, \mathfrak{r}\partial_{\mathfrak{t}} \rangle = \langle \omega_\mu^\nu, \mathfrak{r}\partial_{\mathfrak{t}} \rangle - \overline{\langle \omega_\nu^\mu, \mathfrak{r}\partial_{\mathfrak{t}} \rangle} = \frac{1}{2} (\langle \omega_\mu^\nu, \mathfrak{r}\partial_{\mathfrak{t}} + iJ\mathfrak{r}\partial_{\mathfrak{t}} \rangle - \overline{\langle \omega_\nu^\mu, \mathfrak{r}\partial_{\mathfrak{t}} + iJ\mathfrak{r}\partial_{\mathfrak{t}} \rangle})$$

using that the ω_μ^ν are of type $(0, 1)$. Thus ${}^b\nabla_{\mathfrak{r}\partial_{\mathfrak{t}}} = 0$.

5. LOCAL INVARIANTS

Complex structures have no local invariants: every point of a complex n -manifold has a neighborhood biholomorphic to a ball in \mathbb{C}^n . It is natural to ask the same question about complex b -structures, namely,

is there a local model depending only on dimension for every complex b -structure?

In lieu of a Newlander-Nirenberg theorem, we show that complex b -structures have no local formal invariants at the boundary. More precisely:

Proposition 5.1. *Every $p_0 \in \mathcal{N}$ has a neighborhood V in \mathcal{M} on which there are smooth coordinates x^j , $j = 1, \dots, 2n + 2$ centered at p_0 with x^{n+1} vanishing on $V \cap \mathcal{N}$ such that with*

$$(5.2) \quad \bar{L}_j^0 = \frac{1}{2}(\partial_{x^j} + i\partial_{x^{j+n+1}}), \quad j \leq n, \quad \bar{L}_{n+1}^0 = \frac{1}{2}(x^{n+1}\partial_{x^{n+1}} + i\partial_{x^{2n+2}})$$

there are smooth functions γ_k^j vanishing to infinite order on $V \cap \mathcal{N}$ such that

$$\bar{L}_j = \bar{L}_j^0 + \sum_{k=1}^{n+1} \gamma_j^k L_k^0$$

is a frame for ${}^bT^{0,1}\mathcal{M}$ over V .

The proof will require some preparation. Let $\mathfrak{r} : \mathcal{M} \rightarrow \mathbb{R}$ be a defining function for $\partial\mathcal{M}$. Let $p_0 \in \mathcal{N}$, pick a hypoanalytic chart (z, t) (cf. (4.4)) centered at p_0 with $\mathcal{T}t = 1$. Let $U \subset \mathcal{N}$ be a neighborhood of p_0 contained in the domain of the chart, mapped by it to $B \times (-\delta, \delta) \subset \mathbb{C}^n \times \mathbb{R}$, where B is a ball with center 0 and δ is some small positive number. For reference purposes we state

Lemma 5.3. *On such U , the problem*

$$\bar{\mathbb{D}}\phi = \psi, \quad \psi \in C^\infty(U; \Lambda^{q+1}\bar{\mathcal{V}}^*|_U) \text{ and } \bar{\mathbb{D}}\psi = 0$$

has a solution in $C^\infty(U; \Lambda^q\bar{\mathcal{V}}^*|_U)$.

Extend the functions z^j and t to a neighborhood of p_0 in \mathcal{M} . Shrinking U if necessary, we may assume that in some neighborhood V of p_0 in \mathcal{M} with $V \cap \partial\mathcal{M} = U$, (z, t, \mathfrak{r}) maps V diffeomorphically onto $B \times (-\delta, \delta) \times [0, \varepsilon)$ for some $\delta, \varepsilon > 0$. Since the form $\beta_\mathfrak{r}$ defined in (4.8) is $\bar{\mathbb{D}}$ -closed, there is $\alpha \in C^\infty(U)$ such that

$$-i\bar{\mathbb{D}}\alpha = \beta_\mathfrak{r}.$$

Extend α to V as a smooth function. The section

$$(5.4) \quad \bar{b}\partial(\log \mathfrak{r} + i\alpha) = \frac{\bar{b}\partial \mathfrak{r}}{\mathfrak{r}} + i\bar{b}\partial\alpha$$

of ${}^b\Lambda^{0,1}\mathcal{M}$ over V vanishes on U , since $\beta_\mathfrak{r} + i\bar{\mathbb{D}}\alpha = 0$. So there is a smooth section ϕ of ${}^b\Lambda^{0,1}\mathcal{M}$ over V such that

$$\bar{b}\partial(\log \mathfrak{r} + i\alpha) = \mathfrak{r}e^{i\alpha}\phi.$$

Suppose $\zeta : U \rightarrow \mathbb{C}$ is a solution of $\bar{\mathbb{D}}\zeta = 0$ on U , and extend it to V . Then $\bar{b}\partial\zeta$ vanishes on U , so again we have

$$\bar{b}\partial\zeta = \mathfrak{r}e^{i\alpha}\psi.$$

for some smooth section ψ of ${}^b\Lambda^{0,1}\mathcal{M}$ over V . The following lemma will be applied for f_0 equal to $\log \mathfrak{r} + i\alpha$ or each of the functions z^j .

Lemma 5.5. *Let f_0 be smooth in $V \setminus U$ and suppose that $\bar{b}\partial f_0 = \mathfrak{r}e^{i\alpha}\psi_1$ with ψ_1 smooth on V . Then there is $f : V \rightarrow \mathbb{C}$ smooth vanishing at U such that $\bar{b}\partial(f_0 + f)$ vanishes to infinite order on U .*

Proof. Suppose that f_1, \dots, f_{N-1} are defined on V and that

$$(5.6) \quad \bar{b}\partial \sum_{k=0}^{N-1} (\mathfrak{r}e^{i\alpha})^k f_k = (\mathfrak{r}e^{i\alpha})^N \psi_N$$

holds with ψ_N smooth in V ; by the hypothesis, (5.6) holds when $N = 1$. Using (5.4) we get that $\bar{\partial}(\mathfrak{r}e^{i\alpha}) = (\mathfrak{r}e^{i\alpha})^2\phi$, therefore

$$0 = \bar{\partial}((\mathfrak{r}e^{i\alpha})^N\psi_N) = (\mathfrak{r}e^{i\alpha})^N[\bar{\partial}\psi_N + N\mathfrak{r}e^{i\alpha}\phi \wedge \psi_N],$$

which implies that $\bar{\partial}\psi_N = 0$ on U . With arbitrary f_N we have

$$\bar{\partial} \sum_{k=0}^N (\mathfrak{r}e^{i\alpha})^k f_k = (\mathfrak{r}e^{i\alpha})^N (\psi_N + \bar{\partial}f_N + N\mathfrak{r}e^{i\alpha}f_N\phi).$$

Since $\bar{\mathbb{D}}\psi_N = 0$ and $H_{\bar{\mathbb{D}}}^1(U) = 0$ by Lemma 5.3, there is a smooth function f_N defined in U such that $\bar{\mathbb{D}}f_N = -\psi_N$ in U . So there is χ_N such that $\psi_N + \bar{\partial}f_N = \mathfrak{r}e^{i\alpha}\chi_N$. With such f_N , (5.6) holds with $N+1$ in place of N and some ψ_{N+1} . Thus there is a sequence $\{f_j\}_{j=1}^{\infty}$ such that (5.6) holds for each N . Borel's lemma then gives f smooth with

$$f \sim \sum_{k=1}^{\infty} (\mathfrak{r}e^{i\alpha})^k f_k \quad \text{on } U$$

such that $\bar{\mathbb{D}}(f_0 + f)$ vanishes to infinite order on U . \square

Proof of Proposition 5.1. Apply the lemma with $f_0 = \log \mathfrak{r} + i\alpha$ to get a function f such that $\bar{\partial}(f_0 + f)$ vanishes to infinite order at U . Let

$$x^{n+1} = \mathfrak{r}e^{-\Im\alpha + \Re f}, \quad x^{2n+2} = \Re\alpha + \Im f.$$

These functions are smooth up to U .

Applying the lemma to each of the functions $f_0 = z^j$, $j = 1, \dots, n$ gives smooth functions ζ^j such that $\zeta^j = z^j$ on U and $\bar{\partial}\zeta^j = 0$ to infinite order at U . Define

$$x^j = \Re\zeta^j, \quad x^{j+n+1} = \Im\zeta^j, \quad j = 1, \dots, n.$$

The functions x^j , $j = 1 \dots, 2n+2$ are independent, and the forms

$$\eta^j = {}^b d\zeta^j, \quad j = 1 \dots, n, \quad \eta^{n+1} = \frac{1}{x^{n+1}e^{ix^{2n+2}}} {}^b d[x^{n+1}e^{ix^{2n+2}}]$$

together with their conjugates form a frame for $\mathbb{C}^b T\mathcal{M}$ near p_0 . Let $\eta_{1,0}^j$ and $\eta_{0,1}^j$ be the $(1,0)$ and $(0,1)$ components of η^j according to the complex b -structure of \mathcal{M} . Then

$$\eta_{0,1}^j = \sum_k p_k^j \eta^k + q_k^j \bar{\eta}^k.$$

Since $\eta_{0,1}^j = \bar{\partial}\zeta^j$ vanishes to infinite order at U , the coefficients p_k^j and q_k^j vanish to infinite order at U . Replacing this formula for $\eta_{0,1}^j$ in $\eta^j = \eta_{1,0}^j + \eta_{0,1}^j$ get

$$\sum_k (\delta_k^j - p_k^j) \eta^k - \sum_k q_k^j \bar{\eta}^k = \eta_{1,0}^j.$$

The matrix $I - [p_k^j]$ is invertible with inverse of the form $I + [P_k^j]$ with P_k^j vanishing to infinite order at U . So

$$(5.7) \quad \eta^j - \sum_k \gamma_k^j \bar{\eta}^k = \sum_k (\delta_k^j + P_k^j) \eta_{1,0}^k$$

with suitable γ_k^j vanishing to infinite order on U . Define the vector fields \bar{L}_j^0 as in (5.2). The vector fields

$$\bar{L}_j = \bar{L}_j^0 + \sum_k \gamma_j^k L_k^0, \quad j = 1, \dots, n+1$$

are independent and since $\langle \bar{L}_j^0, \eta^k \rangle = 0$ and $\langle L_j^0, \eta^k \rangle = \delta_j^k$, they annihilate each of the forms on the left hand side of (5.7). So they annihilate the forms $\eta_{1,0}^k$, which proves that the \bar{L}_j form a frame of ${}^bT^{0,1}\mathcal{M}$. \square

6. INDICIAL COMPLEXES

Throughout this section we assume that \mathcal{N} is a connected component of the boundary of a compact manifold \mathcal{M} . Let

$$(6.1) \quad \dots \rightarrow C^\infty(\mathcal{M}; E^q) \xrightarrow{A_q} C^\infty(\mathcal{M}; E^{q+1}) \rightarrow \dots$$

be a b -elliptic complex of operators $A_q \in \text{Diff}_b^1(\mathcal{M}; E^q, E^{q+1})$; the E^q , $q = 0, \dots, r$, are vector bundles over \mathcal{M} .

Note that since A_q is a first order operator,

$$(6.2) \quad A_q(f\phi) = fA_q\phi - i {}^b\sigma(A_q)({}^bdf)(\phi).$$

This formula follows from the analogous formula for the standard principal symbol and the definition of principal b -symbol. It follows from (6.2) and (2.8) that A_q defines an operator

$$A_{b,q} : \text{Diff}^1(\mathcal{N}; E_{\mathcal{N}}^q, E_{\mathcal{N}}^{q+1}).$$

Fix a smooth defining function $\mathfrak{r} : \mathcal{M} \rightarrow \mathbb{R}$ for $\partial\mathcal{M}$, $\mathfrak{r} > 0$ in the interior of \mathcal{M} , let

$$\mathcal{A}_q(\sigma) : \text{Diff}_b^1(\mathcal{N}; E_{\mathcal{N}}^q, E_{\mathcal{N}}^{q+1}), \quad \sigma \in \mathbb{C}$$

denote the indicial family of A_q with respect to \mathfrak{r} , see (A.5). Using (6.2) and defining

$$\Lambda_{\mathfrak{r},q} = {}^b\sigma(A_q)\left(\frac{{}^b d\mathfrak{r}}{\mathfrak{r}}\right),$$

the indicial family of A_q with respect to \mathfrak{r} is

$$(6.3) \quad \mathcal{A}_q(\sigma) = A_{b,q} + \sigma \Lambda_{\mathfrak{r},q} : C^\infty(\mathcal{N}; E_{\mathcal{N}}^q) \rightarrow C^\infty(\mathcal{N}; E_{\mathcal{N}}^{q+1}).$$

Because of (A.4), these operators form an elliptic complex

$$(6.4) \quad \dots \rightarrow C^\infty(\mathcal{N}; E_{\mathcal{N}}^q) \xrightarrow{\mathcal{A}_q(\sigma)} C^\infty(\mathcal{N}; E_{\mathcal{N}}^{q+1}) \rightarrow \dots$$

for each σ and each connected component \mathcal{N} of $\partial\mathcal{M}$. The operators depend on \mathfrak{r} , but the cohomology groups at a given σ for different defining functions \mathfrak{r} are isomorphic. Indeed, if \mathfrak{r}' is another defining function for $\partial\mathcal{M}$, then $\mathfrak{r}' = e^u \mathfrak{r}$ for some smooth real-valued function u , and a simple calculation gives

$$(A_{b,q} + \sigma \Lambda_{\mathfrak{r},q})(e^{i\sigma u} \phi) = e^{i\sigma u} (A_{b,q} + \sigma \Lambda_{\mathfrak{r}',q})\phi.$$

In analogy with the definition of boundary spectrum of an elliptic operator $A \in \text{Diff}_b^m(\mathcal{M}; E, F)$, we have

Definition 6.5. Let \mathcal{N} be a connected component of $\partial\mathcal{M}$. The family of complexes (6.4), $\sigma \in \mathbb{C}$, is the indicial complex of (6.1) at \mathcal{N} . For each $\sigma \in \mathbb{C}$ let $H_{\mathcal{A}(\sigma)}^q(\mathcal{N})$ denote the q -th cohomology group of (6.4) on \mathcal{N} . The q -th boundary spectrum of the complex (6.1) at \mathcal{N} is the set

$$\text{spec}_{b,\mathcal{N}}^q(A) = \{\sigma \in \mathbb{C} : H_{\mathcal{A}(\sigma)}^q(\mathcal{N}) \neq 0\}.$$

The q -th boundary spectrum of A is $\text{spec}_b^q(A) = \bigcup_{\mathcal{N}} \text{spec}_{b,\mathcal{N}}^q(A)$.

The spaces $H_{\mathcal{A}(\sigma)}^q(\mathcal{N})$ are finite-dimensional because (6.4) is an elliptic complex and \mathcal{N} is compact. It is convenient to isolate the behavior of the indicial complex according to the components of the boundary, since the sets $\text{spec}_{b,\mathcal{N}}^q(A)$ can vary drastically from component to component.

Suppose that \mathcal{M} is a complex b -manifold. Recall that since

$${}^b\bar{\partial} \in \text{Diff}_b^1(\mathcal{M}; {}^b\Lambda^{0,q}\mathcal{M}, {}^b\Lambda^{0,q+1}\mathcal{M}),$$

there are induced boundary operators

$${}^b\bar{\partial}_b \in \text{Diff}^1(\mathcal{N}; {}^b\Lambda_{\mathcal{N}}^{0,q}\mathcal{M}, {}^b\Lambda_{\mathcal{N}}^{0,q+1}\mathcal{M})$$

which via the isomorphism (4.1) become the operators of the $\bar{\mathbb{D}}$ -complex (4.5). Combining (2.11) and (2.13) we get

$${}^b\sigma({}^b\bar{\partial})\left(\frac{{}^b d\mathbf{r}}{\mathbf{r}}\right)(\phi) = i\frac{{}^b\bar{\partial}\mathbf{r}}{\mathbf{r}} \wedge \phi$$

and using (4.8) we may identify $\widehat{{}^b\bar{\partial}_b}(\sigma)$, given by (6.3), with the operator

$$(6.6) \quad \bar{\mathbb{D}}(\sigma)\phi = \bar{\mathbb{D}}\phi + i\sigma\beta_{\mathbf{r}} \wedge \phi.$$

If $E \rightarrow \mathcal{M}$ is a holomorphic vector bundle, then the indicial family of

$${}^b\bar{\partial} \in \text{Diff}_b^1(\mathcal{M}; {}^b\Lambda^{0,q}\mathcal{M} \otimes E, {}^b\Lambda^{0,q+1}\mathcal{M} \otimes E)$$

is again given by (6.6), but using the operator $\bar{\mathbb{D}}$ of the complex (4.11).

Returning to the general complex (6.1), fix a smooth positive b -density \mathbf{m} on \mathcal{M} and a Hermitian metric on each E^q . Let $\mathcal{A}_q^*(\sigma)$ be the indicial operator of the formal adjoint, A_q^* , of A_q . The Laplacian \square_q of the complex (6.1) in degree q belongs to $\text{Diff}_b^2(\mathcal{M}; E^q\mathcal{M})$, is b -elliptic, and its indicial operator is

$$\widehat{\square}_q(\sigma) = \mathcal{A}_q^*(\sigma)\mathcal{A}_q(\sigma) + \mathcal{A}_{q-1}(\sigma)\mathcal{A}_{q-1}^*(\sigma).$$

The b -spectrum of \square_q at \mathcal{N} , see Melrose [6], is the set

$$\text{spec}_{b,\mathcal{N}}(\square_q) = \{\sigma \in \mathbb{C} : \widehat{\square}_q(\sigma) : C^\infty(\mathcal{N}; E_{\mathcal{N}}^q) \rightarrow C^\infty(\mathcal{N}; E_{\mathcal{N}}^q) \text{ is not invertible}\}.$$

Note that unless σ is real, $\widehat{\square}_q(\sigma)$ is not the Laplacian of the complex (6.4).

Proposition 6.7. For each q , $\text{spec}_{b,\mathcal{N}}^q(A) \subset \text{spec}_{b,\mathcal{N}}(\square_q)$.

Note that the set $\text{spec}_{b,\mathcal{N}}(\square_q)$ depends on the choice of Hermitian metrics and b -density used to construct the Laplacian, but that the subset $\text{spec}_{b,\mathcal{N}}^q(A)$ is independent of such choices. For a general b -elliptic complex (6.1) it may occur that $\text{spec}_{b,\mathcal{N}}^q(A) \neq \text{spec}_{b,\mathcal{N}}(\square_q)$. In Example 6.13 we show that $\text{spec}_{b,\mathcal{N}}^q(bd) \subset \{0\}$. As is well known, $\text{spec}_{b,\mathcal{N}}(\Delta_q)$ is an infinite set if $\dim \mathcal{M} > 1$. At the end of this

section we will give an example where $\text{spec}_{b,\mathcal{N}}^0(\bar{\partial})$ is an infinite set. A full discussion of $\text{spec}_{b,\mathcal{N}}^q(\bar{\partial})$ for any q and other aspects of the indicial complex of complex b -structures is given in Section 9.

Proof of Proposition 6.7. Since \square_q is b -elliptic, the set $\text{spec}_{b,\mathcal{N}}(\square_q)$ is closed and discrete. Let $H^2(\mathcal{N}; E_{\mathcal{N}}^q)$ be the L^2 -based Sobolev space of order 2. For $\sigma \notin \text{spec}_{b,\mathcal{N}}(\square_q)$ let

$$\mathcal{G}_q(\sigma) : L^2(\mathcal{N}; E_{\mathcal{N}}^q) \rightarrow H^2(\mathcal{N}; E_{\mathcal{N}}^q)$$

be the inverse of $\widehat{\square}_q(\sigma)$. The map $\sigma \mapsto \mathcal{G}_q(\sigma)$ is meromorphic with poles in $\text{spec}_b(\square_q)$. Since

$$\mathcal{A}_q^*(\sigma) = [\mathcal{A}_q(\bar{\sigma})]^*$$

the operators $\widehat{\square}_q(\sigma)$ are the Laplacians of the complex (6.4) when σ is real. Thus for $\sigma \in \mathbb{R} \setminus (\text{spec}_{b,\mathcal{N}}(\square_q) \cup \text{spec}_{b,\mathcal{N}}(\square_{q+1}))$ we have

$$\mathcal{A}_q(\sigma)\mathcal{G}_q(\sigma) = \mathcal{G}_{q+1}(\sigma)\mathcal{A}_q(\sigma), \quad \mathcal{A}_q(\sigma)^*\mathcal{G}_{q+1}(\sigma) = \mathcal{G}_q(\sigma)\mathcal{A}_q^*(\sigma)$$

by standard Hodge theory. Since all operators depend holomorphically on σ , the same equalities hold for $\sigma \in \mathfrak{X} = \mathbb{C} \setminus (\text{spec}_{b,\mathcal{N}}(\square_q) \cup \text{spec}_{b,\mathcal{N}}(\square_{q+1}))$. It follows that

$$\mathcal{A}_q^*(\sigma)\mathcal{A}_q(\sigma)\mathcal{G}_q(\sigma) = \mathcal{G}_q(\sigma)\mathcal{A}_q^*(\sigma)\mathcal{A}_q(\sigma)$$

in \mathfrak{X} . By analytic continuation the equality holds on all of $\mathbb{C} \setminus \text{spec}_{b,\mathcal{N}}(\square_q)$. Thus if $\sigma_0 \notin \text{spec}_{b,\mathcal{N}}(\square_q)$ and ϕ is a $\mathcal{A}_q(\sigma_0)$ -closed section, $\mathcal{A}_q(\sigma_0)\phi = 0$, then the formula

$$\phi = [\mathcal{A}_q^*(\sigma_0)\mathcal{A}_q(\sigma_0) + \mathcal{A}_{q-1}(\sigma_0)\mathcal{A}_{q-1}^*(\sigma_0)]\mathcal{G}_q(\sigma_0)\phi$$

leads to

$$\phi = \mathcal{A}_{q-1}(\sigma_0)[\mathcal{A}_{q-1}^*(\sigma_0)\mathcal{G}_q(\sigma_0)\phi].$$

Therefore $\sigma_0 \notin \text{spec}_{b,\mathcal{N}}^q(A)$. \square

Since \square_q is b -elliptic, the set $\text{spec}_{b,\mathcal{N}}(\square_q)$ is discrete and intersects each horizontal strip $a \leq \Im\sigma \leq b$ in a finite set (Melrose [6]). Consequently:

Corollary 6.8. *The sets $\text{spec}_{b,\mathcal{N}}^q(A)$, $q = 0, 1, \dots$, are closed, discrete, and intersect each horizontal strip $a \leq \Im\sigma \leq b$ in a finite set.*

We note in passing that the Euler characteristic of the complex (6.4) vanishes for each σ . Indeed, let $\sigma_0 \in \mathbb{C}$. The Euler characteristic of the $\mathcal{A}(\sigma_0)$ -complex is the index of

$$\mathcal{A}(\sigma_0) + \mathcal{A}(\sigma_0)^* : \bigoplus_{q \text{ even}} C^\infty(\mathcal{N}; E^q) \rightarrow \bigoplus_{q \text{ odd}} C^\infty(\mathcal{N}; E^q).$$

The operator $\mathcal{A}_q(\sigma)$ is equal to $A_{b,q} + \sigma\Lambda_{\tau,q}$, see (6.3). Thus $\mathcal{A}_q(\sigma)^* = A_{b,q}^* + \bar{\sigma}\Lambda_{\tau,q}^*$, and it follows that for any σ ,

$$\mathcal{A}(\sigma) + \mathcal{A}(\sigma)^* = \mathcal{A}(\sigma_0) + \mathcal{A}(\sigma_0)^* + (\sigma - \sigma_0)\Lambda_{\tau} + (\bar{\sigma} - \bar{\sigma}_0)\Lambda_{\tau}^*$$

is a compact perturbation of $\mathcal{A}(\sigma_0) + \mathcal{A}(\sigma_0)^*$. Therefore, since the index is invariant under compact perturbations, the index of $\mathcal{A}(\sigma) + \mathcal{A}(\sigma)^*$ is independent of σ . Then it vanishes, since it vanishes when $\sigma \notin \bigcup_q \text{spec}_{b,\mathcal{N}}^q(A)$.

Let $\mathfrak{Mero}^q(\mathcal{N})$ be the sheaf of germs of $C^\infty(\mathcal{N}; E^q)$ -valued meromorphic functions on \mathbb{C} and let $\mathfrak{Hol}^q(\mathcal{N})$ be the subsheaf of germs of holomorphic functions. Let $\mathfrak{S}^q(\mathcal{N}) = \mathfrak{Mero}^q(\mathcal{N})/\mathfrak{Hol}^q(\mathcal{N})$. The holomorphic family $\sigma \mapsto \mathcal{A}_q(\sigma)$ gives a

sheaf homomorphism $\mathcal{A}_q : \mathfrak{Mero}^q(\mathcal{N}) \rightarrow \mathfrak{Mero}^{q+1}(\mathcal{N})$ such that $\mathcal{A}_q(\mathfrak{Hol}^q(\mathcal{N})) \subset \mathfrak{Hol}^{q+1}(\mathcal{N})$ and $\mathcal{A}_{q+1} \circ \mathcal{A}_q = 0$, so we have a complex

$$(6.9) \quad \cdots \rightarrow \mathfrak{S}^q(\mathcal{N}) \xrightarrow{\mathcal{A}_q} \mathfrak{S}^{q+1}(\mathcal{N}) \rightarrow \cdots$$

The cohomology sheafs $\mathfrak{H}_A^q(\mathcal{N})$ of this complex contain more refined information about the cohomology of the complex A .

Proposition 6.10. *The sheaf $\mathfrak{H}_A^q(\mathcal{N})$ is supported on $\text{spec}_{b,\mathcal{N}}^q(A)$.*

Proof. Let $\sigma_0 \in \mathbb{C}$ be such that $H_{\mathcal{A}(\sigma_0)}^q(\mathcal{N}) = 0$ and let

$$(6.11) \quad \phi(\sigma) = \sum_{k=1}^{\mu} \frac{\phi_k}{(\sigma - \sigma_0)^k},$$

$\mu > 0$, $\phi_k \in C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{V}}^*)$, represent the \mathcal{A} -closed element $[\phi]$ of the stalk of $\mathfrak{S}^q(\mathcal{N})$ over σ_0 . The condition that $\mathcal{A}_q[\phi] = 0$ means that $\mathcal{A}_q(\sigma)\phi(\sigma)$ is holomorphic, that is,

$$\frac{\mathcal{A}_q(\sigma_0)\phi_\mu}{(\sigma - \sigma_0)^\mu} + \sum_{k=1}^{\mu-1} \frac{\mathcal{A}_q(\sigma_0)\phi_k + \Lambda_{\tau,q}\phi_{k+1}}{(\sigma - \sigma_0)^k} = 0.$$

In particular $\mathcal{A}_q(\sigma_0)\phi_\mu = 0$. Since $H_{\mathcal{A}(\sigma_0)}^q(\mathcal{N}) = 0$, there is $\psi_\mu \in C^\infty(\mathcal{N}; E^{q-1})$ such that $\mathcal{A}_{q-1}(\sigma_0)\psi_\mu = \phi_\mu$. This shows that if $\mu = 1$, then $[\phi]$ is exact, and that if $\mu > 1$, then letting $\phi'(\sigma) = \phi(\sigma) - \mathcal{A}_{q-1}(\sigma)\psi_\mu/(\sigma - \sigma_0)^\mu$, that ϕ is cohomologous to an element $[\phi']$ represented by a sum as in (6.11) with $\mu - 1$ instead of μ . By induction, $[\phi]$ is exact. \square

Definition 6.12. The cohomology sheafs $\mathfrak{H}_A^q(\mathcal{N})$ of the complex (6.9) will be referred to as the indicial cohomology sheafs of the complex A . If $[\phi] \in \mathfrak{h}_A^q(\mathcal{N})$ is a nonzero element of the stalk over σ_0 , the smallest μ such that there is a meromorphic function (6.11) representing $[\phi]$ will be called the order of the pole of $[\phi]$.

The relevancy of this notion of pole lies in that it predicts, for any given cohomology class of the complex A , the existence of a representative with the most regular leading term (the smallest power of log that must appear in the expansion at the boundary). We will see later (Proposition 9.5) that for the b -Dolbeault complex, under a certain geometric assumption, the order of the pole of $[\phi] \in \mathfrak{H}_{\bar{\partial}}^q(\mathcal{N}) \setminus 0$ is 1.

Example 6.13. For the b -de Rham complex one has $\text{spec}_{b,\mathcal{N}}^q(b^d) \subset \{0\}$ and

$$H_{\mathcal{D}(0)}^q(\mathcal{N}) = H_{\text{dR}}^q(\mathcal{N}) \oplus H_{\text{dR}}^{q-1}(\mathcal{N})$$

for each component \mathcal{N} of $\partial\mathcal{M}$, and that every element of the stalk of $\mathfrak{H}_{b^d}^q(\mathcal{N})$ over 0 has a representative with a simple pole. By way of the residue we get an isomorphism from the stalk over 0 onto $H_{\text{dR}}^q(\mathcal{N})$.

Since the map (2.4) is surjective with kernel spanned by $\tau\partial\tau$, the dual map

$$(6.14) \quad \text{ev}_{\mathcal{N}}^* : T^*\mathcal{N} \rightarrow {}^bT_{\mathcal{N}}^*\mathcal{M}$$

is injective with image the annihilator, \mathcal{H} , of $\tau\partial\tau$. Let $\mathbf{i}_{\tau\partial\tau} : {}^b\Lambda_{\mathcal{N}}^q\mathcal{M} \rightarrow {}^b\Lambda_{\mathcal{N}}^{q-1}\mathcal{M}$ denote interior multiplication by $\tau\partial\tau$. Then $\Lambda^q\mathcal{H} = \ker(\mathbf{i}_{\tau\partial\tau} : {}^b\Lambda_{\mathcal{N}}^q\mathcal{M} \rightarrow {}^b\Lambda_{\mathcal{N}}^{q-1}\mathcal{M})$. The isomorphism (6.14) gives isomorphisms

$$\text{ev}_{\mathcal{N}}^* : \Lambda^q\mathcal{N} \rightarrow \mathcal{H}^q$$

for each q . Fix a defining function \mathfrak{r} for \mathcal{N} and let $\Pi : {}^b\Lambda_{\mathcal{N}}^q \mathcal{M} \rightarrow {}^b\Lambda_{\mathcal{N}}^q \mathcal{M}$ be the projection on \mathcal{H}^q according to the decomposition

$${}^b\Lambda_{\mathcal{N}}^q \mathcal{M} = \mathcal{H}^q \oplus \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge \mathcal{H}^{q-1},$$

that is,

$$\Pi\phi = \phi - \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge \mathbf{i}_{\mathfrak{r}\partial_{\mathfrak{r}}}\phi.$$

If $\phi^0 \in C^\infty(\mathcal{N}, \mathcal{H}^q)$ and $\phi^1 \in C^\infty(\mathcal{N}, \mathcal{H}^{q-1})$, then

$${}^b d(\phi^0 + \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge \phi^1) = \Pi {}^b d\phi^0 + \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge (-\Pi {}^b d\phi^1).$$

Since

$$\mathfrak{r}^{-i\sigma} {}^b d\mathfrak{r}^{i\sigma} \phi = {}^b d\phi + i\sigma \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge \phi,$$

the indicial operator $\mathcal{D}(\sigma)$ of ${}^b d$ is

$$\mathcal{D}(\sigma)(\phi_0 + \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge \phi^1) = \Pi {}^b d\phi^0 + \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge (i\sigma\phi^0 - \Pi {}^b d\phi^1).$$

If $\mathcal{D}(\sigma)(\phi_0 + \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge \phi^1) = 0$, then of course $\Pi {}^b d\phi^0 = 0$ and $i\sigma\phi^0 = \Pi {}^b d\phi^1$, and it follows that if $\sigma \neq 0$, then

$$(\phi_0 + \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge \phi^1) = \mathcal{D}(\sigma) \frac{1}{i\sigma} \phi^1.$$

Thus all cohomology groups of the complex $\mathcal{D}(\sigma)$ vanish if $\sigma \neq 0$, i.e., $\text{spec}_{b,\mathcal{N}}^q({}^b d) \subset \{0\}$.

It is not hard to verify that

$$\Pi {}^b d \text{ev}_{\mathcal{N}}^* = \text{ev}_{\mathcal{N}}^* d.$$

Since

$$\mathfrak{r}^{-i\sigma} {}^b d\mathfrak{r}^{i\sigma} \phi = {}^b d\phi + i\sigma \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge \phi,$$

the indicial operator of ${}^b d$ at $\sigma = 0$ can be viewed as the operator

$$\begin{bmatrix} d & 0 \\ 0 & -d \end{bmatrix} : \begin{array}{c} \Lambda^q \mathcal{N} \\ \oplus \\ \Lambda^{q-1} \mathcal{N} \end{array} \rightarrow \begin{array}{c} \Lambda^q \mathcal{N} \\ \oplus \\ \Lambda^{q-1} \mathcal{N} \end{array}.$$

From this we get the cohomology groups of $\mathcal{D}(0)$ in terms of the de Rham cohomology of \mathcal{N} :

$$H_{\mathcal{D}(0)}^q(\mathcal{N}) = H_{\text{dR}}^q(\mathcal{N}) \oplus H_{\text{dR}}^{q-1}(\mathcal{N}).$$

Thus the groups $H_{\mathcal{D}(0)}^q(\mathcal{N})$ do not vanish for $q = 0, 1, \dim \mathcal{M} - 1, \dim \mathcal{M}$ but may vanish for other values of q .

We now show that every element of the stalk of $\mathfrak{H}_{b,d}^q(\mathcal{N})$ over 0 has a representative with a simple pole at 0. Suppose that

$$(6.15) \quad \phi(\sigma) = \sum_{k=1}^{\mu} \frac{1}{\sigma^k} \left(\phi_k^0 + \frac{{}^b d\mathfrak{r}}{\mathfrak{r}} \wedge \phi_k^1 \right)$$

is such that $\mathcal{D}(\sigma)\phi(\sigma)$ is holomorphic. Then

$$\sum_{k=1}^{\mu} \frac{1}{\sigma^k} \left(d\phi_k^0 - \frac{b d\mathbf{r}}{\mathbf{r}} \wedge d\phi_k^1 \right) + \frac{b d\mathbf{r}}{\mathbf{r}} \wedge \left(\sum_{k=1}^{\mu-1} \frac{i}{\sigma^k} \phi_{k+1}^0 \right) = 0,$$

hence $d\phi_1^0 = 0$, $d\phi_{\mu}^1 = 0$ and $\phi_k^0 = -id\phi_{k-1}^1$, $k = 2, \dots, \mu$. Let

$$\psi(\sigma) = -i \sum_{k=2}^{\mu+1} \frac{1}{\sigma^k} \phi_{k-1}^1.$$

Then

$$\begin{aligned} \mathcal{D}(\sigma)\psi(\sigma) &= -i \sum_{k=2}^{\mu+1} \frac{1}{\sigma^k} d\phi_{k-1}^1 + \frac{b d\mathbf{r}}{\mathbf{r}} \wedge \sum_{k=2}^{\mu+1} \frac{1}{\sigma^{k-1}} \phi_{k-1}^1 \\ &= \sum_{k=2}^{\mu} \frac{1}{\sigma^k} \phi_k^0 + \frac{b d\mathbf{r}}{\mathbf{r}} \wedge \sum_{k=1}^{\mu} \frac{1}{\sigma^k} \phi_k^1 \end{aligned}$$

so

$$\phi(\sigma) - \mathcal{D}(\sigma)\psi(\sigma) = \frac{1}{\sigma} \phi_1^0.$$

The map that sends the class of the $\mathcal{D}(\sigma)$ -closed element (6.15) to the class of ϕ_1^0 in $H_{\text{dR}}^q(\mathcal{N})$ is an isomorphism.

Example 6.16. As we just saw, the boundary spectrum of the ${}^b d$ complex in degree 0 is just $\{0\}$. In contrast, $\text{spec}_{b, \mathcal{N}}^0({}^b \bar{\partial})$ may be an infinite set. We illustrate this in the context of Example 4.10. The functions

$$z^\alpha = (z^1)^{\alpha_1} \dots (z^{n+1})^{\alpha_{n+1}},$$

where the α_j are nonnegative integers, are CR functions that satisfy

$$\mathcal{T}z^\alpha = i \left(\sum \tau_j \alpha_j \right) z^\alpha.$$

This implies that

$$\bar{\mathbb{D}}z^\alpha + i \left(-i \sum \tau_j \alpha_j \right) \beta z^\alpha = 0$$

with β as in Example 4.10, so the numbers $\sigma_\alpha = (-i \sum \tau_j \alpha_j)$ belong to $\text{spec}_{b, \mathcal{N}}^0({}^b \bar{\partial})$.

For the sake of completeness we also show that if $\sigma \in \text{spec}_{b, \mathcal{N}}^0({}^b \bar{\partial})$, then $\sigma = \sigma_\alpha$ for some α as above. To see this, suppose that $\zeta : S^{2n+1} \rightarrow \mathbb{C}$ is not identically zero and satisfies

$$\bar{\mathbb{D}}\zeta + i\sigma\zeta\beta = 0$$

for some $\sigma \neq 0$. Then ζ is smooth, because the principal symbol of $\bar{\mathbb{D}}$ on functions is injective. Since $\langle \beta, \mathcal{T} \rangle = -i$,

$$T\zeta + \sigma\zeta = 0.$$

Thus $\zeta(\mathbf{a}_t(p)) = e^{-\sigma t} \zeta(p)$ for any p . Since $|\zeta(\mathbf{a}_t(p))|$ is bounded as a function of t and ζ is not identically 0, σ must be purely imaginary. Since ζ is a CR function, it extends uniquely to a holomorphic function $\tilde{\zeta}$ on $B = \{z \in \mathbb{C}^{n+1} : \|z\| < 1\}$, necessarily smooth up to the boundary. Let $\zeta_t = \zeta \circ \mathbf{a}_t$. This is also a smooth CR function, so it has a unique holomorphic extension $\tilde{\zeta}_t$ to B . The integral curve through $z_0 = (z_0^1, \dots, z_0^{n+1})$ of the vector field \mathcal{T} is

$$t \mapsto \mathbf{a}_t(z_0) = (e^{i\tau_1 t} z_0^1, \dots, e^{i\tau_{n+1} t} z_0^{n+1})$$

Extending the definition of \mathbf{a}_t to allow arbitrary $z \in \mathbb{C}^{n+1}$ as argument we then have that $\tilde{\zeta}_t = \tilde{\zeta} \circ \mathbf{a}_t$. Then

$$\partial_t \tilde{\zeta}_t + \sigma \tilde{\zeta}_t = 0$$

gives

$$\tilde{\zeta}(z) = \sum_{\{\alpha: \boldsymbol{\tau} \cdot \alpha = i\sigma\}} c_\alpha z^\alpha$$

for $|z| < 1$, where $\boldsymbol{\tau} = (\tau_1, \dots, \tau_{n+1})$. Thus $\sigma = -i \sum \tau_j \alpha_j$ as claimed. Note that $\Im \sigma$ is negative (positive) if the τ_j are positive (negative) and $\alpha \neq 0$.

7. UNDERLYING CR COMPLEXES

Again let $\mathbf{a} : \mathbb{R} \times \mathcal{N} \rightarrow \mathcal{N}$ be the flow of \mathcal{T} . Let $\mathcal{L}_{\mathcal{T}}$ denote the Lie derivative with respect to \mathcal{T} on de Rham q -forms or vector fields and let $\mathbf{i}_{\mathcal{T}}$ denote interior multiplication by \mathcal{T} of de Rham q -forms or of elements of $\Lambda^q \bar{\mathcal{V}}^*$.

The proofs of the following two lemmas are elementary.

Lemma 7.1. *If α is a smooth section of the annihilator of $\bar{\mathcal{V}}$ in $\mathbb{C}T^*\mathcal{N}$, then $(\mathcal{L}_{\mathcal{T}}\alpha)|_{\bar{\mathcal{V}}} = 0$. Consequently, for each $p \in \mathcal{N}$ and $t \in \mathbb{R}$, $d\mathbf{a}_t : \mathbb{C}T_p\mathcal{N} \rightarrow \mathbb{C}T_{\mathbf{a}_t(p)}\mathcal{N}$ maps $\bar{\mathcal{V}}_p$ onto $\bar{\mathcal{V}}_{\mathbf{a}_t(p)}$.*

It follows that there is a well defined smooth bundle homomorphism $\mathbf{a}_t^* : \Lambda^q \bar{\mathcal{V}}^* \rightarrow \Lambda^q \bar{\mathcal{V}}^*$ covering \mathbf{a}_{-t} . In particular, one can define the Lie derivative $\mathcal{L}_{\mathcal{T}}\phi$ with respect to \mathcal{T} of an element in $\phi \in C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{V}}^*)$. The usual formula holds:

Lemma 7.2. *If $\phi \in C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{V}}^*)$, then $\mathcal{L}_{\mathcal{T}}\phi = \mathbf{i}_{\mathcal{T}}\bar{\mathbb{D}}\phi + \bar{\mathbb{D}}\mathbf{i}_{\mathcal{T}}\phi$. Consequently, for each t and $\phi \in C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{V}}^*)$, $\mathbb{D}\mathbf{a}_t^*\phi = \mathbf{a}_t^*\bar{\mathbb{D}}\phi$.*

For any defining function \mathbf{r} of \mathcal{N} in \mathcal{M} , $\bar{\mathcal{K}}_{\mathbf{r}} = \ker \beta_{\mathbf{r}}$ is a CR structure of CR codimension 1: indeed, $\mathcal{K}_{\mathbf{r}} \cap \bar{\mathcal{K}}_{\mathbf{r}} \subset \text{span}_{\mathbb{C}} \mathcal{T}$ but since $\langle \beta_{\mathbf{r}}, \mathcal{T} \rangle$ vanishes nowhere, we must have $\bar{\mathcal{K}} \cap \mathcal{K} = 0$. Since $\mathcal{K} \oplus \bar{\mathcal{K}} \oplus \text{span}_{\mathbb{C}} \mathcal{T} = \mathbb{C}T\mathcal{N}$, the CR codimension is 1. Finally, if $V, W \in C^\infty(\mathcal{N}; \bar{\mathcal{K}}_{\mathbf{r}})$, then

$$\langle \beta_{\mathbf{r}}, [V, W] \rangle = V\langle \beta_{\mathbf{r}}, W \rangle - W\langle \beta_{\mathbf{r}}, V \rangle - 2\bar{\mathbb{D}}\beta(V, W),$$

Since the right hand side vanishes, $[V, W]$ is again a section of $\bar{\mathcal{K}}_{\mathbf{r}}$.

Since $\bar{\mathcal{V}} = \bar{\mathcal{K}}_{\mathbf{r}} \oplus \text{span}_{\mathbb{C}} \mathcal{T}$, the dual of $\bar{\mathcal{K}}_{\mathbf{r}}$ is canonically isomorphic to the kernel of $\mathbf{i}_{\mathcal{T}} : \bar{\mathcal{V}}^* \rightarrow \mathbb{C}$. We will write $\bar{\mathcal{K}}^*$ for this kernel. More generally, $\Lambda^q \bar{\mathcal{K}}_{\mathbf{r}}^*$ and the kernel, $\Lambda^q \bar{\mathcal{K}}^*$, of $\mathbf{i}_{\mathcal{T}} : \Lambda^q \bar{\mathcal{V}}^* \rightarrow \Lambda^{q-1} \bar{\mathcal{V}}^*$ are canonically isomorphic. The vector bundles $\Lambda^q \bar{\mathcal{K}}^*$ are independent of the defining function \mathbf{r} . We regard the $\bar{\partial}_b$ -operators of the CR structure as operators

$$C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^*) \rightarrow C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{K}}^*).$$

They do depend on \mathbf{r} but we will not indicate this in the notation.

To get a formula for $\bar{\partial}_b$, let

$$\tilde{\beta}_{\mathbf{r}} = \frac{i}{i - \mathbf{a}_{\mathbf{r}}} \beta_{\mathbf{r}}$$

(so that $\langle i\tilde{\beta}_{\mathbf{r}}, \mathcal{T} \rangle = 1$). The projection $\Pi_{\mathbf{r}} : \Lambda^q \bar{\mathcal{V}}^* \rightarrow \Lambda^q \bar{\mathcal{V}}^*$ on $\Lambda^q \bar{\mathcal{K}}^*$ according to the decomposition

$$(7.3) \quad \Lambda^q \bar{\mathcal{V}}^* = \Lambda^q \bar{\mathcal{K}}^* \oplus i\tilde{\beta}_{\mathbf{r}} \wedge \Lambda^{q-1} \bar{\mathcal{K}}^*$$

is

$$(7.4) \quad \Pi_{\tau}\phi = \phi - i\tilde{\beta}_{\tau} \wedge \mathbf{i}_{\mathcal{T}}\phi.$$

Lemma 7.5. *With the identification of $\Lambda^q \overline{\mathcal{K}}_{\tau}^*$ with $\Lambda^q \overline{\mathcal{K}}^*$ described above, the $\bar{\partial}_b$ -operators of the CR structure $\overline{\mathcal{K}}_{\tau}$ are given by*

$$(7.6) \quad \bar{\partial}_b\phi = \Pi_{\tau}\mathbb{D}\phi \quad \text{if } \phi \in C^{\infty}(\mathcal{N}, \Lambda^q \overline{\mathcal{K}}^*),$$

Proof. Suppose that (z, t) is a hypoanalytic chart for $\overline{\mathcal{V}}$ on some open set U , with $\mathcal{T}t = 1$. So $\partial_{\bar{z}^{\mu}}$, $\mu = 1, \dots, n$, $\mathcal{T} = \partial_t$ is a frame for $\overline{\mathcal{V}}$ over U with dual frame $\mathbb{D}\bar{z}^{\mu}$, $\mathbb{D}t$. If

$$\beta_{\tau} = \sum_{\mu=1}^n \beta_{\mu} \mathbb{D}\bar{z}^{\mu} + \beta_0 \mathbb{D}t.$$

then

$$\bar{L}_{\mu} = \partial_{\bar{z}^{\mu}} - \frac{\beta_{\mu}}{\beta_0} \partial_t, \quad \mu = 1, \dots, n$$

is a frame for $\overline{\mathcal{K}}_{\tau}$ over U . Let $\bar{\eta}^{\mu}$ denote the dual frame (for $\overline{\mathcal{K}}_{\tau}^*$). Since the \bar{L}_{μ} commute, $\bar{\partial}_b \bar{\eta}^{\mu} = 0$, so if $\phi = \sum'_{|I|=q} \phi_I \bar{\eta}^I$, then (with the notation as in eg. Folland and Kohn [2])

$$\bar{\partial}_b\phi = \sum'_{|J|=q+1} \sum'_{|I|=q} \sum_{\mu} \epsilon_J^{\mu I} \bar{L}_{\mu} \phi_I \bar{\eta}^J.$$

On the other hand, the frame of $\overline{\mathcal{V}}^*$ dual to the frame \bar{L}_{μ} , $\mu = 1, \dots, n$, \mathcal{T} of $\overline{\mathcal{V}}$ is $\mathbb{D}\bar{z}^{\mu}$, $i\tilde{\beta}_{\tau}$, and the identification of $\overline{\mathcal{K}}_{\tau}^*$ with $\overline{\mathcal{K}}^*$ maps the η^{μ} to the $\mathbb{D}\bar{z}^{\mu}$. So, as a section of $\Lambda^q \overline{\mathcal{V}}^*$,

$$\phi = \sum'_{|I|=q} \phi_I \mathbb{D}\bar{z}^I$$

and

$$\mathbb{D}\phi = \sum'_{|J|=q+1} \sum'_{|I|=q} \epsilon_J^{\mu I} \bar{L}_{\mu} \phi_I \mathbb{D}\bar{z}^J + i\tilde{\beta}_{\tau} \wedge \sum'_{|I|=q} \mathcal{T}\phi_I \mathbb{D}\bar{z}^I.$$

Thus $\Pi_{\tau}\mathbb{D}\phi$ is the section of $\Lambda^{q+1} \overline{\mathcal{K}}^*$ associated with $\bar{\partial}_b\phi$ by the identifying map. \square

Using (7.4) in (7.6) and the fact that $\mathbf{i}_{\mathcal{T}}\mathbb{D}\phi = \mathcal{L}_{\mathcal{T}}\phi$ if $\phi \in C^{\infty}(\mathcal{N}; \Lambda^q \overline{\mathcal{K}}^*)$ we get

$$(7.7) \quad \bar{\partial}_b\phi = \mathbb{D}\phi - i\tilde{\beta}_{\tau} \wedge \mathcal{L}_{\mathcal{T}}\phi \quad \text{if } \phi \in C^{\infty}(\mathcal{N}, \Lambda^q \overline{\mathcal{K}}^*).$$

The \mathbb{D} operators can be expressed in terms of the $\bar{\partial}_b$ operators. Suppose $\phi \in C^{\infty}(\mathcal{N}; \Lambda^q \overline{\mathcal{V}}^*)$. Then $\phi = \phi^0 + i\tilde{\beta}_{\tau} \wedge \phi^1$ with unique $\phi^0 \in C^{\infty}(\mathcal{N}; \Lambda^q \overline{\mathcal{K}}^*)$ and $\phi^1 \in C^{\infty}(\mathcal{N}; \Lambda^{q-1} \overline{\mathcal{K}}^*)$, and

$$\mathbb{D}\phi^0 = \bar{\partial}_b\phi^0 + i\tilde{\beta}_{\tau} \wedge \mathcal{L}_{\mathcal{T}}\phi^0,$$

see (7.7). Using

$$\mathbb{D}\tilde{\beta}_{\tau} = \frac{\mathbb{D}a_{\tau}}{i - a_{\tau}} \wedge \tilde{\beta}_{\tau}$$

and (7.7) again we get

$$\mathbb{D}(i\tilde{\beta}_{\tau} \wedge \phi^1) = i\tilde{\beta}_{\tau} \wedge \left(-\frac{\mathbb{D}a_{\tau}}{i - a_{\tau}} \wedge \phi^1 - \mathbb{D}\phi^1 \right) = i\tilde{\beta}_{\tau} \wedge \left(-\frac{\bar{\partial}_b a_{\tau}}{i - a_{\tau}} \wedge \phi^1 - \bar{\partial}_b \phi^1 \right).$$

This gives

$$(7.8) \quad \mathbb{D} = \begin{bmatrix} \bar{\partial}_b & 0 \\ \mathcal{L}_{\mathcal{T}} & -\bar{\partial}_b - \frac{\bar{\partial}_b a_{\mathfrak{r}}}{i - a_{\mathfrak{r}}} \end{bmatrix} : \begin{array}{c} C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^*) \\ \oplus \\ C^\infty(\mathcal{N}; \Lambda^{q-1} \bar{\mathcal{K}}^*) \end{array} \rightarrow \begin{array}{c} C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{K}}^*) \\ \oplus \\ C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^*) \end{array}.$$

Since \mathcal{T} itself is \mathcal{T} -invariant, $\mathbf{i}_{\mathcal{T}} \mathbf{a}_t^* = \mathbf{a}_t^* \mathbf{i}_{\mathcal{T}}$: the subbundle $\bar{\mathcal{K}}^*$ of $\bar{\mathcal{V}}^*$ is invariant under \mathbf{a}_t^* for each t . This need not be true of $\bar{\mathcal{K}}_{\mathfrak{r}}$, i.e., the statement that for all t , $d\mathbf{a}_t(\bar{\mathcal{K}}_{\mathfrak{r}}) \subset \bar{\mathcal{K}}_{\mathfrak{r}}$, equivalently,

$$L \in C^\infty(\mathcal{M}; \bar{\mathcal{K}}_{\mathfrak{r}}) \implies [\mathcal{T}, L] \in C^\infty(\mathcal{M}; \bar{\mathcal{K}}_{\mathfrak{r}}),$$

may fail to hold. Since $\mathbb{D}\beta_{\mathfrak{r}} = 0$, the formula

$$0 = \mathcal{T}\langle \beta_{\mathfrak{r}}, L \rangle - L\langle \beta_{\mathfrak{r}}, \mathcal{T} \rangle - \langle \beta_{\mathfrak{r}}, [\mathcal{T}, L] \rangle$$

with $L \in C^\infty(\mathcal{N}; \bar{\mathcal{K}}_{\mathfrak{r}})$ gives that $\bar{\mathcal{K}}_{\mathfrak{r}}$ is invariant under $d\mathbf{a}_t$ if and only if $L\mathbf{a}_{\mathfrak{r}} = 0$ for each CR vector field, that is, if and only if $\mathbf{a}_{\mathfrak{r}}$ is a CR function. This proves the equivalence between the first and last statements in the following lemma. The third statement is the most useful.

Lemma 7.9. *Let \mathfrak{r} be a defining function for \mathcal{N} in \mathcal{M} and let $\bar{\partial}_b$ denote the operators of the associated CR complex. The following are equivalent:*

- (1) *The function $\mathbf{a}_{\mathfrak{r}}$ is CR;*
- (2) *$\mathcal{L}_{\mathcal{T}} \tilde{\beta}_{\mathfrak{r}} = 0$;*
- (3) *$\mathcal{L}_{\mathcal{T}} \bar{\partial}_b - \bar{\partial}_b \mathcal{L}_{\mathcal{T}} = 0$;*
- (4) *$\bar{\mathcal{K}}_{\mathfrak{r}}$ is \mathcal{T} -invariant.*

Proof. From $\beta_{\mathfrak{r}} = (a_{\mathfrak{r}} - i)i\tilde{\beta}_{\mathfrak{r}}$ and $\mathcal{L}_{\mathcal{T}}\beta_{\mathfrak{r}} = \mathbb{D}\mathbf{a}_{\mathfrak{r}}$ we obtain

$$\mathbb{D}\mathbf{a}_{\mathfrak{r}} = (\mathcal{L}_{\mathcal{T}}\mathbf{a}_{\mathfrak{r}})i\tilde{\beta}_{\mathfrak{r}} + (a_{\mathfrak{r}} - i)i\mathcal{L}_{\mathcal{T}}\tilde{\beta}_{\mathfrak{r}},$$

so

$$\bar{\partial}_b \mathbf{a}_{\mathfrak{r}} = \mathbb{D}\mathbf{a}_{\mathfrak{r}} - (\mathcal{L}_{\mathcal{T}}\mathbf{a}_{\mathfrak{r}})i\tilde{\beta}_{\mathfrak{r}} = (a_{\mathfrak{r}} - i)i\mathcal{L}_{\mathcal{T}}\tilde{\beta}_{\mathfrak{r}}.$$

Thus $\mathbf{a}_{\mathfrak{r}}$ is CR if and only if $\mathcal{L}_{\mathcal{T}}\tilde{\beta}_{\mathfrak{r}} = 0$.

Using $\mathcal{L}_{\mathcal{T}}\mathbb{D} = \mathbb{D}\mathcal{L}_{\mathcal{T}}$ and the definition of $\bar{\partial}_b$ we get

$$\mathcal{L}_{\mathcal{T}}\bar{\partial}_b\phi = \mathcal{L}_{\mathcal{T}}(\mathbb{D}\phi - i\tilde{\beta}_{\mathfrak{r}} \wedge \mathcal{L}_{\mathcal{T}}\phi) = \bar{\partial}_b\mathcal{L}_{\mathcal{T}}\phi - i(\mathcal{L}_{\mathcal{T}}\tilde{\beta}_{\mathfrak{r}}) \wedge \mathcal{L}_{\mathcal{T}}\phi$$

for $\phi \in C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^*)$. Thus $\mathcal{L}_{\mathcal{T}}\bar{\partial}_b - \bar{\partial}_b\mathcal{L}_{\mathcal{T}} = 0$ if and only if $\mathcal{L}_{\mathcal{T}}\tilde{\beta}_{\mathfrak{r}} = 0$. \square

Lemma 7.10. *Suppose that $\bar{\mathcal{V}}$ admits a \mathcal{T} -invariant metric. Then there is a defining function \mathfrak{r} for \mathcal{N} in \mathcal{M} such that $\mathbf{a}_{\mathfrak{r}}$ is constant. If \mathfrak{r} and \mathfrak{r}' are defining functions such that $\mathbf{a}_{\mathfrak{r}}$ and $\mathbf{a}_{\mathfrak{r}'}$ are constant, then $\mathbf{a}_{\mathfrak{r}} = \mathbf{a}_{\mathfrak{r}'}$. This constant will be denoted \mathbf{a}_{av} .*

Proof. Let h be a metric as stated. Let $\mathcal{H}^{0,1}$ be the subbundle of $\bar{\mathcal{V}}$ orthogonal to \mathcal{T} . This is \mathcal{T} -invariant, and since the metric is \mathcal{T} -invariant, $\mathcal{H}^{0,1}$ has a \mathcal{T} -invariant metric. This metric gives canonically a metric on $\mathcal{H}^{1,0} = \overline{\mathcal{H}^{0,1}}$. Using the decomposition $\text{CTN} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1} \oplus \text{span}_{\mathbb{C}} \mathcal{T}$ we get a \mathcal{T} -invariant metric on CTN for which the decomposition is orthogonal. This metric is induced by a Riemannian metric g . Let \mathbf{m}_0 be the corresponding Riemannian density, which is \mathcal{T} -invariant because g is. Since \mathbb{D} , h , and \mathbf{m}_0 are \mathcal{T} -invariant, so are the formal adjoint \mathbb{D}^* of \mathbb{D} and the Laplacians of the \mathbb{D} -complex, and if G denotes the Green's operators for these Laplacians, then G is also \mathcal{T} -invariant, as is the orthogonal projection Π on

the space of $\bar{\mathbb{D}}$ -harmonic forms. Arbitrarily pick a defining function \mathfrak{r} for \mathcal{N} in \mathcal{M} . Then

$$a_{\mathfrak{r}} - G\bar{\mathbb{D}}^* \bar{\mathbb{D}} a_{\mathfrak{r}} = \Pi a_{\mathfrak{r}}$$

where $\Pi a_{\mathfrak{r}}$ is a constant function by Lemma 4.6. Since $\beta_{\mathfrak{r}}$ is $\bar{\mathbb{D}}$ -closed, $\bar{\mathbb{D}} a_{\mathfrak{r}} = \mathcal{L}_{\mathcal{T}} \beta_{\mathfrak{r}}$. Thus $G\bar{\mathbb{D}}^* \bar{\mathbb{D}} a_{\mathfrak{r}} = \mathcal{T} G\bar{\mathbb{D}}^* \beta_{\mathfrak{r}}$, and since $a_{\mathfrak{r}}$ is real valued and \mathcal{T} is a real vector field,

$$a_{\mathfrak{r}} - \mathcal{T} \mathfrak{R} G\bar{\mathbb{D}}^* \beta_{\mathfrak{r}} = \mathfrak{R} \Pi a_{\mathfrak{r}}.$$

Extend the function $u = \mathfrak{R} G\bar{\mathbb{D}}^* \beta_{\mathfrak{r}}$ to \mathcal{M} as a smooth real-valued function. Then $\mathfrak{r}' = e^{-u} \mathfrak{r}$ has the required property.

Suppose that $\mathfrak{r}, \mathfrak{r}'$ are defining functions for \mathcal{N} in \mathcal{M} such that $a_{\mathfrak{r}}$ and $a_{\mathfrak{r}'}$ are constant. Then these functions are equal by Proposition 4.9. \square

Note that if for some \mathfrak{r} , the subbundle $\bar{\mathcal{K}}_{\mathfrak{r}}$ is \mathcal{T} -invariant and admits a \mathcal{T} invariant Hermitian metric, then there is a \mathcal{T} -invariant metric on $\bar{\mathcal{V}}$.

Suppose now that $\rho : F \rightarrow \mathcal{M}$ is a holomorphic vector bundle over \mathcal{M} . Using the operators

$$\bar{\mathbb{D}} : C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{V}}^* \otimes F_{\mathcal{N}}) \rightarrow C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{V}}^* \otimes F_{\mathcal{N}}),$$

see (4.12), define operators

$$(7.11) \quad \dots \rightarrow C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^* \otimes F_{\mathcal{N}}) \xrightarrow{\bar{\partial}_b} C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{K}}^* \otimes F_{\mathcal{N}}) \rightarrow \dots$$

by

$$\bar{\partial}_b \phi = \Pi_{\mathfrak{r}} \bar{\mathbb{D}} \phi, \quad \phi \in C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^* \otimes F_{\mathcal{N}})$$

where $\Pi_{\mathfrak{r}}$ means $\Pi_{\mathfrak{r}} \otimes I$ with $\Pi_{\mathfrak{r}}$ defined by (7.4). The operators (7.11) form a complex. Define also

$$\mathcal{L}_{\mathcal{T}} = \mathfrak{i}_{\mathcal{T}} \bar{\mathbb{D}} + \bar{\mathbb{D}} \mathfrak{i}_{\mathcal{T}}$$

where $\mathfrak{i}_{\mathcal{T}}$ stands for $\mathfrak{i}_{\mathcal{T}} \otimes I$. Then

$$\mathfrak{i}_{\mathcal{T}} \mathcal{L}_{\mathcal{T}} = \mathcal{L}_{\mathcal{T}} \mathfrak{i}_{\mathcal{T}}, \quad \mathcal{L}_{\mathcal{T}} \bar{\mathbb{D}} = \bar{\mathbb{D}} \mathcal{L}_{\mathcal{T}}.$$

The first of these identities implies that the image of $C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^* \otimes F_{\mathcal{N}})$ by $\mathcal{L}_{\mathcal{T}}$ is contained in $C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^* \otimes F_{\mathcal{N}})$. With these definitions, $\bar{\mathbb{D}}$ as an operator

$$\bar{\mathbb{D}} : \begin{array}{ccc} C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^* \otimes F_{\mathcal{N}}) & & C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{K}}^* \otimes F_{\mathcal{N}}) \\ & \oplus & \oplus \\ C^\infty(\mathcal{N}; \Lambda^{q-1} \bar{\mathcal{K}}^* \otimes F_{\mathcal{N}}) & \rightarrow & C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}^* \otimes F_{\mathcal{N}}) \end{array}.$$

is given by the matrix in (7.8) with the new meanings for $\bar{\partial}_b$ and $\mathcal{L}_{\mathcal{T}}$.

Assume that there is a \mathcal{T} -invariant Riemannian metric on \mathcal{N} , that \mathfrak{r} has been chosen so that $a_{\mathfrak{r}}$ is constant, that $\bar{\mathcal{K}}_{\mathfrak{r}}$ is orthogonal to \mathcal{T} , and that \mathcal{T} has unit length. Then the term involving $\bar{\partial}_b a_{\mathfrak{r}}$ in the matrix (7.8) is absent, and since $\mathbb{D}^2 = 0$,

$$\mathcal{L}_{\mathcal{T}} \bar{\partial}_b = \bar{\partial}_b \mathcal{L}_{\mathcal{T}}.$$

Write $h_{\bar{\mathcal{V}}^*}$ for the metric induced on the bundles $\Lambda^q \bar{\mathcal{V}}^*$ or $\Lambda^q \bar{\mathcal{K}}^*$.

If $\eta_\mu, \mu = 1, \dots, k$ is a local frame of $F_{\mathcal{N}}$ over an open set $U \subset \mathcal{N}$ and ϕ is a local section of $\Lambda^q \bar{\mathcal{V}}^* \otimes F_{\mathcal{N}}$ over U , then for some smooth sections ϕ^μ of $\Lambda^q \bar{\mathcal{V}}^*$ and ω_μ^ν of $\bar{\mathcal{V}}^*$ over U ,

$$\phi = \sum_{\mu} \phi^\mu \otimes \eta_\mu, \quad \bar{\mathbb{D}} \sum_{\mu} \phi^\mu \otimes \eta_\mu = \sum_{\nu} (\bar{\mathbb{D}} \phi^\nu + \sum_{\mu} \omega_\mu^\nu \wedge \phi^\mu) \otimes \eta_\nu.$$

This gives

$$\bar{\partial}_b \sum_{\mu} \phi^{\mu} \otimes \eta_{\mu} = \sum_{\nu} (\bar{\partial}_b \phi^{\nu} + \sum_{\mu} \Pi_{\tau} \omega_{\mu}^{\nu} \wedge \phi^{\mu}) \otimes \eta_{\nu}$$

and

$$\mathcal{L}_{\mathcal{T}} \sum_{\mu} \phi^{\mu} \otimes \eta_{\mu} = \sum_{\nu} (\mathcal{L}_{\mathcal{T}} \phi^{\nu} + \sum_{\mu} \langle \omega_{\mu}^{\nu}, \mathcal{T} \rangle \phi^{\mu}) \otimes \eta_{\nu}.$$

Suppose now that h_F is a Hermitian metric on F . With this metric and the metric $h_{\bar{\mathcal{V}}^*}$ we get Hermitian metrics h on each of the bundles $\bigwedge^q \bar{\mathcal{V}}^* \otimes F_{\mathcal{N}}$. If η_{μ} is an orthonormal frame of $F_{\mathcal{N}}$ and $\phi = \sum \phi^{\mu} \otimes \eta_{\mu}$, $\psi = \sum \psi^{\mu} \otimes \eta_{\mu}$ are sections of $\bigwedge^q \bar{\mathcal{V}}^* \otimes F_{\mathcal{N}}$, then

$$h(\phi, \psi) = \sum_{\nu} h_{\bar{\mathcal{V}}^*}(\phi^{\nu}, \psi^{\nu}).$$

Therefore

$$\begin{aligned} h(\mathcal{L}_{\mathcal{T}} \phi, \psi) + h(\phi, \mathcal{L}_{\mathcal{T}} \psi) &= \sum_{\nu} h_{\bar{\mathcal{V}}^*}(\mathcal{L}_{\mathcal{T}} \phi^{\nu} + \sum_{\mu} \langle \omega_{\mu}^{\nu}, \mathcal{T} \rangle \phi^{\mu}, \psi^{\nu}) + \sum_{\mu} h_{\bar{\mathcal{V}}^*}(\phi^{\mu}, \mathcal{L}_{\mathcal{T}} \psi^{\mu} + \langle \omega_{\nu}^{\mu}, \mathcal{T} \rangle \psi^{\nu}) \\ &= \sum_{\nu} \mathcal{T} h_{\bar{\mathcal{V}}^*}(\phi^{\nu}, \psi^{\nu}) + \sum_{\mu, \nu} (\langle \omega_{\mu}^{\nu}, \mathcal{T} \rangle + \overline{\langle \omega_{\nu}^{\mu}, \mathcal{T} \rangle}) h_{\bar{\mathcal{V}}^*}(\phi^{\mu}, \psi^{\nu}) \\ &= \mathcal{T} h(\phi, \psi) + \sum_{\mu, \nu} (\langle \omega_{\mu}^{\nu}, \mathcal{T} \rangle + \overline{\langle \omega_{\nu}^{\mu}, \mathcal{T} \rangle}) h_{\bar{\mathcal{V}}^*}(\phi^{\mu}, \psi^{\nu}). \end{aligned}$$

Thus $\mathcal{T} h(\phi, \psi) = h(\mathcal{L}_{\mathcal{T}} \phi, \psi) + h(\phi, \mathcal{L}_{\mathcal{T}} \psi)$ if and only if

$$(7.12) \quad \langle \omega_{\mu}^{\nu}, \mathcal{T} \rangle + \overline{\langle \omega_{\nu}^{\mu}, \mathcal{T} \rangle} = 0 \text{ for all } \mu, \nu.$$

This condition is (4.17); just note that by the definition of $\bar{\mathbb{D}}$, the forms $(\Phi^*)^{-1} \omega_{\mu}^{\nu}$ in (4.17) are the forms that we are denoting ω_{μ}^{ν} here. Thus (7.12) holds if and only if h_F is an exact Hermitian metric, see Definition (4.18).

Consequently,

Lemma 7.13. *The statement*

$$(7.14) \quad \mathcal{T} h(\phi, \psi) = h(\mathcal{L}_{\mathcal{T}} \phi, \psi) + h(\phi, \mathcal{L}_{\mathcal{T}} \psi) \quad \forall \phi, \psi \in C^{\infty}(\mathcal{N}; \bigwedge^q \bar{\mathcal{V}}^* \otimes F_{\mathcal{N}})$$

holds if and only the Hermitian metric h_F is exact.

8. SPECTRUM

Suppose that $\bar{\mathcal{V}}$ admits an invariant Hermitian metric. Let τ be a defining function for \mathcal{N} in \mathcal{M} such that a_{τ} is constant. By Lemma (7.9) $\bar{\mathcal{K}}_{\tau}$ is \mathcal{T} -invariant, so the restriction of the metric to this subbundle gives a \mathcal{T} -invariant metric; we use the induced metric on the bundles $\bigwedge^q \bar{\mathcal{K}}^*$ in the following. As in the proof of Lemma 7.10, there is a \mathcal{T} -invariant density \mathfrak{m}_0 on \mathcal{N} .

Let $\rho : F \rightarrow \mathcal{M}$ be a Hermitian holomorphic vector bundle, assume that the Hermitian metric of F is exact, so with the induced metric h on the vector bundles $\bigwedge^q \bar{\mathcal{V}}^* \otimes F_{\mathcal{N}}$, (7.14) holds. We will write F in place of $F_{\mathcal{N}}$.

Let $\bar{\partial}_b^*$ be the formal adjoint of the $\bar{\partial}$ operator (7.11) with respect to the inner on the bundles $\bigwedge^q \bar{\mathcal{K}}^* \otimes F$ and the density \mathfrak{m}_0 , and let $\square_{b,q} = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$ be the

formal $\bar{\partial}_b$ -Laplacian. Since $-i\mathcal{L}_\mathcal{T}$ is formally selfadjoint and commutes with $\bar{\partial}_b$, $\mathcal{L}_\mathcal{T}$ commutes with $\square_{b,q}$. Let

$$\mathcal{H}_{\bar{\partial}_b}^q(\mathcal{N}; F) = \ker \square_{b,q} = \{\phi \in L^2(\mathcal{N}; \wedge^q \bar{\mathcal{K}}^* \otimes F) : \square_{b,q}\phi = 0\}$$

and let

$$\text{Dom}_q(\mathcal{L}_\mathcal{T}) = \{\phi \in \mathcal{H}_{\bar{\partial}_b}^q(\mathcal{N}; F) \text{ and } \mathcal{L}_\mathcal{T}\phi \in \mathcal{H}_{\bar{\partial}_b}^q(\mathcal{N}; F)\}.$$

The spaces $\mathcal{H}_{\bar{\partial}_b}^q(\mathcal{N}; F)$ may be of infinite dimension, but in any case they are closed subspaces of $L^2(\mathcal{N}; \wedge^q \bar{\mathcal{K}}^* \otimes F)$, so they may be regarded as Hilbert spaces on their own right. If $\phi \in \mathcal{H}_{\bar{\partial}_b}^q(\mathcal{N}; F)$, the condition $\mathcal{L}_\mathcal{T}\phi \in \mathcal{H}_{\bar{\partial}_b}^q(\mathcal{N}; F)$ is equivalent to the condition

$$\mathcal{L}_\mathcal{T}\phi \in L^2(\mathcal{N}; \wedge^q \bar{\mathcal{K}}^* \otimes F).$$

So we have a closed operator

$$(8.1) \quad -i\mathcal{L}_\mathcal{T} : \text{Dom}_q(\mathcal{L}_\mathcal{T}) \subset \mathcal{H}_{\bar{\partial}_b}^q(\mathcal{N}; F) \rightarrow \mathcal{H}_{\bar{\partial}_b}^q(\mathcal{N}; F).$$

The fact that $\square_{b,q} - \mathcal{L}_\mathcal{T}^2$ is elliptic, symmetric, and commutes with $\mathcal{L}_\mathcal{T}$ implies that (8.1) is a selfadjoint Fredholm operator with discrete spectrum (see [13, Theorem 2.5]).

Definition 8.2. Let $\text{spec}_0^q(-i\mathcal{L}_\mathcal{T})$ be the spectrum of the operator (8.1), and let $\mathcal{H}_{\bar{\partial}_b, \tau}^q(\mathcal{N}; F)$ be the eigenspace of $-i\mathcal{L}_\mathcal{T}$ in $\mathcal{H}_{\bar{\partial}_b}^q(\mathcal{N}; F)$ corresponding to the eigenvalue τ .

Let τ denote the principal symbol of $-i\mathcal{T}$. Then the principal symbol of $\mathcal{L}_\mathcal{T}$ acting on sections of $\wedge^q \bar{\mathcal{K}}^*$ is τI . Because $\square_{b,q} - \mathcal{L}_\mathcal{T}^2$ is elliptic, $\text{Char}(\square_{b,q})$, the characteristic variety of $\square_{b,q}$, lies in $\tau \neq 0$. Let

$$\text{Char}^\pm(\square_{b,q}) = \{\nu \in \text{Char}(\square_{b,q}) : \tau(\nu) \gtrless 0\}.$$

By [13, Theorem 4.1], if $\square_{b,q}$ is microlocally hypoelliptic on $\text{Char}^\pm(\square_{b,q})$, then

$$\{\tau \in \text{spec}_0^q(-i\mathcal{L}_\mathcal{T}) : \tau \gtrless 0\}$$

is finite. We should perhaps point out that $\text{Char}(\square_{b,q})$ is equal to the characteristic variety, $\text{Char}(\bar{\mathcal{K}}_r)$, of the CR structure.

As a special case consider the situation where F is the trivial line bundle. Let θ_τ be the real 1-form on \mathcal{N} which vanishes on $\bar{\mathcal{K}}_\tau$ and satisfies $\langle \theta_\tau, \mathcal{T} \rangle = 1$; thus θ_τ is smooth, spans $\text{Char}(\bar{\mathcal{K}}_\tau)$, and has values in $\text{Char}^+(\bar{\mathcal{K}}_\tau)$. The Levi form of the structure is

$$\text{Levi}_{\theta_\tau}(v, w) = -id\theta_\tau(v, \bar{w}), \quad v, w \in \mathcal{K}_{\tau, p}, \quad p \in \mathcal{N}.$$

Suppose that $\text{Levi}_{\theta_\tau}$ is nondegenerate, with k positive and $n-k$ negative eigenvalues. It is well known that then $\square_{b,q}$ is microlocally hypoelliptic at $\nu \in \text{Char} \mathcal{K}_\tau$ for all q except if $q = k$ and $\tau(\nu) < 0$ or if $q = n-k$ and $\tau(\nu) > 0$.

Then the already mentioned Theorem 4.1 of [13] gives:

Theorem 8.3 ([13, Theorem 6.1]). *Suppose that $\bar{\mathcal{V}}$ admits a Hermitian metric and that for some defining function τ such that \mathbf{a}_τ is constant, $\text{Levi}_{\theta_\tau}$ is nondegenerate with k positive and $n-k$ negative eigenvalues. Then*

- (1) $\text{spec}_0^q(-i\mathcal{L}_\mathcal{T})$ is finite if $q \neq k, n-k$;
- (2) $\text{spec}_0^k(-i\mathcal{L}_\mathcal{T})$ contains only finitely many positive elements, and
- (3) $\text{spec}_0^{n-k}(-i\mathcal{L}_\mathcal{T})$ contains only finitely many negative elements.

9. INDICIAL COHOMOLOGY

Suppose that there is a \mathcal{T} -invariant Hermitian metric \tilde{h} on $\bar{\mathcal{V}}$. By Lemma 7.10 there is a defining function \mathfrak{r} such that $\langle \beta_{\mathfrak{r}}, \mathcal{T} \rangle$ is constant, equal to $a_{\text{av}} - i$. Therefore $\bar{\mathcal{K}}_{\mathfrak{r}}$ is \mathcal{T} -invariant. Let h be the metric on $\bar{\mathcal{V}}$ which coincides with \tilde{h} on $\bar{\mathcal{K}}_{\mathfrak{r}}$, makes the decomposition $\bar{\mathcal{V}} = \bar{\mathcal{K}}_{\mathfrak{r}} \oplus \text{span}_{\mathbb{C}} \mathcal{T}$ orthogonal, and for which \mathcal{T} has unit length. The metric h is \mathcal{T} -invariant. We fix \mathfrak{r} and such a metric, and let \mathfrak{m}_0 be the Riemannian measure associated with h . The decomposition (7.3) of $\Lambda^q \bar{\mathcal{V}}^*$ is an orthogonal decomposition.

Recall that $\bar{\mathcal{D}}(\sigma)\phi = \bar{\mathbb{D}}\phi + i\sigma\beta_{\mathfrak{r}} \wedge \phi$. Since $a_{\mathfrak{r}} = a_{\text{av}}$ is constant (in particular CR),

$$\bar{\mathcal{D}}(\sigma)(\phi^0 + i\tilde{\beta}_{\mathfrak{r}} \wedge \phi^1) = \bar{\partial}_b \phi_0 + i\tilde{\beta}_{\mathfrak{r}} \wedge [(\mathcal{L}_{\mathcal{T}} + (1 + ia_{\text{av}})\sigma)\phi^0 - \bar{\partial}_b \phi^1]$$

if $\phi^0 \in C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}_{\mathfrak{r}}^*)$ and $\phi^1 \in C^\infty(\mathcal{N}; \Lambda^{q-1} \bar{\mathcal{K}}_{\mathfrak{r}}^*)$. So $\bar{\mathcal{D}}(\sigma)$ can be regarded as the operator

$$(9.1) \quad \bar{\mathcal{D}}(\sigma) = \begin{bmatrix} \bar{\partial}_b & 0 \\ \mathcal{L}_{\mathcal{T}} + (1 + ia_{\text{av}})\sigma & -\bar{\partial}_b \end{bmatrix} : \begin{array}{c} C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}_{\mathfrak{r}}^*) \\ \oplus \\ C^\infty(\mathcal{N}; \Lambda^{q-1} \bar{\mathcal{K}}_{\mathfrak{r}}^*) \end{array} \rightarrow \begin{array}{c} C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{K}}_{\mathfrak{r}}^*) \\ \oplus \\ C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}_{\mathfrak{r}}^*) \end{array}.$$

Since the subbundles $\Lambda^q \bar{\mathcal{K}}_{\mathfrak{r}}$ and $\tilde{\beta} \wedge \Lambda^{q-1} \bar{\mathcal{K}}_{\mathfrak{r}}$ are orthogonal with respect to the metric induced by h on $\Lambda^q \bar{\mathcal{V}}$, the formal adjoint of $\bar{\mathcal{D}}(\sigma)$ with respect to this metric and the density \mathfrak{m}_0 is

$$\bar{\mathcal{D}}(\sigma)^* = \begin{bmatrix} \bar{\partial}_b^* & -\mathcal{L}_{\mathcal{T}} + (1 - ia_{\text{av}})\bar{\sigma} \\ 0 & -\bar{\partial}_b^* \end{bmatrix} : \begin{array}{c} C^\infty(\mathcal{N}; \Lambda^{q+1} \bar{\mathcal{K}}_{\mathfrak{r}}^*) \\ \oplus \\ C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}_{\mathfrak{r}}^*) \end{array} \rightarrow \begin{array}{c} C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}_{\mathfrak{r}}^*) \\ \oplus \\ C^\infty(\mathcal{N}; \Lambda^{q-1} \bar{\mathcal{K}}_{\mathfrak{r}}^*) \end{array}$$

where $\bar{\partial}_b^*$ is the formal adjoint of $\bar{\partial}_b$. So the Laplacian, $\square_{\bar{\mathcal{D}}(\sigma), q}$, of the $\bar{\mathcal{D}}(\sigma)$ -complex is the diagonal operator with diagonal entries $P_q(\sigma)$, $P_{q-1}(\sigma)$ where

$$P_q(\sigma) = \square_{b,q} + (\mathcal{L}_{\mathcal{T}} + (1 + ia_{\text{av}})\sigma)(-\mathcal{L}_{\mathcal{T}} + (1 - ia_{\text{av}})\bar{\sigma})$$

acting on $C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}_{\mathfrak{r}}^*)$ and $P_{q-1}(\sigma)$ is the ‘‘same’’ operator, acting on sections of $\Lambda^{q-1} \bar{\mathcal{K}}_{\mathfrak{r}}^*$; recall that $\mathcal{L}_{\mathcal{T}}$ commutes with $\bar{\partial}_b$ and since $\mathcal{L}_{\mathcal{T}}^* = -\mathcal{L}_{\mathcal{T}}$, also with $\bar{\partial}_b^*$, and that a_{av} is constant. Note that $P_q(\sigma)$ is an elliptic operator.

Suppose that $\phi \in C^\infty(\mathcal{N}; \Lambda^q \bar{\mathcal{K}}_{\mathfrak{r}}^*)$ is a nonzero element of $\ker P_q(\sigma)$; the complex number σ is fixed. Since $P_q(\sigma)$ is elliptic, $\ker P_q(\sigma)$ is a finite dimensional space, invariant under $-i\mathcal{L}_{\mathcal{T}}$ since the latter operator commutes with $P_q(\sigma)$. As an operator on $\ker P_q(\sigma)$, $-i\mathcal{L}_{\mathcal{T}}$ is selfadjoint, so there is a decomposition of $\ker P_q(\sigma)$ into eigenspaces of $-i\mathcal{L}_{\mathcal{T}}$. Thus

$$\phi = \sum_{j=1}^N \phi_j, \quad -i\mathcal{L}_{\mathcal{T}}\phi_j = \tau_j \phi_j$$

where the τ_j are distinct real numbers and $\phi_j \in \ker P_q(\sigma)$, $\phi_j \neq 0$. In particular,

$$\square_{b,q}\phi_j + (\mathcal{L}_{\mathcal{T}} + (1 + ia_{\text{av}})\sigma)(-\mathcal{L}_{\mathcal{T}} + (1 - ia_{\text{av}})\bar{\sigma})\phi_j = 0,$$

for each j , that is,

$$\square_{b,q}\phi_j + |i\tau_j + (1 + ia_{\text{av}})\sigma|^2 \phi_j = 0.$$

Since $\square_{b,q}$ is a nonnegative operator and $\phi_j \neq 0$, $i\tau_j + (1 + ia_{\text{av}})\sigma = 0$ and $\phi_j \in \ker \square_{b,q}$. Since σ is fixed, all τ_j are equal, which means that $N = 1$. Conversely, if

$\phi \in C^\infty(\mathcal{N}; \bigwedge^q \overline{\mathcal{K}}_\mathfrak{r}^*)$ belongs to $\ker \square_{b,q}$ and $-i\mathcal{L}_\mathcal{T}\phi = \tau\phi$, then $P_q(\sigma)\phi = 0$ with σ such that $\tau = (i - a_{\text{av}})\sigma$.

Let $\mathcal{H}_{\overline{\mathcal{D}}(\sigma)}^q(\mathcal{N})$ be the kernel of $\square_{\overline{\mathcal{D}}(\sigma),q}$.

Theorem 9.2. *Suppose that $\overline{\mathcal{V}}$ admits a \mathcal{T} -invariant metric and let \mathfrak{r} be a defining function for \mathcal{N} in \mathcal{M} such that $\langle \beta_\mathfrak{r}, \mathcal{T} \rangle = a_{\text{av}} - i$ is constant. Then*

$$\text{spec}_{b,\mathcal{N}}^q(\overline{b}\overline{\partial}) = (i - a_{\text{av}})^{-1} \text{spec}_0^q(-i\mathcal{L}_\mathcal{T}) \cup (i - a_{\text{av}})^{-1} \text{spec}_0^{q-1}(-i\mathcal{L}_\mathcal{T}),$$

and if $\sigma \in \text{spec}_{b,\mathcal{N}}^q(\overline{b}\overline{\partial})$, then, with the notation in Definition 8.2

$$\mathcal{H}_{\overline{\mathcal{D}}(\sigma)}^q(\mathcal{N}) = \mathcal{H}_{\overline{\partial}_b, \tau(\sigma)}^q(\mathcal{N}) \oplus \mathcal{H}_{\overline{\partial}_b, \tau(\sigma)}^{q-1}(\mathcal{N})$$

with $\tau(\sigma) = (i - a_{\text{av}})\sigma$.

If the CR structure $\overline{\mathcal{K}}_\mathfrak{r}$ is nondegenerate, Proposition 8.3 gives more specific information on $\text{spec}_{b,\mathcal{N}}^q(\overline{b}\overline{\partial})$. In particular,

Proposition 9.3. *With the hypotheses of Theorem 9.2, suppose that $\text{Levi}_{\theta_\mathfrak{r}}$ is nondegenerate with k positive and $n - k$ negative eigenvalues. If $k > 0$, then $\text{spec}_{b,\mathcal{N}}^0 \subset \{\sigma \in \mathbb{C} : \Im\sigma \leq 0\}$, and if $n - k > 0$, then $\text{spec}_{b,\mathcal{N}}^0(\overline{b}\overline{\partial}) \subset \{\sigma \in \mathbb{C} : \Im\sigma \geq 0\}$.*

Remark 9.4. The b -spectrum of the Laplacian of the $\overline{b}\overline{\partial}$ -complex in any degree can be described explicitly in terms of the joint spectra $\text{spec}(-i\mathcal{L}_\mathcal{T}, \square_{b,q})$. We briefly indicate how. With the metric h and defining function \mathfrak{r} as in the first paragraph of this section, suppose that h is extended to a metric on ${}^bT^{0,1}\mathcal{M}$. This gives a Riemannian b -metric on \mathcal{M} that in turn gives a b -density \mathfrak{m} on \mathcal{M} . With these we get formal adjoints $\overline{b}\overline{\partial}^*$ whose indicial families $\overline{\mathcal{D}}^*(\sigma)$ are related to those of $\overline{b}\overline{\partial}$ by

$$\overline{\mathcal{D}}^*(\sigma) = \widehat{\overline{b}\overline{\partial}^*}(\sigma) = [\widehat{\overline{b}\overline{\partial}}(\overline{\sigma})]^* = \overline{\mathcal{D}}(\overline{\sigma})^*.$$

By (9.1),

$$\overline{\mathcal{D}}^*(\sigma) = \begin{bmatrix} \overline{\partial}_b^* & -\mathcal{L}_\mathcal{T} + (1 - ia_{\text{av}})\sigma \\ 0 & -\overline{\partial}_b^* \end{bmatrix}.$$

Using this one obtains that the indicial family of the Laplacian \square_q of the $\overline{b}\overline{\partial}$ -complex in degree q is a diagonal operator with diagonal entries $P'_q(\sigma)$, $P'_{q-1}(\sigma)$ with

$$P'_q(\sigma) = \square_{b,q} + (\mathcal{L}_\mathcal{T} + (1 + ia_{\text{av}})\sigma)(-\mathcal{L}_\mathcal{T} + (1 - ia_{\text{av}})\sigma)$$

and the analogous operator in degree $q - 1$. The set $\text{spec}_b(\square_q)$ is the set of values of σ for which either $P'_q(\sigma)$ or $P'_{q-1}(\sigma)$ is not injective. These points can be written in terms of the points $\text{spec}(-i\mathcal{L}_\mathcal{T}, \square_b)$ as asserted. In particular one gets

$$\text{spec}_b(\square_q) \subset \{\sigma : |\Re\sigma| \leq |a_{\text{av}}| |\Im\sigma|\}$$

with $\text{spec}_{b,\mathcal{N}}^q(\overline{b}\overline{\partial})$ being a subset of the boundary of the set on the right.

We now discuss the indicial cohomology sheaf of $\overline{b}\overline{\partial}$, see Definition 6.12. We will show:

Proposition 9.5. *Let $\sigma_0 \in \text{spec}_{b,\mathcal{N}}^q(\overline{b}\overline{\partial})$. Every element of the stalk of $\mathfrak{S}_{\overline{b}\overline{\partial}}^q(\mathcal{N})$ at σ_0 has a representative of the form*

$$\frac{1}{\sigma - \sigma_0} \begin{bmatrix} \phi^0 \\ 0 \end{bmatrix}$$

where $\phi^0 \in \mathcal{H}_{\overline{\partial}_b, \tau_0}^q(\mathcal{N})$, $\tau_0 = (i - a_{\text{av}})\sigma_0$.

Proof. Let

$$(9.6) \quad \phi(\sigma) = \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} \phi_k^0 \\ \phi_k^1 \end{bmatrix}$$

represent an element in the stalk at σ_0 of the sheaf of germs of $C^\infty(\mathcal{N}; \wedge^q \bar{\mathcal{V}}^* \otimes F)$ -valued meromorphic functions on \mathbb{C} modulo the subsheaf of holomorphic elements. Letting $\alpha = 1 + ia_{av}$ we have

$$\bar{\mathcal{D}}(\sigma)\phi(\sigma) = \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \left[(\mathcal{L}_{\mathcal{T}} + \alpha\sigma_0)\phi_k^0 - \bar{\partial}_b\phi_k^1 \right] + \sum_{k=0}^{\mu-1} \frac{\alpha}{(\sigma - \sigma_0)^k} \begin{bmatrix} 0 \\ \phi_{k+1}^0 \end{bmatrix},$$

so the condition that $\bar{\mathcal{D}}(\sigma)\phi(\sigma)$ is holomorphic is equivalent to

$$(9.7) \quad \bar{\partial}_b\phi_k^0 = 0, \quad k = 1, \dots, \mu$$

and

$$(9.8) \quad \begin{aligned} (\mathcal{L}_{\mathcal{T}} + \alpha\sigma_0)\phi_\mu^0 - \bar{\partial}_b\phi_\mu^1 &= 0, \\ (\mathcal{L}_{\mathcal{T}} + \alpha\sigma_0)\phi_k^0 - \bar{\partial}_b\phi_k^1 + \alpha\phi_{k+1}^0 &= 0, \quad k = 1, \dots, \mu - 1. \end{aligned}$$

Let $P_{q'} = \square_{b,q'} - \mathcal{L}_{\mathcal{T}}^2$ in any degree q' . For any $(\tau, \lambda) \in \mathbb{R}^2$ and q' let

$$\mathcal{E}_{\tau,\lambda}^{q'} = \{ \psi \in C^\infty(\mathcal{N}; \wedge^{q'} \bar{\mathcal{V}}^* \otimes F) : P_{q'}\psi = \lambda\psi, \quad -i\mathcal{L}_{\mathcal{T}}\psi = \tau\psi \}.$$

This space is zero if (τ, λ) is not in the joint spectrum $\Sigma^{q'} = \text{spec}^{q'}(-i\mathcal{L}_{\mathcal{T}}, P_{q'})$. Each ϕ_k^i decomposes as a sum of elements in the spaces $\mathcal{E}_{\tau,\lambda}^{q-i}$, $(\tau, \lambda) \in \Sigma^{q-i}$. Suppose that already $\phi_k^i \in \mathcal{E}_{\tau,\lambda}^{q-i}$:

$$P_{q-i}\phi_k^i = \lambda\phi_k^i, \quad -i\mathcal{L}_{\mathcal{T}}\phi_k^i = \tau\phi_k^i, \quad i = 0, 1, \quad k = 1, \dots, \mu.$$

Then (9.8) becomes

$$(9.9) \quad \begin{aligned} (i\tau + \alpha\sigma_0)\phi_\mu^0 - \bar{\partial}_b\phi_\mu^1 &= 0, \\ (i\tau + \alpha\sigma_0)\phi_k^0 - \bar{\partial}_b\phi_k^1 + \alpha\phi_{k+1}^0 &= 0, \quad k = 1, \dots, \mu - 1. \end{aligned}$$

If $\tau \neq \tau_0$, then $i\tau + \alpha\sigma_0 \neq 0$, and we get $\phi_k^0 = \bar{\partial}_b\psi_k^0$ for all k with

$$\psi_k^0 = \sum_{j=0}^{\mu-k} \frac{(-\alpha)^j}{(i\tau + \alpha\sigma_0)^{j+1}} \phi_{k+j}^1.$$

Trivially

$$(\mathcal{L}_{\mathcal{T}} + \alpha\sigma_0)\psi_\mu^0 = \phi_\mu^1$$

and also

$$(\mathcal{L}_{\mathcal{T}} + \alpha\sigma_0)\psi_k^0 + \alpha\psi_{k+1}^0 = \phi_k^1, \quad k = 1, \dots, \mu - 1,$$

so

$$\phi(\sigma) - \bar{\mathcal{D}}(\sigma) \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} \psi_k^0 \\ 0 \end{bmatrix} = 0$$

modulo an entire element.

Suppose now that the ϕ_k^i are arbitrary and satisfy (9.7)-(9.8). The sum

$$(9.10) \quad \phi_k^i = \sum_{(\tau,\lambda) \in \Sigma^{q-i}} \phi_{k,\tau,\lambda}^i, \quad \phi_{k,\tau,\lambda}^i \in \mathcal{E}_{\tau,\lambda}^{q-i}$$

converges in C^∞ , indeed for each N there is $C_{i,k,N}$ such that

$$(9.11) \quad \sup_{p \in \mathcal{N}} \|\phi_{k,\tau,\lambda}^i(p)\| \leq C_{i,k,N}(1+\lambda)^{-N} \quad \text{for all } \tau, \lambda.$$

Since $\overline{\mathcal{D}}(\sigma)$ preserves the spaces $\mathcal{E}_{\tau,\lambda}^q \oplus \mathcal{E}_{\tau,\lambda}^{q-1}$, the relations (9.9) hold for the $\phi_{k,\tau,\lambda}^i$ for each (τ, λ) . Therefore, with

$$(9.12) \quad \psi_k^0 = \sum_{\substack{(\tau,\lambda) \in \Sigma^{q-1} \\ \tau \neq \tau_0}} \sum_{j=0}^{\mu-k} \frac{(-\alpha)^j}{(i\tau + \alpha\sigma_0)^{j+1}} \phi_{k+j,\tau,\lambda}^1$$

we have formally that

$$\phi(\sigma) - \overline{\mathcal{D}}(\sigma) \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} \psi_k \\ 0 \end{bmatrix} = \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} \tilde{\phi}_k^0 \\ \tilde{\phi}_k^1 \end{bmatrix}$$

with

$$(9.13) \quad \tilde{\phi}_k^i = \sum_{\substack{(\tau,\lambda) \in \Sigma^{q-1} \\ \tau = \tau_0}} \phi_{k,\tau,\lambda}^i, \quad \phi_{k,\tau,\lambda}^i \in \mathcal{E}_{\tau,\lambda}^{q-i}.$$

However, the convergence in C^∞ of the series (9.12) is questionable since there may be a sequence $\{(\tau_\ell, \lambda_\ell)\}_{\ell=1}^\infty \subset \text{spec}(-i\mathcal{L}_{\mathcal{T}}, P_{q-1})$ of distinct points such that $\tau_\ell \rightarrow \tau_0$ as $\ell \rightarrow \infty$, so that the denominators $i\tau_\ell + \alpha\sigma_0$ in the formula for ψ_k^0 tend to zero so fast that for some nonnegative N , $\lambda_\ell^{-N}/(i\tau_\ell + \alpha\sigma_0)$ is unbounded. To resolve this difficulty we will first show that $\phi(\sigma)$ is $\overline{\mathcal{D}}(\sigma)$ -cohomologous (modulo holomorphic terms) to an element of the same form as $\phi(\sigma)$ for which in the series (9.10) the terms $\phi_{k,\tau,\lambda}^i$ vanish if $\lambda - \tau^2 > \varepsilon$; the number $\varepsilon > 0$ is chosen so that

$$(9.14) \quad (\tau_0, \lambda) \in \Sigma^q \cup \Sigma^{q-1} \implies \lambda = \tau_0^2 \text{ or } \lambda \geq \tau_0^2 + \varepsilon.$$

Recall that $\text{spec}^{q'}(-i\mathcal{L}_{\mathcal{T}}, P_{q'}) \subset \{(\tau, \lambda) : \lambda \geq \tau^2\}$.

For any $V \subset \bigcup_{q'} \Sigma^{q'}$ let

$$\Pi_V^{q'} : L^2(\mathcal{N}; \wedge^{q'} \overline{\mathcal{V}}^* \otimes F) \rightarrow L^2(\mathcal{N}; \wedge^{q'} \overline{\mathcal{V}}^* \otimes F)$$

be the orthogonal projection on $\bigoplus_{(\tau,\lambda) \in V} \mathcal{E}_{\tau,\lambda}^{q'}$. If $\psi \in C^\infty(\mathcal{N}; \wedge^{q'} \overline{\mathcal{V}}^* \otimes F)$, then the series

$$\Pi_V^{q'} \psi = \sum_{(\tau,\lambda) \in V} \psi_{\tau,\lambda}, \quad \psi_{\tau,\lambda} \in \mathcal{E}_{\tau,\lambda}^{q'}$$

converges in C^∞ . It follows that $\square_{b,q'}$ and $\mathcal{L}_{\mathcal{T}}$ commute with $\Pi_V^{q'}$ and that $\overline{\partial}_b \Pi_V^{q'} = \Pi_V^{q'+1} \overline{\partial}_b$. Since the $\Pi_V^{q'}$ are selfadjoint, also $\overline{\partial}_b^* \Pi_V^{q'+1} = \Pi_V^{q'} \overline{\partial}_b^*$.

Let

$$U = \{(\tau, \lambda) \in \Sigma^q \cup \Sigma^{q-1} : \lambda < \tau^2 + \varepsilon\}, \quad U^c = \Sigma^q \cup \Sigma^{q-1} \setminus U.$$

Then, for any sequence $\{(\tau_\ell, \lambda_\ell)\} \subset U$ of distinct points we have $|\tau_\ell| \rightarrow \infty$ as $\ell \rightarrow \infty$. Define

$$G_{U^c}^{q'} \psi = \sum_{(\tau,\lambda) \in U^c} \frac{1}{\lambda - \tau^2} \psi_{\tau,\lambda}$$

In this definition the denominators $\lambda - \tau^2$ are bounded from below by ε , so $G_{U^c}^{q'}$ is a bounded operator in L^2 and maps smooth sections to smooth sections because

the components of such sections satisfy estimates as in (9.11). The operators are analogous to Green operators: we have

$$(9.15) \quad \square_{b,q'} G_{U^c}^{q'} = G_{U^c}^{q'} \square_{b,q'} = I - \Pi_U^{q'}$$

so if $\bar{\partial}_b \psi = 0$, then

$$(9.16) \quad \square_{b,q'} G_{U^c}^{q'} \psi = \bar{\partial}_b \bar{\partial}_b^* G_{U^c}^{q'} \psi$$

since $\bar{\partial}_b G_{U^c}^{q'} = G_{U^c}^{q'+1} \bar{\partial}_b$.

Write $\phi(\sigma)$ in (9.6) as

$$\phi(\sigma) = \Pi_{U^c} \phi(\sigma) + \Pi_U \phi(\sigma)$$

where

$$\Pi_{U^c} \phi(\sigma) = \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} \Pi_{U^c}^q \phi_k^0 \\ \Pi_{U^c}^{q-1} \phi_k^1 \end{bmatrix}, \quad \Pi_U \phi(\sigma) = \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} \Pi_U^q \phi_k^0 \\ \Pi_U^{q-1} \phi_k^1 \end{bmatrix}.$$

Since $\bar{\mathcal{D}}(\sigma) \phi(\sigma)$ is holomorphic, so are $\bar{\mathcal{D}}(\sigma) \Pi_{U^c} \phi(\sigma)$ and $\bar{\mathcal{D}}(\sigma) \Pi_U \phi(\sigma)$.

We show that $\Pi_{U^c} \phi(\sigma)$ is exact modulo holomorphic functions. Using (9.7), (9.15), and (9.16), $\Pi_{U^c}^q \phi_k^0 = \bar{\partial}_b^* \bar{\partial}_b \Pi_{U^c}^q \phi_k^0$. Then

$$\Pi_{U^c} \phi(\sigma) - \bar{\mathcal{D}}(\sigma) \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} \bar{\partial}_b^* G_{U^c}^q \Pi_{U^c}^q \phi_k^0 \\ 0 \end{bmatrix} = \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} 0 \\ \hat{\phi}_k^1 \end{bmatrix}$$

modulo a holomorphic term for some $\hat{\phi}_k^1$ with $\Pi_{U^c}^{q-1} \hat{\phi}_k^1 = \hat{\phi}_k^1$. The element on the right is $\bar{\mathcal{D}}(\sigma)$ -closed modulo a holomorphic function, so its components satisfy (9.7), (9.8), which give that the $\hat{\phi}_k^1$ are $\bar{\partial}_b$ -closed. Using again (9.15) and (9.16) we see that $\Pi_{U^c} \phi(\sigma)$ represent an exact element.

We may thus assume that $\Pi_{U^c}^q \phi(\sigma) = 0$. If this is the case, then the series (9.12) converges in C^∞ , so $\phi(\sigma)$ is cohomologous to the element

$$\tilde{\phi}(\sigma) = \sum_{k=1}^{\mu} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} \tilde{\phi}_k^0 \\ \tilde{\phi}_k^1 \end{bmatrix}$$

where the $\tilde{\phi}_k^i$ are given by (9.13) and satisfy $\Pi_{U^c}^{q-i} \tilde{\phi}_k^i = 0$. By (9.14), $\tilde{\phi}_k^i \in \mathcal{E}_{\tau_0, \tau_0^2}^{q-i}$. In particular, $\square_{b,q-i} \tilde{\phi}_k^i = 0$.

Assuming now that already $\phi_k^i \in \mathcal{E}_{\tau_0, \tau_0^2}^{q-i}$, the formulas (9.9) give (since $\tau = \tau_0$ and $i\tau_0 + \alpha\sigma_0 = 0$)

$$\bar{\partial}_b \phi_\mu^1 = 0, \quad \phi_k^0 = \bar{\partial}_b \frac{1}{\alpha} \phi_{k-1}^1, \quad k = 2, \dots, \mu.$$

Then

$$\phi(\sigma) - \frac{1}{\alpha} \bar{\mathcal{D}}(\sigma) \sum_{k=2}^{\mu+1} \frac{1}{(\sigma - \sigma_0)^k} \begin{bmatrix} \phi_{k-1}^1 \\ 0 \end{bmatrix} = \frac{1}{\sigma - \sigma_0} \begin{bmatrix} \phi_1^0 \\ 0 \end{bmatrix}$$

with $\square_{b,q} \phi_1^0 = 0$. □

APPENDIX A. TOTALLY CHARACTERISTIC DIFFERENTIAL OPERATORS

We review here some basic definitions and notation concerning totally characteristic differential operators.

Let $E, F \rightarrow \mathcal{M}$ be vector bundles and let $\text{Diff}^m(\mathcal{M}; E, F)$ be the space of differential operators $C^\infty(\mathcal{M}; E) \rightarrow C^\infty(\mathcal{M}; F)$ of order m . Then

(A.1) $\text{Diff}_b^m(\mathcal{M}; E, F)$, the space of totally characteristic differential operators of order m , consists of those elements $P \in \text{Diff}^m(\mathcal{M}; E, F)$ with the property

$$\mathfrak{r}^{-\nu} P \mathfrak{r}^\nu \in \text{Diff}^m(\mathcal{M}; E, F), \quad \nu = 1, \dots, m$$

i.e., $\mathfrak{r}^{-\nu} P \mathfrak{r}^\nu$ has coefficients smooth up to the boundary.

Let $\pi : T^*\mathcal{M} \rightarrow \mathcal{M}$ and ${}^b\pi : {}^bT^*\mathcal{M} \rightarrow \mathcal{M}$ be the canonical projections. Suppose $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$. Since P is in particular a differential operator, it has a principal symbol

$$\sigma(P) \in C^\infty(T^*\mathcal{M}; \text{Hom}(\pi^*E, \pi^*F)).$$

The fact that P is totally characteristic implies that $\sigma(P)$ lifts to a section

$${}^b\sigma(P) \in C^\infty({}^bT^*\mathcal{M}; \text{Hom}({}^b\pi^*E, {}^b\pi^*F)),$$

the principal b -symbol of P , characterized by

$$(A.2) \quad {}^b\sigma(P)(\text{ev}^*\xi) = \sigma(P)(\xi).$$

If $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$, then P induces a differential operator

$$(A.3) \quad P_b \in \text{Diff}_b^m(\mathcal{M}; E_{\partial\mathcal{M}}, F_{\partial\mathcal{M}}),$$

as follows. If $\phi \in C^\infty(\partial\mathcal{M}; E_{\partial\mathcal{M}})$, let $\tilde{\phi} \in C^\infty(\mathcal{M}; E)$ be an extension of ϕ and let

$$P_b\phi = (P\tilde{\phi})|_{\partial\mathcal{M}}.$$

The condition (A.1) ensures that $P\tilde{\phi}|_{\partial\mathcal{M}}$ is independent of the extension of ϕ used. Clearly if P and Q are totally characteristic differential operators, then so is PQ , and

$$(A.4) \quad (PQ)_b = P_b Q_b.$$

The indicial family of $P \in \text{Diff}_b^m(\mathcal{M}; E, F)$ is defined as follows. Fix a defining function \mathfrak{r} for $\partial\mathcal{M}$. Then for any $\sigma \in \mathbb{C}$,

$$P(\sigma) = \mathfrak{r}^{-i\sigma} P \mathfrak{r}^{i\sigma} \in \text{Diff}_b^m(\mathcal{M}; E, F).$$

Let

$$(A.5) \quad \widehat{P}(\sigma) = P(\sigma)_b.$$

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