1. Let $\mathcal{A}$ be the algebra of complex-valued functions on $\mathbb{R}$ whose elements are linear combinations of the functions $1, \cos nx, \sin nx, n \in \mathbb{N}$. Show that $\mathcal{A}$ is dense in the subspace
\[
C_{2\pi}(\mathbb{R}) = \{ f \in C(\mathbb{R}) : f \text{ is periodic of period } 2\pi \}
\]of $C_b(\mathbb{R})$. [Trick: View the problem on $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$.]

2. Let $F_b$ be the vector space of all bounded, not necessarily continuous, functions $\mathbb{R} \to \mathbb{R}$ with the uniform norm. Let $H : X \to \mathbb{R}$ be the Heaviside function:
\[
H(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{if } x < 0
\end{cases}
\]
Find the distance from $H$ to the subspace of $F_b$ of continuous functions.

In the rest of the problems, $X$ is a non-compact locally compact Hausdorff space.
Write $K \subseteq X$ if $X$ is a compact subset of $X$.

3. For any $K \subseteq X$ let $C_c(K) = \{ f \in C(X) : \text{supp } f \subseteq K \}$, a subspace of $C_b(X)$. Show that $C_c(K)$ is closed in $C_b(X)$.

Let $C_c(X) = \bigcup_{K \subseteq X} C_c(K)$. The inductive limit topology on $C_c(X)$ is the topology in which the open sets are those subsets $U \subseteq C_c(X)$ such that $U \cap C_c(K)$ is open in $C_c(K)$ for every $K \subseteq X$. (This topology may be larger than necessary, see Problem 6.) Use the inductive limit topology in the following three problems.

4. Let $\iota_K : C_c(K) \to C_c(X)$ be the inclusion map. Show: A function $\lambda : C_c(X) \to \mathbb{C}$ is continuous if and only if $f \circ \iota_K$ is continuous for every $K \subseteq X$.

5. Say that a sequence $\{ u_n \}_{n=1}^{\infty} \subseteq C_c(X)$ is Cauchy in $C_c(X)$ if for every open neighborhood of $0$ there is $n_0$ such that $u_m - u_n \in U$ if $n, m \geq n_0$. Show that $\{ u_n \}$ has a limit in the topology of uniform convergence on compact sets. [Trick: For each $K \subseteq X$ consider the sets
\[
V_\varepsilon = \{ u \in C_c(X) : \sup_{x \in K} |u(x)| < \varepsilon \},
\]
with arbitrary $\varepsilon > 0$.]

6. Let $X$ be $\sigma$-compact. Let $\{ u_n \}_{n=1}^{\infty} \subseteq C_c(X)$ be a Cauchy sequence in $C_c(X)$. Show that there is $K \subseteq X$ such that $\text{supp } u_n \subseteq K$ for all $n$, that $u \in C_c(K)$, and that $u_n$ converges to $u$ in $C_c(K)$. (This is also true with the smaller topology defined in Problem 6.) [Trick: What if the conclusion is false?]

7. Let $X$ be a locally compact Hausdorff space. Give $C_c(X)$ the topology generated by the convex sets $U \subseteq C_c(X)$ such that $\iota_K^{-1}(U)$ is an open subset of $C_c(K)$ for every $K \subseteq X$. Show: A linear function $\lambda : C_c(X) \to \mathbb{C}$ is continuous if and only if $f \circ \iota_K$ is continuous for every $K \subseteq X$.