Additional problems

1. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(Y, \mathcal{N})$ a measurable space, and $f : X \to Y$ measurable. Define $\nu : \mathcal{N} \to [0, \infty]$ by

$$\nu(E) = \mu(f^{-1}(E)).$$

Show that $\nu$ is a measure and that if $\nu$ is $\sigma$-finite, then so is $\mu$.

2. Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. For $n \in \mathbb{N}$ define

$$x_{n,k} = a + \frac{(b - a)k}{n}$$

and let

$$f_n = \sum_{k=0}^{n-1} f(x_{n,k}) \chi_{[x_{n,k}, x_{n,k+1})}$$

Show that $f_n \to f$ almost everywhere.

3. Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that its Riemann integral is equal to its Lebesgue integral.

4. Let $f : \mathbb{R} \to \mathbb{C}$ be a function of class $C^2$ and period $2\pi$. Define

$$c_n = \int_{-\pi}^{\pi} e^{-iny} f(y) \, dy.$$

Use $e^{-iny} = -\frac{1}{in} \frac{d}{dy} e^{-iny} \ (n \neq 0)$ to show that

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

converges absolutely.