1. Let \((X, \mathcal{M}, \mu)\) be a measure space. Let \(\{E_k\}_{k=1}^\infty\) be a family of measurable subsets of \(X\) and let \(\{c_k\}_{k=1}^\infty\) be a sequence of nonnegative numbers. Show that 
\[
f = \sum_{k=1}^\infty c_k \chi_{E_k}
\]
is measurable and 
\[
\int f \, d\mu = \sum_{k=1}^\infty c_k \mu(E_k).
\]

2. Let \((X, \mathcal{M}, \mu)\) be a measure space. Suppose \(f \in L^+(X, \mathcal{M})\) and \(\int f < \infty\). Show that 
\[
E = \{x : f(x) = \infty\}
\]
has measure 0 and that 
\[
Y = \{x : f(x) > 0\}
\]
is \(\sigma\)-finite. (This is a proposition and a problem in our textbook.)

3. Let \((X, \mathcal{M}, \mu)\) be a measure space, let \(\{f_n\}_{n=1}^\infty\) be a sequence in \(L^+(X, \mathcal{M})\) which is monotonically decreasing, with each \(f_n\) finite-valued (the value \(\infty\) is not attained). Let 
\[
f = \lim_{n \to \infty} f_n(x)
\]
(which is measurable and belongs to \(L^+(X, \mathcal{M})\)). Assuming that \(\int f_{n_0}\) is finite for some \(n_0\), show that 
\[
\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu.
\]

4. Let \((X, \mathcal{M}, \mu)\) be a measure space. Let \(\{f_n\}_{n=1}^\infty\) be a sequence in \(L^+(X, \mathcal{M})\) (not necessarily monotonically increasing). Let 
\[
E = \{x \in X : \exists k \forall n \geq k \quad f_n(x) \leq f_{n+1}(x)\}
\]
Show that \(E\) is measurable, that \(\chi_E f_n\) converges pointwise, and that 
\[
\lim_{n \to \infty} \int \chi_E f_n \, d\mu = \int \lim_{n \to \infty} \chi_E f_n \, d\mu.
\]

5. Define \(L^k(t)\) for \(k = 1, 2, \ldots\) and real numbers \(t \geq c_k\) by letting \(L^1(t) = \log(t)\) for \(t \geq c_1 = e\) and \(L^{k+1}(t) = \log(L^k(t))\) for \(t > c_{k+1} = e^{c_k}\) (the \(c_k\) are such that \(L^k(t) \geq 1\) if \(t \geq c_k\)). Each of the sequences 
\[
\{L^k(n)\}_{n=|c_k|}^\infty
\]
is increasing but their rate of increase drops as \(k\) increases. Give an example of a sequence of simple nonnegative functions \(\phi_n : [0, 1] \to \mathbb{R}^+\) which is pointwise strictly increasing (\(\phi_{n+1}(x) > \phi_n(x)\) for all \(x\)) and such that for every \(k\) the set 
\[
\{x : L^{k+1}(n) \leq \phi_n(x) \leq L^k(n) \text{ if } n \geq c_{k+1}\}
\]
has positive Lebesgue measure for every positive integer \(k\).