Topic 9. Harmonic polynomials

A polynomial $p(x)$ in n variables $x_1, \dots, x_n$ is harmonic if it satisfies

$$(\frac{\partial^2}{x_1^2} + \cdots + \frac{\partial^2}{x_n^2})p(x) = 0$$

The task is to compute the dimension of the space of homogeneous harmonic polynomials of degree m. This space is defined in the next paragraph.

A monomial of degree m in the n variables $x_1, \dots, x_n$ is a product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with nonnegative integers α_j such that $\alpha_1 + \cdots + \alpha_n = m$. We say that $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multiindex and that m is its length. We also write $|\alpha|$ for its length and x^α for $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. A homogeneous polynomial of degree m in n variables is a linear combination of homogeneous monomials of degree m:

$$p(x) = \sum_{|\alpha|=m} p_\alpha x^\alpha. \tag{1}$$

The set of homogeneous polynomials of degree m, denoted Π_m, is a finite-dimensional vector space. The set of monomials $\{x^\alpha\}_{|\alpha|=m}$ is a basis. For convenience we define $\Pi_m = 0$ if $m < 0$.

The following is useful multiindex notation: Let $\alpha, \beta \in \mathbb{N}_0^n$.

(1) $\alpha| = \alpha_1 + \cdots + \alpha_n$;
(2) $\beta \le \alpha$ *if* $\forall j: \ \beta_j \le \alpha_j$;
(3) $\alpha! = \alpha_1! \dots \alpha_n!$;
(4) If $\beta \le \alpha$, *then* $\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.
(5) $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

To get a sense of how things work in the problems below, analyze the cases $n = 2$ or $n = 3$. However the paper is with general n. This material is based on

Neri, Umberto, *Singular integrals*, Lecture Notes in Mathematics, Vol. 200. Springer-Verlag, Berlin-New York, 1971.

***Note**: Problem 9.1 by itself can be used for one paper and Problems 9.2–9.5, together, for another. You may hand in partial work, it is not necessary to work everything out before you send me your work.*

9.1. Find the dimension of Π_m, that is, find $\#\{\alpha : |\alpha| = m\}$.

To do this, show that

$$\frac{1}{1-tx_1} \cdots \frac{1}{1-tx_n} = \sum_{m=0}^{\infty} (\sum_{|\alpha|=m} x^\alpha) t^m \tag{2}$$

so that setting $x_1 = x_2 = \cdots = x_n = 1$ then the coefficient of t^m is $\#\{\alpha : |\alpha| = m\}$. The task is then to find the Taylor coefficients d_m of

$$\frac{1}{(1-t)^n} = \sum_{m=0}^{\infty} d_m t^m. \tag{3}$$

To find these, use that

$$\frac{1}{1-t} = \sum_{m=0}^{\infty} t^m$$

if $|t| < 1$ and

$$\frac{1}{(1-t)^n} = \frac{1}{(n-1)!} \frac{d^{n-1}}{dt^{n-1}} \frac{1}{1-t}.$$

(which is proved by induction on n).
A formula for the d_m can be found without first establishing the validity of (3). However, to prove (3) use

$$\frac{1}{1-tx_k} = \sum_{\alpha_k=0}^{\infty} (tx_k)^{\alpha_k}$$

which converges if $|tx_k| < 1$. One possible way of getting the formula is by splitting the series as

$$\frac{1}{1-tx_k} = \sum_{\alpha_k=0}^{N} (tx_k)^{\alpha_k} + \sum_{\alpha_k=N+1}^{\infty} (tx_k)^{\alpha_k}.$$

The second sum factors as $(tx_k)^{N+1}\sum_{\alpha_k=0}^{\infty}(tx_k)^{\alpha_k}$. Writing it as $(tx_k)^{N+1}h_N(tx_k)$, the product in (3) becomes

$$\begin{aligned}
&\frac{1}{1-tx_1}\cdots\frac{1}{1-tx_n}\\
&\quad = \Big(\sum_{\alpha_1=0}^{N}(tx_1)^{\alpha_1} + (tx_1)^{N+1}h_N(tx_1)\Big)\cdots\Big(\sum_{\alpha_1=0}^{N}(tx_n)^{\alpha_k} + (tx_n)^{N+1}h_N(tx_n)\Big)\\
&\quad = \Big(\sum_{\alpha_1=0}^{N}(tx_1)^{\alpha_1}\Big)\cdots\Big(\sum_{\alpha_1=0}^{N}(tx_n)^{\alpha_n}\Big) + t^{N+1}f_N(t,x_1,\ldots,x_n)
\end{aligned}$$

for some function f_N. We don't care about what f_N is exactly, only that there is the factor t^{N+1} in front of it. Multiplying out the finite sums gives

$$\begin{aligned}
&\Big(\sum_{\alpha_1=0}^{N}(tx_1)^{\alpha_1}\Big)\cdots\Big(\sum_{\alpha_1=0}^{N}(tx_n)^{\alpha_n}\Big)\\
&\qquad = \sum_{m=0}^{N}\sum_{\alpha_1+\cdots+\alpha_n=m}(tx_1)^{\alpha_1}\cdots(tx_n)^{\alpha_n} + t^{N+1}g_N(t,x_1,\ldots,x_n)\\
&\qquad\qquad \sum_{m=0}^{N}\Big(\sum_{\alpha_1+\cdots+\alpha_n=m}(x_1)^{\alpha_1}\cdots(x_n)^{\alpha_n}\Big)t^m + t^{N+1}g_N(t,x_1,\ldots,x_n).
\end{aligned}$$

with some other function g_N.

9.2. Fix some $\alpha \in \mathbb{N}_0^n$. The operator ∂_x^α defines a linear map $\Pi_m \to \Pi_{m-|\alpha|}$,

$$\partial_x^\alpha\Big(\sum_{|\beta|=m} q_\beta x^\beta\Big) = \Big(\sum_{|\beta|=m} q_\beta \partial_x^\alpha x^\beta\Big)$$

Write a formula for $\partial_x^\alpha x^\beta$.

9.3. If $p(x) \in \Pi_m$ is given by (1), define

$$p(\partial_x) = \sum_{|\alpha|=m} p_\alpha \partial_x^\alpha.$$

Define $\langle p(x), q(x)\rangle$ for $p, q \in \Pi_m$ by

$$\langle p(x), q(x)\rangle = p(\partial_x)q(x).$$

Show that this defines an inner product on Π_m.

9.4. Suppose $p(x) \in \Pi_\ell$, $q(x) \in \Pi_m$ and $r(x) \in \Pi_{m-\ell}$. Then

$$\langle r(x), p(\partial_x)q(x)\rangle = \langle r(x)p(x), q(x)\rangle$$

9.5. Suppose $p(x) \in \Pi_\ell$ is not 0. Then $p(\partial_x) : \Pi_m \to \Pi_{m-\ell}$ is onto. To prove this, suppose that on the contrary it is not onto. Since the image is a proper subspace, there is a nonzero $r(x) \in \Pi_{m-\ell}$ orthogonal to the image of $p(\partial_x) : \Pi_m \to \Pi_{m-\ell}$:

$$\langle r(x), p(\partial_x)q(x)\rangle = 0 \quad \text{for every } q \in \Pi_m.$$

Using the previous problem argue that this implies $p = 0$.

9.6. Let

$$\Delta = \frac{\partial^2}{x_1^2} + \cdots + \frac{\partial^2}{x_n^2},$$

This Δ is $p(\partial_x)$ with $p(x) = \sum x_j^2$. Show by hand (without using the previous problem) that $\Delta : \Pi_2 \to \Pi_0$ is onto.

9.7. For general $m \geq 2$, find the dimension of the kernel of

$$\Delta : \Pi_m \to \Pi_{m-2}.$$

Use the formula from linear algebra that relates the dimensions of the domain, the kernel, and the image of a linear transformation.