Math 4096, Fall 2019

Topic 8. Tensor and exterior algebras

All vector spaces will be over $\mathbb{R}$. Let $\mathcal{X}$ be a vector space and let

$$X = \{\lambda : \mathcal{X} \to \mathbb{R} : \lambda \text{ is linear}\}.$$

Thus X is the dual space of $\mathcal{X}$. We are not using $\mathcal{X}^*$ as notation because later the $*$ will clutter the notation too much. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be vector spaces. A map $\beta : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is bilinear if for every $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$, the maps

$$x \mapsto \beta(x, y_0), \quad y \mapsto \beta(x_0, y)$$

are linear as maps into $\mathcal{Z}$. The set of all bilinear maps $\beta : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ (i.e., $\mathcal{Z} = \mathbb{R}$) is denoted $X \otimes Y$.

Each of the following three problems is one unit (the first two are long).

8.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces over $\mathbb{R}$, denote by X and Y the respective dual spaces.

(i) Discuss how $X \otimes Y$ can be made into a vector space in a natural way. (Give rigorous definitions of addition and scalar multiplication, then prove some of the properties (not all, please!) that characterize the algebraic operations of a vector space.)

Suppose $\lambda \in X$, $\eta \in Y$. Define $\lambda \otimes \eta : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ by

$$(\lambda \otimes \eta)(x, y) = \lambda(x)\eta(y)$$

(ii) Verify that with this definition, $\lambda \otimes \eta \in X \otimes Y$.

(iii) Let $\pi : X \times Y \to X \otimes Y$ be defined by

$$\pi(\lambda, \eta) = \lambda \otimes \eta.$$

Show that π is bilinear.

(iv) Suppose that $E \subset X$ and $F \subset Y$ are linearly independent sets. Show that

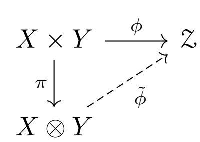
$$\{\lambda \otimes \eta : \lambda \in E,\ \eta \in F\}$$

is an independent subset of $X \otimes Y$.

(v) Assuming that $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional, show that $X \otimes Y$ is finite-dimensional and find its dimension.

(vi) Let $\mathcal{Z}$ be a vector space over $\mathbb{R}$. Let $\phi : X \times Y \to \mathcal{Z}$ be some bilinear map. Show that there is a linear map $\tilde{\phi} : X \otimes Y \to \mathcal{Z}$ such that

$$\phi = \tilde{\phi} \circ \pi.$$



The map π was defined in part (iii). The diagram helps to visualize the problem.

(vii) Show that $\tilde{\phi}$ is unique.

8.2. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be real vector spaces, let X, Y and Z denote the respective dual spaces. Show:

(i) $X \otimes Y$ is isomorphic to $Y \otimes X$.

(ii) $(X \otimes Y) \otimes Z$ is isomophic to $X(\otimes Y \otimes Z)$.

(iii) $(X \otimes Y) \otimes Z$ is isomorphic to the vector space of trilinear maps

$$\tau : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}.$$

In all the above cases there is a canonical isomorphism. Denote X by $X^{\otimes 1}$. If k is an integer ≥ 1 and $X^{\otimes k}$ has been defined, define

$$X^{\otimes(k+1)} = (X^{\otimes k}) \otimes X.$$

(Thus $X^{\otimes 2} = X \otimes X$, $X^{\otimes 3} = (X \otimes X) \otimes X$ and so on; because of (ii), we omit the parentheses when the meaning is clear.) Define also $X^{\otimes 0} = \mathbb{R}$.

(iv) Show that if $k > 2$ then $X^{\otimes k}$ is isomorphic to the space of k-linear maps $\prod_{i=1}^{k} \mathcal{X} \to \mathbb{R}$. (Use induction, the cases $k = 1$ and $k = 2$ being true by the definitions. Play with examples when $k = 3$ of 4 to get a sense of what is happening.)

Thus, for example, if $\lambda_1, \ldots, \lambda_k \in X$, then

$$(x_1, x_2, \ldots, x_k) \mapsto \lambda_1(x_1)\lambda_2(x_2)\ldots\lambda_k(x_k)$$

being k-linear, defines an element of $X^{\otimes k}$. This element is denoted

$$\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_k.$$

Given $\mu \in X^{\otimes k}$ and $\nu \in X^{\otimes \ell}$, define $\mu \otimes \nu$ if $k, \ell \geq 1$ by the formula

$$(\mu \otimes \nu)(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_\ell) = \mu(x_1, x_2, \ldots, x_k)\nu(x_{k+1}, \ldots, x_\ell)$$

(v) Assuming that $\mathcal{X}$ is finite-dimensional, show that $X^{\otimes k}$ is also finite-dimensional, and find its dimension. (The cases $k = 0$ and $k = 1$ are clear, so assume $k \geq 2$. If $e_1, \ldots, e_n$ is a basis of $\mathcal{X}$ and $e_1^*, \ldots, e_n^*$ is the dual basis, consider the elements

$$e_{i_1} \otimes e_{i_2} \otimes e_{i_k}$$

for all possible k-tumles $(i_1, i_2, \ldots, i_k) \in \prod_1^k \{1, 2, \ldots, n\}$ of the integesr between 1 and n.)

When $k = 0$, $\mu \in \mathbb{R}$ (by definition of $X^{\otimes 0}$). In this case, whatever $\ell \geq 0$,

$$\mu \otimes \nu = \mu\nu$$

the product of ν by the number μ. Likewise if $\ell = 0$ and k is arbitrary. This gives bilinear maps

$$(X^{\otimes k}) \times (X^{\otimes \ell}) \to X^{\otimes(k+\ell)}$$

for any $k, \ell \geq 0$ and an associative operation,

$$(\mu \otimes \nu) \otimes \eta = \mu \otimes (\nu \otimes \eta).$$

This will be clear if elements are interpreted as multilinear functions. Putting all together we can give the vector space

$$X^{\otimes *} = \mathbb{R} \oplus X \oplus X^{\otimes 2} \oplus \cdots = \bigoplus_{k=0}^{\infty} X^{\otimes k}$$

a product by setting, when $\mu_k \in X^{\otimes k}$, $k = 0, \ldots, K$, and $\nu_\ell \in X^{\otimes \ell}$, $\ell = 0, \ldots, L$,

$$(\sum_{k=0}^{K} \mu_k) \otimes (\sum_{\ell=0}^{L} \nu_\ell) = \sum_{k,\ell} \mu_k \otimes \nu_\ell.$$

This makes $X^{\otimes *}$ into a unital (has multiplicative unit, $1 \in \mathbb{R}$), associative, non-commutative algebra over $\mathbb{R}$. This is the tensor algebra of X.

8.3. We continue with the notation and concepts of the previous problems. Viewing elements of $X^{\otimes k}$ as k-linear functions, $k \geq 1$ as proved in Part (iv) of Problem 8.2, define, for each $\mu \in X^{\otimes k}$,

$$\operatorname{Alt}(\beta) : \prod_{i=1}^{k} \mathcal{X} \to \mathbb{R}$$

by:

if $x_1, x_2, \ldots, x_k \in \mathcal{X}$, then

$$\operatorname{Alt}(\beta)(x_1, \ldots x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma} \mu(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}).$$

Here S_k is the group of permutations of $\{1, 2, \ldots, k\}$ and $(-1)^{\sigma}$ is the sign of the permutation σ. An element $\mu \in X^{\otimes k}$ is skew-symmetric if when viewed as a k-linear map it satisfies

$$\mu(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = (-1)^{\sigma} \mu(x_1, x_1, \ldots, x_k) \text{ for all } \sigma \in S_k.$$

(i) Show that if $\beta \in X^{\otimes k}$, then $\operatorname{Alt}(\beta)$ is k-linear (hence again an element of $\bigoplus^k X$) and skew-symmetric.

(ii) Show that if $\mu \in X^{\otimes k}$ is skew-symmetric, then $\operatorname{Alt}(\mu) = \mu$.

It is easy to verify that $\operatorname{Alt} : X^{\otimes k} \to X^{\otimes k}$ is linear, so its image, a subset of $X^{\otimes k}$, is actually a vector subspace. Write $\bigwedge^k X$ for that subspace. If $\lambda_1, \ldots, \lambda_k \in X$, then $\operatorname{Alt}(\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_k)$ is denoted

$$\lambda_1 \wedge \lambda_2 \wedge \cdots \wedge \lambda_k.$$

(iii) Suppose λ, η, $\theta \in \bigoplus^1 X$ (that is , they are linear functions $\mathcal{X} \to \mathbb{R}$). Find
  - (a) $\lambda \wedge \eta$ and $\eta \wedge \lambda$ (how are these two bilinear functions related?)
  - (b) $\lambda \wedge \eta \wedge \theta$.

(iv) Let $\lambda_j \in X$, $j = 1 \ldots, k$. Show that $\lambda_1 \wedge \lambda_2 \wedge \cdots \wedge \lambda_k = 0$ if for some i and j with $i \neq j$, $\lambda_i = \lambda_j$

(v) Assuming that $\mathcal{X}$ is finite-dimensional, find $\dim \bigwedge^k X$. What is $\bigwedge^k X$ if $k > \dim \mathcal{X}$?

If $\alpha \in \bigwedge^k X$ and $\beta \in \bigwedge^{\ell} X$, then in particular $\alpha \in X^{\otimes k}$ and $\beta \in X^{\otimes \ell}$, so we can form $\alpha \otimes \beta$. Define

$$\alpha \wedge \beta = \operatorname{Alt}(\alpha \otimes \beta).$$

(vi) Show that $\beta \wedge \alpha = (-1)^{k\ell} \alpha\beta$.

With the above definition of $\wedge$ we can give the vector space

$$\mathbb{R} \oplus X \oplus \bigwedge^2 X \oplus \cdots$$

a product operation that (like the tensor algebra) results in this space being an algebra. This algebra, denoted $\bigwedge^* X$ (the $*$ refers to having all powers, not to some dual relation) is the exterior algebra of X. The direct sum above includes only finitely many terms, according to your solution of Part (v).

8.4. Let $W_1, \ldots, W_n$ be vector spaces over $\mathbb{R}$. Let $W_0 = \mathbb{R}$ and for notational convenience define $W_{\ell} = 0$ for $\ell > n$. Suppose we are given bilinear maps

$$\wedge : W_k \times W_{\ell} \to W_{k+\ell}, \quad W_k \times W_{\ell} \ni (\beta, \gamma) \mapsto \alpha \wedge \beta \in W_{k+\ell}$$

such that for all $\alpha \in W_k$, $\beta \in W_{\ell}$, $\gamma \in W_m$ the following holds:

(1) If $k=0$ (so $\alpha \in \mathbb{R}$) then $\alpha \wedge \beta = \alpha\beta$ (multiplication of the vector β by the number α).
(2) $\alpha \wedge \beta = 0$ if $k+\ell > n$.
(3) $\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha$.
(4) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

Let

$$G = \{1 + \sum_{k=1}^{n} w_k : w_k \in W_k\}$$

Given $v, w \in G$, $v = (1 + \sum_{k=1}^{n} v_k)$ and $w = (1 + \sum_{k=1}^{n} w_k)$, let

$$v \wedge w = \sum_{m=0}^{n} \sum_{k+\ell=m} v_k \wedge w_\ell.$$

Show that G is a group under this operation. (The issue is mostly existence of inverse). Try first with some small n, for instance $n=5$, multiply the two elements paying attention to the rules (especially (1) at this stage), then equate the result to some other element of G, say $u = 1 + \sum_{k=1}^{n} u_k$ ($n=5$ in this case), and figure out the w_k in terms of the components of v and of u.