Topic 7. Estimates, induction

7.1. Let $k \in \mathbb{N}$, define $s(m) : \mathbb{N} \to \mathbb{N}$ by

$$s(m) = \sum_{n=1}^{m} n^k \tag{1}$$

Thus, when $k = 1$, $s_1(m) = 1 + 2 + \cdots + m = \dfrac{m(m+1)}{2}$. Find polynomials $a(x)$ and $b(x)$ with real coefficients such that

$$b_k(m) \leq s_k(m) \leq a_k(m) \text{ for all } m.$$

7.2. Let $p(x) = c_1 X + c_2 x^2 + c_3 x^3$, let $s(m)$ be given by (1) with $k = 2$ (so, sum of squares of integers). The three equations

$$p(1) = s(1), \quad p(2) = s(2), \quad p(3) = s(3)$$

form a 3×3 linear system of equations with c_1, c_2, c_3 as unknowns. Solve the system to get a specific polynomial with which $p(m) = s(m)$ when $m \in \{1, 2, 3\}$. Then use induction to show that in fact $p(m) = s(m)$ for all $m \in \mathbb{N}$.

7.3. The polynomial

$$p(x) = \frac{1}{30}(6x^5 + 15x^4 + 10x^3 - x)$$

satisfies

$$p(m) = \sum_{n=1}^{m} n^4 \tag{2}$$

for $m = 1, \dots, 5$. Show that (2) holds for any $m \in \mathbb{N}$.

7.4. Define

$$t(m) = \sum_{n=1}^{m} (2n-1)^2.$$

For a given m this is the sum of the squares of the odd integers up to $2m - 1$. Show that with

$$p(x) = \frac{1}{3}(4x^3 - x)$$

we have

$$t(m) = p(m)$$

for all m.

7.5. Estimate $n!$ (for $n \in \mathbb{N}$) from above and from below by estimating $\log n!$.

7.6. Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ an infinitely differentiable function (i.e., $f \in C^\infty(I)$). Fix $x_0 \in I$. Show that for any $x \in I$

(1) $f(x) = f(x_0) + \displaystyle\int_0^1 f'\big(x_0 + t(x - x_0)\big)\, dt\, (x - x_0)$;

(2) $f(x) = f(x_0) + f'(x_0)(x - x_0) + \displaystyle\int_0^1 (1-t) f''\big(x_0 + t(x - x_0)\big)\, dt\, (x - x_0)^2$.

(3) Use induction to show that for any $N \in \mathbb{N}$,

$$f(x) = \sum_{\ell=0}^{N} \frac{1}{\ell!} f^{(\ell)}(x_0)(x-x_0)^{\ell} + \frac{1}{N!} \int_0^1 (1-t)^N f^{(N+1)}\big(x_0+t(x-x_0)\big)\, dt\, (x-x_0)^{(N+1)}$$

where $f^{(\ell)}$ is the ℓ-th derivative of f.