Topic 6. Ideals

Let $\mathcal{R}$ be the ring of polynomials with complex coefficients. We will on occasion think of elements of $\mathcal{R}$ as functions $\mathbb{C} \to \mathbb{C}$.

6.1. Show: If p and p_0 are polynomials with $p_0 \neq 0$, then there are polynomials q and r with $r = 0$ or $\deg r < \deg p_0$ such that $p = qp_0 + r$.

6.2. Let $\mathcal{I} \subset \mathcal{R}$ be an ideal. Show that there is $p_0 \in \mathcal{I}$ such that for every $p \in \mathcal{I}$ there is $q \in \mathcal{R}$ such that $p = qp_0$. (This is: every ideal in $\mathcal{R}$ is principal. The polynomial p_0 is a generator of the ideal, and sometimes we writes $\mathcal{I} = (p_0)$. Often one uses as generator the generator whose leading coefficient is 1.)

6.3. Let $a_1, \dots, a_N \in \mathbb{C}$, let $m_1, \dots, m_n \in \mathbb{N}_0$ (non-negative integers). Let

$$\mathcal{I} = \{p \in \mathcal{R} : p \text{ vanishes at } a_j \text{ to order } m_j\}$$

Show that $\mathcal{I}$ is an ideal, and find one of its generators. The case where all m_k are 0 is not excluded. (To say that p vanishes to order m at a is to say that $d^k p/dz^k = 0$ at a when $k < m$.)

6.4. An ideal $\mathcal{I} \subset \mathcal{R}$ is said to be prime if

$$\forall p, q \in \mathcal{R} : pq \in \mathcal{I} \implies p \in \mathcal{I} \text{ or } q \in \mathcal{I}$$

Suppose $\mathcal{I}$ is prime and find a generator p_0. What is the cardinality of $\{z : p_0(z) = 0\}$?

6.5. Let $\mathcal{I}$, $\mathcal{J} \subset \mathcal{R}$ be ideals. Let

$$\mathcal{K} = \{\sum_{\not\infty} p_j q_j : p_j \in \mathcal{I},\ q_j \in \mathcal{J}\}$$

Show that $\mathcal{K}$ is an ideal. This ideal is denoted $\mathcal{I}\mathcal{J}$. Here $\not\infty$ means that the sum is over finitely many j's

6.6. An ideal $\mathcal{I} \subset \mathcal{R}$ is said to be nontrivial if it is different from $\{0\}$ and $\mathcal{R}$. Suppose $\mathcal{I}$, $\mathcal{J} \subset \mathcal{R}$ are nontrivial. Show that $\mathcal{I}\mathcal{J}$ is not prime.

6.7. An ideal $\mathcal{M} \subset \mathcal{R}$ is said to be maximal if it is nontrivial and has the property

$$\text{if } \mathcal{I} \subset \mathcal{R} \text{ is an ideal and } \mathcal{M} \subset \mathcal{I}, \text{ then } \mathcal{I} = \mathcal{R}.$$

Find a maximal ideal of $\mathcal{R}$ and an ideal which is not maximal.

6.8. If $\mathcal{I} \subset \mathcal{R}$ is an ideal that contains the constant polynomials, then $\mathcal{I} = \mathcal{R}$.

6.9. Let $\mathcal{I}_1$, $\mathcal{I}_2 \subset \mathcal{R}$ be ideals. Let

$$\mathcal{K} = \{a_1 p_1 + a_2 p_2 : a_1, a_2 \in \mathcal{R},\ p_1 \in \mathcal{I}_1,\ p_2 \in \mathcal{I}_2\}$$

Show that $\mathcal{K}$ is an ideal. This is the ideal generated by $\mathcal{I}_1$ and $\mathcal{I}_2$, denoted $\mathcal{I} + \mathcal{J}$.

6.10. Let p_1, $p_2 \in \mathcal{R}$ be nonzero polynomials. Show there is a unique polynomial p with leading coefficient 1 such that $p_1 = q_1 p$ and $p_2 = q_2 p$ for some (unique) polynomials p_1, p_2.

6.11. Let p_1, $p_2 \in \mathcal{R}$ be nonzero polynomials without common zeros. Show there are polynomials a_1, a_2 such that

$$a_1 p_1 + a_2 p_2 = 1.$$