Topic 5. Directed sets, nets, Riemann sums

1. Directed sets

A directed set is a set $\mathcal{D}$ together with a relation $\leq$ such that:

(1) for all $p \in \mathcal{D}$: $p \leq p$;
(2) for all $p, q, r \in \mathcal{D}$, if $p \leq q$ and $q \leq r$, then $p \leq r$
(3) for all $p, q \in \mathcal{D}$ there is $r \in D$ such that $p \leq r$ and $q \leq r$.

Both $\mathbb{N}$ and $\mathbb{Z}$ with the usual ordering are directed sets. Two further examples, the second more to the point than the first, are as follows.

5.1. Let S be some set, let $\mathcal{D} = \{P : P \subset S\}$, the family of subsets of S, with $\leq$ defined by

$$\forall P, Q \in \mathcal{D} : P \leq Q \iff P \subseteq Q.$$

Show that $\mathcal{D}$ with the ordering just defined is a directed set.

5.2. Let $a, b \in \mathbb{R}$ with $a < b$, let $I = [a, b]$. Let $\mathcal{P}$ be the set of finite partitions of I: the set whose elements are finite subsets $\{t_0, t_1, \ldots, t_n\}$ of I containing a and b, with the labeling such that $a = t_0 < t_1 < \cdots < t_n = b$. Define the relation $\leq$ for elements of $\mathcal{P}$ by:

$$\forall P, Q \in \mathcal{P} : P \leq Q \iff P \subseteq Q.$$

Show that $\mathcal{P}$ with the ordering just defined is a directed set.

2. Nets

A net is a generalization of a sequence. A sequence $\{x_n\}_{n=1}^{\infty}$ in a set A can be viewed as the function $\mathbb{N} \to A$ defined by the rule $n \mapsto x_n$. In the case of a net, $\mathbb{N}$ is replaced by a directed set $\mathcal{D}$: A net in A is a function $\mathcal{D} \to A$. Such a function assigns to each element p of $\mathcal{D}$ an element x_p of A. As with sequences we may write it as $\{x_p\}_{p \in \mathcal{D}}$.

Convergence of a net of, for instance, real numbers, is defined by following the pattern of the definition of convergence in the case of a sequence:

> Let $\mathcal{D}$ be a directed set. A net $\{x_p\}_{p \in \mathcal{D}}$ in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$ if
>
> for all $\varepsilon > 0$ there is $p_0 \in \mathcal{D}$ such that $p \in \mathcal{D}$ and $p_0 \leq p \implies |x - x_p| < \varepsilon$.

If $\{x_p\}_{p \in \mathcal{D}}$ converges to x we write $x_p \to x$ (with $p \in \mathcal{D}$ left implicit), or else $\lim_{p \in \mathcal{P}} x_p = x$.

5.3. Let $\mathcal{D}$ be a directed set and $\{x_p\}_{p \in \mathcal{D}}$ a net of real numbers with the properties:

(1) $p, q \in \mathcal{D}$ and $p \leq q$ implies $x_p \leq x_q$ (we say that $\{x_p\}_{p \in \mathcal{D}}$ is monotonically increasing), and
(2) there is $b \in R$ such that for all $p \in \mathcal{D}$, $x_p \leq b$ (we say that the net s bounded from above).

Show that $\{x_p\}_{p \in \mathcal{D}}$ converges. Hint: consider the *set* $X = \{x_p : p \in \mathcal{D}\}$, let x be its supremum. Show that x is the limit. Imitate the proof for the analogous statement for sequences.

3. Riemann sums

Let I be as in Problem 5.2 above, $\mathcal{P}$ the set of finite partitions of I with the order defined in that problem, and $f : I \to \mathbb{R}$ a bounded function: there is β such that $|f(x)| \le \beta$ for all $x \in I$. Given a partition $P \in \mathcal{P}$, $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, let $M_i = \sup\{f(x) : t_{i-1} \le x \le t_i\}$, define R_P as

$$R_P = \sum_{i=1}^{n} \sup M_i(t_i - t_{i-1}).$$

This assignment of an element of $\mathbb{R}$ to each element of $\mathcal{P}$ is by definition a net. The following sequence of problems leads to the proof that $\{R_P\}_{P\in\mathcal{P}}$ converges.

5.4. Let $P \in \mathcal{P}$ be some partition, let $s \in I$ be a point not in P, and let Q be the partition of I obtained by adding s to P. Thus if $s \in (t_{i_0-1}, t_{i_0})$, then Q is

$$Q = \{a = t_0 < t_1 \cdots < t_{i_0-1} < s < t_{i_0} < \cdots < t_n = b\}$$

Show that

$$R_P \ge R_Q.$$

5.5. Let $P, Q \in \mathcal{P}$ with $P \le Q$. Show that $R_P \ge R_Q$, that is, show that $\{R_P\}_{P\in\mathcal{P}}$ is monotonically decreasing. Hint: use the previous problem noting that Q is obtained from P by adding to P the points of $Q \backslash P$, one at a time.

5.6. Show that $\{R_P\}_{P\in\mathcal{P}}$ converges.