Topic 4. A bound that implies injectivity

4.1. Say that a function f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

has property P_t if $\sum_{n=2}^{\infty} n|a_n| \le t$. Prove:

(a) If P_t holds for some $t < \infty$, then f is continuous on the closed unit disc, i.e., on $\{z \in \mathbb{C} : |z| \le 1\}$.

(b) If P_1 holds, then f is one-to-one on the closed unit disc.

This is part of the problem submitted by W. Rudin, Problem E3325, Amer. Math. Monthly **96** (1989), no. 5, p. 445. Part (a) is an application of convergence theorems in Advanced Calculus. For part (b) use the following sequence of problems.

4.2. Let $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ have norm ≤ 1. Then $\|\vec{v} + \vec{w}\| \le 2$ by the triangle inequality. Show that $\|\vec{v}+\vec{w}\| = 2$ if and only if $\vec{v} = \vec{w}$ and $\|\vec{v}\| = 1$. The norm is the Euclidean norm: $\|v\|^2 = \vec{v} \cdot \vec{v}$.

4.3. Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ have norm ≤ 1. Then $\|\vec{v}_1 + \cdots + \vec{v}_k\| \le k$. Show that $\|\vec{v}_1 + \cdots + \vec{v}_k\| = k$ if and only if $\vec{v}_1 = \vec{v}_j$ for all j and all vectors have norm 1. The norm is again the euclidean norm.

4.4. Let z, $w \in \mathbb{C}$ with $|z|$, $|w| \le 1$. Show that

$$\big|\sum_{\ell=0}^{n-1} z^\ell w^{n-\ell-1}\big| \le n$$

with equality if and only if $z = w$ and $|z| = 1$.

4.5. Let m be an integer ≥ 2. Let $a_n \in \mathbb{C}$, $n = 2, 3, \dots, m$. Suppose $\sum_{n=2}^{m} n|a_n| = t$. Show that

$$\big|\sum_{n=2}^{m} a_n \big(\sum_{\ell=0}^{n-1} z^{n-\ell-1} w^\ell\big)\big| \le t \text{ if } |z|,\ |w| \le 1$$

and that if equality holds and some $a_n \ne 0$, then $z = w$.

4.6. With the notation and assumptions of the previous problem, let

$$h(z) = \sum_{n=2}^{m} a_n z^n, \quad z \in \mathbb{C}.$$

Show that

$$h(z) - h(w) = (z - w) \sum_{n=2}^{m} a_n \big(\sum_{\ell=0}^{n-1} z^{n-\ell-1} w^\ell\big)$$

4.7. Continuing with the notation of the previous two problem, let $f(z) = z + h(z)$ and assume $t \le 1$. Suppose $|z|, |w| \le 1$ and $f(z) = f(w)$. Show that then $z = w$. Hints: $f(z) - f(w)$ has $(z - w)$ as a factor. Analyze whether $z \ne w$ is possible in the presence of the fact that $f(z) - f(w) = 0$. (Use Problems 4.4–4.6.) Now solve part (b) of Problem 4.1.