Topic 2. Miscellaneous problems in geometry

2.1. Let $C = \{(x, y, z) \in \mathbb{R}^3 : x^1 + y^2 + z^2 = 1,\ z = 0\}$. This is the unite circle in the xy plane viewed as a subset of $\mathbb{R}^3$. Find an algebraic formula for

$$\mathrm{dist}((x, y, z), C) = \min_{(x', y', z' \in C} \|(x, y, z) - (x', y', z')\|.$$

2.2. Consider the fluctuations in distance between an observer located at (h, k) and an object orbiting along the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a > b > 0$. If the observer is at or near the origin, then on each orbit the distance will have two relative maxima and two relative minima. If the observer is far from the origin, there will be only one of each. Determine the equation of the curve that marks the boundary between the two cases. (J. Delany, Problem E 3317, Amer. Math. Monthly **96** (1989), no. 3, p. 254.)

2.3. Let $\mathcal{T} \subset \mathbb{R}^3$ be the the surface of rotation constructed as follows. Let R and r be positive numbers with $R > r$. Let $\mathcal{C}$ be the circle on the xz plane with center $(R, 0, 0)$ and radius r. Then $\mathcal{T}$ is obtained by rotating $\mathcal{C}$ around the z axis. Find a polynomial equation, i.e., $p(x, y, z) = 0$, for $\mathcal{T}$.

2.4. Let the distance between points p and q of the torus of Problem 2.3 be given by the shortest length of a differentiable curve in $\mathcal{T}$ joining these points. A curve that realizes that minimum is a geodesic. Find a differential equation (a system of ordinary differential equations) for the geodesics, then use Mathematica, Maple or MATLAB to find numerial solutions for various starting points and velocities (with, say, $r = 1$ and $R = 2$. It may be helpful to parametrized $\mathcal{T}$ by

$$F(\phi, \theta) = (\cos(\theta)(R + r\cos(\phi)), \sin(\theta)(R + r\cos(\phi)), R + r\sin(\phi)]), \quad \phi, \theta \in [0, 2\pi].$$