Topic 13. Differentials, grad, curl, div

1. Differential forms.

All functions will be infinitely differentiable defined on some open subset U of $\mathbb{R}^3$, for instance

$$U = \{(x, y, z) : x^2 + y^2 + z^2 < 1\}.$$

We will deal with vector spaces Ω^0, Ω^1, Ω^2, Ω^3 defined as follows:

Ω^0*: The vector space consisting of all (infinitely differentiable) functions $f : U \to \mathbb{R}$.*

Ω^1*: The vector space of differential 1-forms. These are expressions $f\,dx + g\,dy + h\,dz$, where $f, g, h : U \to \mathbb{R}$ are functions.*

Ω^2*: All differential 2-forms, expressions $f\,dx \wedge dy + g\,dx \wedge dz + h\,dy \wedge dz$, where again $f, g, h : U \to \mathbb{R}$ are functions.*

Ω^3*: Differential 3-forms, expressions $f\,dx \wedge dy \wedge dz$, where $f : U \to \mathbb{R}$ is a function.*

For a given k-form α, the number k is its degree. If $\alpha \in \Omega^k$ we say that α is a k-form. Collectively all these elementnes are differential forms. The symbol $\wedge$ is read wedge. *It represents the wedge product.*

2. Algebraic rules.

The rules for manipulating these are as follows. You can add functions (0-differential forms) and multiply them the usual way. You can add forms of the same degree: just collect terms if and when convenient. You can multiply functions times 1-forms just distributing the product over the sum:

$$p(f\,dx + g\,dy + h\,dz) = pf\,dx + pg\,dy + ph\,dz$$

Likewise functions times 2-forms or 3-forms (in the latter case trivially).

More generally you can multiply forms of some degree times forms of the same or another degree freely, writing for instance $\alpha \wedge \beta$ for the product of α and β. The rules for this kind of multiplication allow for distributivity of product over sum, as in

$$(\alpha_1 + \alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \alpha_2 \wedge \beta.$$

However, $\wedge$ has the property that if α and β are 1-forms, then $\alpha \wedge \beta = -\beta \wedge \alpha$. For example, $dx \wedge dy = -dy \wedge dx$. In general, $\wedge$ is distributive and associative, but not commutative

3. Differentiation

There are operators from Ω^k to Ω^{k+1}, all denoted d, as follows.

(0) The differential of a 0-form $f \in \Omega^0$ (a function) is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

(1) The differential of a 1-form $\alpha = f\,dx + g\,dy + h\,dz$ is computed using the differentials of the coefficients and the wedge product:

$$d\alpha = df \wedge dx + dg \wedge dy + dh \wedge dz$$

(2) Similarly, if $\alpha = f\,dx \wedge dy + g\,dx \wedge dz + h\,dy \wedge dz$ (a 2-form):

$$d\alpha = df \wedge dx \wedge dy + dg \wedge dx \wedge dz + dh \wedge dy \wedge dz$$

(3) Finally, if $\alpha = f dx \wedge dy \wedge dz$, then $d\alpha = 0$.

13.1. Computations using the differential.

(i) Let $\alpha = f\,dx + g\,dy + h\,dz$, an element of Ω^1. Compute

$$d\alpha - \Big[\Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)dx\wedge dy + \Big(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\Big)dx\wedge dz + \Big(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\Big)dy\wedge dz\Big]$$

(ii) Let $\alpha = f\,dx\wedge dy + g\,dx\wedge dz + h\,dy\wedge dz$. Compute $d\alpha$. The result is a function times $dx\wedge dy\wedge dz$.

(iii) Let f be a function, α a differential 1-from. Compute $d(f\alpha)$.

(iv) For each of $k = 0, 1, 2, 3$, let $\alpha \in \Omega^k$. Compute $d(d\alpha)$.

(v) Let α and β be 1-forms. Compute

$$d(\alpha\wedge\beta) - [(d\alpha)\wedge\beta - \alpha\wedge d\beta]$$

(vi) Let $\alpha \in \Omega^1$, $\beta \in \Omega^2$. Compute

$$\alpha\wedge\beta - \beta\wedge\alpha.$$

4. $\star$ operator.

In the next problems we use an operator, denoted $\star$ (the Hodge star operator, called star for short), that takes a k-form and produces a $(3-k)$-form. It has the following properties:

(a) If α and β are differential forms of the same degree and f and g are functions, then $\star(f\alpha + g\beta) = f\star\alpha + g\star\beta$.

(b) Furthermore,

$$\begin{cases} \star(1) = dx\wedge dy\wedge dz \\ \star dx = dy\wedge dz,\ \star dy = dz\wedge dx,\ \star dz = dx\wedge dy \\ \star(dy\wedge dz) = dx,\ \star(dz\wedge dx) = dy,\ \star(dx\wedge dy) = dz \\ \star(dx\wedge dy\wedge dz) = 1. \end{cases}$$

The 1 should be viewed as the constant function 1.

13.2. These problems will give you some practice on how $\star$ works:

(i) As a warm up exercise, compute $\star\alpha$ forα an arbitrary k-form (do $k = 0, 1, 2, 3$).

(ii) Let α be a k-form. Compute $\star d\alpha$ and $d\star\alpha$.

(iii) Let f be a function. Compute $\star d\star df$.

5. From vector fields to 1-forms and back.

In the following problem we pass from vector fields to differential 1 forms and back via the following procedure: Given the vector field $V = f\boldsymbol{i} + g\boldsymbol{j} + h\boldsymbol{k}$, construct a differential 1-form by the rule

$$F(f\boldsymbol{i} + g\boldsymbol{j} + h\boldsymbol{k}) = f\,dx + g\,dy + h\,dz.$$

The inverse procedure produces a vector field starting with a 1-form.

13.3. Grad, curl, div.

(i) Let f be a function. Compute $F^{-1}(df)$.

(ii) Let $V = f\boldsymbol{i} + g\boldsymbol{j} + h\boldsymbol{k}$. Compute

$$F^{-1}\star dF(V)$$

explicitly. Note that $F(V)$ is a 1-form, $dF(V)$ is a 2-form, $\star dF(V)$ is again a 1-form, and $F^{-1}\star dF(V)$ is vector field.

(iii) Let $V = f\boldsymbol{i} + g\boldsymbol{j} + h\boldsymbol{k}$. Compute

$$\star d\star F(V).$$

(iv) Write formulas for grad, curl, and div using F, $\star$, and d.