## Topic 12. Characteristic and minimal polynomials

*Let $A$ be an $n \times n$ matrix. Depending on the problem, the entries may be in the field of rational, real, or complex numbers; we will write $\mathbb{F}$ for the field. The characteristic polynomial of $A$ is*

$$p(x) = \det(xI - A)$$

*where $I$ is the identity matrix. Thus*

$$p_A(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0.$$

*General fact: replacing $A$ for $x$ in $p(x)$ yields $0$. That is, the computation*

$$p_A(A) = A^n + p_{n-1}A^{n-1} + \cdots + p_1 A + p_0 I$$

*gives $0$. Note that the constant term in $p_A(x)$ is viewed as $p_0 x^0$ and that $A^0 = I$ by definition if $A \neq 0$.*

**12.1.**

($i$) Let $A$ be an $n \times n$ matrix with coefficients in the field $\mathbb{F}$, let

$$\mathcal{I}_A = \{q \in \mathbb{F}[x] : q(A) = 0\}$$

Show that $\mathcal{I}_A$ is an ideal in $\mathbb{F}[x]$.

*Since every ideal in $\mathbb{F}[x]$ is principal, there is a polynomial that generates $\mathcal{I}_A$. Let $m_A(x)$ be the monic generator (the generator whose leading coefficient is $1$).*

($ii$) Why does the minimal polynomial of a matrix $A$ divide the characteristic polynomial of the matrix?

($iii$) Let $a$, $b \in \mathbb{F}$, let

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{bmatrix}$$

Compute the characteristic polynomial of $A$, call it $p_A(X)$, then compute $p_A(A)$.

($iv$) Continuing with $A$ as in Part ($iii$), compute

$$\dim\ker(A - aI) \text{ and } \dim\ker(A - bI)$$

($v$) Still with $A$ as in Part ($iii$), compute the minimal polynomial of $A$.

($vi$) Again with $a$, $b \in \mathbb{F}$, let

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{bmatrix}$$

This matrix is slightly different from that of Part ($iii$). Compute the characteristic polynomial of $A$, verify that its characteristic polynomial, $p_a(x)$, annihilates $A$, then find the minimal polynomial of $A$, and finally, find bases of

$$\ker(A - aI) \text{ and } \ker(A - bI).$$

**12.2.** Let

$$A = \begin{bmatrix} 3 & 2 & 0 \\ -5 & -3 & 0 \\ 10 & 4 & -1 \end{bmatrix}$$

Compute $p_A(x)$ and $m_A(x)$. Hint: The number $-1$ is one a root of $p_A(x)$ and $m_A(x)$ divides $p_A(x)$.

**12.3.** Show that any polynomial over $\mathbb{C}$ is the characteristic polynomial of some matrix with complex entries.