Topic 10. Euler's partition function

Euler's p-function is the function $\mathbb{N} \to \mathbb{N}$ such that $p(n)$ is the number of ways n can be written as a sum of positive integers. Here you will build up rigorously the generating function of $p(n)$, the function

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k}. \tag{1}$$

The first problem deals with infinite products (similarly to series, but products) and the definition of (1) particular , the one following it with products of series, and the last ones with a Taylor series whose coefficients turn out to be the $p(n)$.

Note: *Each problem is one unit, cannot be split into several papers.*

10.1. Infinite products

(i) Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers. Define

$$p_m = \prod_{k=1}^{m} a_k. \tag{2}$$

Show that if $\lim\limits_{m\to\infty} p_m$ exists and is nonzero, then $\lim\limits_{m\to\infty} a_m = 1$.

(ii) Suppose now that $\{a_k\}_{k=1}^{\infty}$ be a sequence of real positive numbers. Define p_m as in (2). Show

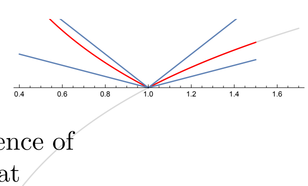
(a) If $p_m \to p$ and $p > 0$, then the series $\sum\limits_{k=1}^{\infty} \log a_k$ converges to $\log p$.

(b) If $\sum\limits_{k=1}^{\infty} \log a_k$ converges, then $\{p_m\}_{m=1}^{\infty}$ converges and the limit is positive.

As with series, if the sequence whose terms are the p_m in (2) converges, we write $\prod_{n=1}^{\infty} a_n$ for the limit. The products in (2) are the partial products

(iii) The natural logarithm satisfies the inequalities

$$\frac{1}{2}|x-1| \leq |\log x| \leq \frac{3}{2}|x-1| \quad \text{if} \quad \frac{1}{2} < x < \frac{3}{2}.$$



as illustrated by the graph. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers and use these inequalities to show that

$$\sum_{k=1}^{\infty} |\log a_k| \text{ converges} \iff \sum_{k=1}^{\infty} |a_k - 1| \text{ converges.}$$

Note that the convergence of either series implies that $1/2 < a_k < 3/2$ once k is large enough.

(iv) Let $|x| < 1$ be fixed. Show that

$$g(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

converges.

We now want to multiply out the first N factors of g, then use

$$\frac{1}{1-x^k} = \sum_{\ell=0}^{\infty} x^{k\ell}$$

to write the resulting series as the sum of two terms, $\alpha_N(x) + x^{N+1}\beta_N(x)$, with α_N collecting all terms with powers $\leq N$, the other being just the remainder. We will then see that the coefficients of α_N are the $p(n)$ which takes us closer to our goal. But we will also need to show that the rest of the factors of g contribute a term of the form $1 + x^{N+1}\gamma_N(x)$, so that

$$\begin{aligned} g(x) &= \big(\alpha_N(x) + x^{N+1}\beta_N(x)\big)\big(1 + x^{N+1}\gamma_N(x)\big) \\ &= \alpha_N(x) + x^{N+1}\big(\beta_N(x)(1 + x^{N+1}\gamma_N(x)) + \alpha_N(x)\gamma_N(x)\big) \end{aligned}$$

which will imply that these other factors don't affect the coefficients of α_N.

10.2. Multiplication of series.

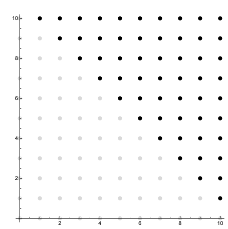
(i) Let $\{a_k\}_{k=0}^{\infty}$, $\{b_\ell\}_{\ell=0}^{\infty}$ be such that both series

$$\sum_{k=0}^{\infty} |a_k|, \quad \sum_{\ell=0}^{\infty} |b_\ell|$$

converge. Let $c_m = \displaystyle\sum_{k+\ell=m} a_k b_\ell$. Show that

$$\sum_{m=0}^{\infty} |c_m|$$

converges and that



$$\sum_{k=0}^{\infty} a_k \sum_{\ell=0}^{\infty} b_\ell = \sum_{m=0}^{\infty} c_m \tag{3}$$

To do this find first

$$\Big(\sum_{k=0}^{N} a_k\Big)\Big(\sum_{\ell=0}^{N} b_\ell\Big) - \sum_{m=0}^{N} c_m$$

then show that the absolute value of this difference tends to 0 as N tends to ∞. This will prove both statements. The darker dots in the graph correspond to the indices appearing in the above expression when $N = 10$. The series on the right in (3) is the Cauchy product of the two series in the left.

(ii) Consider now N sequences $\{a_{k_1}^{(1)}\}_{k_1=0}^{\infty}, \{a_{k_2}^{(2)}\}_{k_2=0}^{\infty}, \ldots, \{a_{k_N}^{(N)}\}_{k_N=0}^{\infty}$ such that

$$\sum_{k_j=0}^{\infty} |a_{k_j}^{(j)}|$$

converges. Show that

$$\Big(\sum_{k_1=0}^{\infty} a_{k_1}^{(1)}\Big) \cdots \Big(\sum_{k_N=0}^{\infty} a_{k^N}^{(N)}\Big) = \sum_{\ell=0}^{\infty} c_\ell,$$

where

$$c_\ell = \sum_{k_1+\cdots+k_N=\ell} a_{k_1}^{(1)} \cdots a_{k_N}^{(N)}.$$

This probably lends itself to a proof by induction (*iii*) Fix $N \in \mathbb{N}$, suppose $|x| < 1$. Show that in

$$\prod_{k=0}^{N} \frac{1}{1-x^k} = \Big(\sum_{\ell_1=0}^{\infty} x^{\ell_1}\Big) \cdots \Big(\sum_{\ell_N=0}^{\infty} x^{N\ell_N}\Big) = \sum_{n=0}^{\infty} d_{n,N} x^n \tag{4}$$

the coefficient $d_{n,N}$ is the cardinality of

$$S_{N,n} = \{(k_1, \dots, k_N) \in \mathbb{N}_0^N : k_1 + 2k_2 + \cdots + Nk_N = n\}.$$

This is the number of ways that n can be written as the sum of positive integers, none of which is bigger than N. This does not preclude that, for instance, N appears multiple times, as in $10 = 5 + 5$ when $N = 5$.

(*iv*) Show that

$$\#S_{N,n} = \#S_{N+1,n} \qquad \text{if } N \geq n.$$

Split the series on the right in (4) *as*

$$\sum_{n=0}^{\infty} d_{n,N} x^n = \sum_{n=0}^{N} d_{n,N} + \sum_{n=N+1}^{\infty} d_{n,N} x^n,$$

define

$$\alpha_N(x) = \sum_{n=0}^{N} d_{n,N} x^n, \qquad \beta_N(x) = x^{N+1} \sum_{n=N+1}^{\infty} d_{n,N} x^{n-N-1}$$

Because of (iv) the numbers $d_{n,N}$ depend only in n, not on N, as soon as $N > n$. So we can define d_n to be $d_{n,n}$ and so

$$\alpha_N(x) = \sum_{n=0}^{N} d_n x^n$$

which says that $\alpha_N(x)$ depends on N only through the upper limit of the sum.

If $(k_1, \dots, k_N) \in S_{N,n}$, then n is the sum of k_1 1's, k_2 2's,..., k_N N's. For example, k can be written in 5 ways: $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, and 4. This can be organized as

k_1	k_2	k_3	k_4
4	0	0	0
2	1	0	0
1	0	1	0
0	2	0	0
0	0	0	1

Another example: the set $S_{N,5}$ with $N \geq$ consists of the points

$$(5,0,0,0,0), (3,1,0,0,0), (1,2,0,0,0),$$
$$(2,0,1,0,0), (0,1,1,0,0), (1,0,0,1,0), (0,0,0,0,1).$$

(*v*) Using

$$\frac{1}{1-x^k} = 1 + \frac{x^k}{1-x^k}$$

and that for some function $h(s)$ defined for $|s| < 1$ (which also happens to be given by a convergent power series)

$$\log(1+s) = sh(s), \quad h(s) = \int_0^1 \frac{1}{1+ts}\, dt,$$

show that

$$\log(\prod_{k=N+1}^{\infty} \frac{1}{1-x^k}) = \sum_{k=N+1}^{\infty} \frac{x^k}{1-x^k} h\big(\frac{x^k}{1-x^k}\big) \tag{5}$$

if $|x| < 1$. The issue is to show convergence of the series for such x.

(vi) The series in (5) factors as

$$x^{N+1} \sum_{k=N+1}^{\infty} \frac{x^{k-N-1}}{1-x^k} h\big(\frac{x^k}{1-x^k}\big) = x^{N+1} f_N(x).$$

Show that

$$e^{x^{n+1} f_n(x)} = 1 + x^{N+1}\gamma_N(x)$$

for some function $\gamma(x)$.

10.3. Now we have

$$g(x) = \alpha_N(x) + x^{N+1}\big(\beta_N(x)(1 + x^{N+1}\gamma_N(x)) + \alpha_N(x)\gamma_N(x)\big)$$

for every N with differentiable functions α_N, $\beta_N(x)$, $\gamma_N(x)$ (which in fact are given by convergent power series). The Taylor expansion of $g(x)$ at $x = 0$ up to order N is the polynomial α_N. Show that the Taylor expanson of $g(x)$ based at $x = 0$ is

$$\sum_{n=0}^{\infty} d_n x^n.$$

One possible approach is to prove that there is $C > 0$ such that

$$|\big(\beta_N(x)(1 + x^{N+1}\gamma_N(x)) + \alpha_N(x)\gamma_N(x)\big)| \leq C \quad \text{if} |x| < 1/2$$

Then the factor x^{N+1} will give that

$$|x^{N+1}\big(\beta_N(x)(1 + x^{N+1}\gamma_N(x)) + \alpha_N(x)\gamma_N(x)| \to 0 \quad \text{as } N \to \infty.$$