Topic 1. Miscellaneous problems in analysis

1.1. Consider the set $C^{0,1/2}([0,1])$ consisting of all real-valued functions on $[0,1]$ such that

$$\|f\| = |f(0)| + \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^{1/2}} : x \neq y\right\}$$

is finite. This is a vector space over $\mathbb{R}$. Show that $\|f\|$ is a norm and that $C^{0,1/2}([0,1])$ with that norm is complete.

1.2. Let $f \in C(\mathbb{R})$. Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ defined by

$$f_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(x + \frac{k}{n})$$

converges uniformly on each finite interval $[a,b]$.

1.3. Show that

$$\lim_{n\to\infty} \sum_{k=0}^{2n} \frac{k}{k^2+n^2} = \frac{1}{2}\log 5.$$

1.4. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Fix $x_1, \ldots, x_n \in [a,b]$. Show that there is $z \in [a,b]$ such that

$$f(z) = \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

1.5. Let $\mathcal{R}$ be a finite-dimensional vector subspace of $C(\mathbb{R})$ that is closed under multiplication:

$$f, g \in \mathcal{R} \implies fg \in \mathcal{R}.$$

Show that $\mathcal{R}$ consists of only constant functions.

1.6. This "problem" was removed.

1.7. Consider the fluctuations in distance between an observer located at (h,k) and an object orbiting along the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a > b > 0$. If the observer is at or near the origin, then on each orbit the distance will have two relative maxima and two relative minima. If the observer is far from the origin, there will be only one of each. Determine the equation of the curve that marks the boundary between the two cases. [J. Delany, Problem E3317, Amer. Math. Monthly **96** (1989), no. 3, p. 254.]

1.8. Let $f_n(x) = n\sin\left(\frac{x}{n}\right)$. Prove that:

(a) f_n converges uniformly on any finite interval.
Hint: make good use of $\sin x \approx x$ when x is small.

(b) f_n does not converge uniformly on $\mathbb{R}$.

1.9. Let $h : \mathbb{R} \to \mathbb{R}$ be differentiable at 0. Define $f_n : \mathbb{R} \to \mathbb{R}$ for $n \in \mathbb{N}$ by

$$f_n(x) = nh(x/n).$$

Assuming $h(0) = 0$, show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on any finite interval.