1. Compute the Jacobian matrix of the following functions:
   (1) \( f(x_1, x_2) = (x_1^2 + 2x_2, x_1 - x_2) \)
   (2) \( f(x_1, x_2, x_3) = (x_1x_2x_3, x_2x_3, x_3) \)

2. For each of the two functions in the previous problem, determine for which \( x_0 = (x_0^1, x_0^2) \) or \( (x_0^1, x_0^2, x_0^3) \) will the Inverse Function Theorem guarantee the existence of an inverse to the function in a neighborhood of \( x_0 \).

3. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x_1, x_2) = x_1x_2 - x_1^2 \). Pick an arbitrary number \( c \neq 0 \). Use the implicit function theorem to show that the level set
   \[ L = \{(x_1, x_2) : f(x_1, x_2) = c\} \]

   has the following property:
   Every \( p_0 \in L \) has a neighborhood \( W \subset \mathbb{R}^2 \) such that \( L \cap W \) is either
   the graph of a function of \( x_1 \) or a function of \( x_2 \).

   (Show that if \( f(p_0) = c \), then one of
   \[ \frac{\partial f}{\partial x_1} \bigg|_{p_0}, \quad \frac{\partial f}{\partial x_2} \bigg|_{p_0} \]
   is nonzero [use \( c \neq 0 \)], then use the Implicit Function Theorem.)

4. Let \( f \) be the function in the previous problem, pick \( c \neq 0 \). The expression \( f(x_1, x_2) - c \) can be viewed as a polynomial in \( x_1 \) with coefficients depending on \( x_2 \), or a polynomial in \( x_2 \) with coefficients depending on \( x_1 \). either way it is a polynomial of degree at most 2, so the equation \( f(x_1, x_2) = c \) can be solved explicitly. Combine this with your analysis in Problem 3 to get \( L \) explicitly near any of its points.
Theorem (Inverse Function Theorem). Let $\Omega \subset \mathbb{R}^n$ be an open set, $x_0$ a point in $\Omega$, $f : \Omega \to \mathbb{R}^n$ a function of class $C^k$ with $k \geq 1$. Assume that the Jacobian matrix of $f$ at $x_0$ is invertible. Then there are neighborhoods $U \subset \Omega$ of $x_0$ and $V \subset \mathbb{R}^n$ of $f(x_0)$ and a $C^k$ function $g : V \to U$ such that
\[ f \circ g = I_V, \quad g \circ f \big|_U = I_U. \]

Theorem (Implicit Function Theorem). Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set, $(x_0, y_0) \in \Omega$, $F : \Omega \to \mathbb{R}^m$ a function of class $C^k$ with $k \geq 1$, let $c = F(x_0, y_0)$. Assume that the Jacobian matrix of $y \mapsto F(x_0, y)$ at $y_0$ is invertible. Then there are neighborhoods $U \subset \mathbb{R}^n$ of $x_0$ and $V \subset \mathbb{R}^m$ of $y_0$ with $U \times V \subset \Omega$ and a $C^k$ function $f : U \to V$ such that
\begin{enumerate}
  \item $F(x, f(x)) = c$ for all $x \in U$.
  \item If $(x, y) \in U \times V$ and $F(x, y) = 0$, then $y = f(x)$.
\end{enumerate}