Set-up: Let \( M \) be a 2-manifold in \( \mathbb{R}^3 \) parametrized by \( \Phi : U \to W, U \subset \mathbb{R}^2 \) and \( W \subset M \) open sets. The function

\[
\Phi(\xi, \eta) = (x(\xi, \eta), y(\xi, \eta), z(\xi, \eta)),
\]

as a function into \( \mathbb{R}^3 \), is assumed to be smooth, invertible, with smooth inverse \( \Psi : W \to U \). Let

\[
\vec{\Xi}_0(\xi, \eta) = \left( \frac{\partial x}{\partial \xi}(\xi, \eta), \frac{\partial y}{\partial \xi}(\xi, \eta), \frac{\partial z}{\partial \xi}(\xi, \eta) \right)
\]

and

\[
\vec{H}_0(\xi, \eta) = \left( \frac{\partial x}{\partial \eta}(\xi, \eta), \frac{\partial y}{\partial \eta}(\xi, \eta), \frac{\partial z}{\partial \eta}(\xi, \eta) \right).
\]

These functions of \((\xi, \eta)\) are vector fields on \( M \) along \( \Phi \) in the “old” notation, which we will use in this homework. If the orientation of \( M \) is given by

\[
\vec{N}(p) = \frac{1}{\|\vec{\Xi}_0(\xi, \eta) \times \vec{H}_0(\xi, \eta)\|} \vec{\Xi}_0(\xi, \eta) \times \vec{H}_0(\xi, \eta)
\]

then the area form \( dA \) written with respect to the parametrization \( \Phi \) is \( h(\xi, \eta) \, d\xi \wedge d\eta \) where \( h \) is the determinant of

\[
\begin{vmatrix}
\frac{\partial x}{\partial \xi}(\xi, \eta) & \frac{\partial y}{\partial \xi}(\xi, \eta) & \frac{\partial z}{\partial \xi}(\xi, \eta) \\
\frac{\partial x}{\partial \eta}(\xi, \eta) & \frac{\partial y}{\partial \eta}(\xi, \eta) & \frac{\partial z}{\partial \eta}(\xi, \eta) \\
n_1(\xi, \eta) & n_2(\xi, \eta) & n_3(\xi, \eta)
\end{vmatrix}.
\]

where the \( n_i(\xi, \eta) \) are the components of \( \vec{N}(\xi, \eta) \).

1. Let \( \vec{u}, \vec{v} \) be vectors in \( \mathbb{R}^3 \) and \( \vec{w} = \vec{u} \times \vec{v} \). Verify that (with the obvious notation)

\[
\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \|\vec{w}\|^2.
\]

Assuming that \( \vec{u} \) and \( \vec{v} \) are linearly independent, let \( n = \frac{1}{||\vec{n}||} \vec{w} \) have components \((n_1, n_2, n_3)\). Deduce that

\[
\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = ||\vec{w}||.
\]

Finally, deduce that

\[
dA = \|\vec{\Xi}_0(\xi, \eta) \times \vec{H}_0(\xi, \eta)\| \, d\xi \wedge d\eta.
\]

(Except for the meaning of \( \wedge \) this last formula is how Calculus III books represent \( dA \).)

2. Let \( M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 1 - y^2 - z^2 = 0\} \). This surface is parametrized by

\[
\Phi(\xi, \eta) = (\xi, \sqrt{1+\xi^2} \cos \eta, \sqrt{1+\xi^2} \sin \eta).
\]

Using the notation of the set-up above, compute \( \vec{\Xi}_0, \vec{H}_0, \) and \( \vec{N} \) and find \( dA \) using Problem 1. Verify that \( \vec{H}_0 \) is orthogonal to \( \vec{\Xi}_0 \).